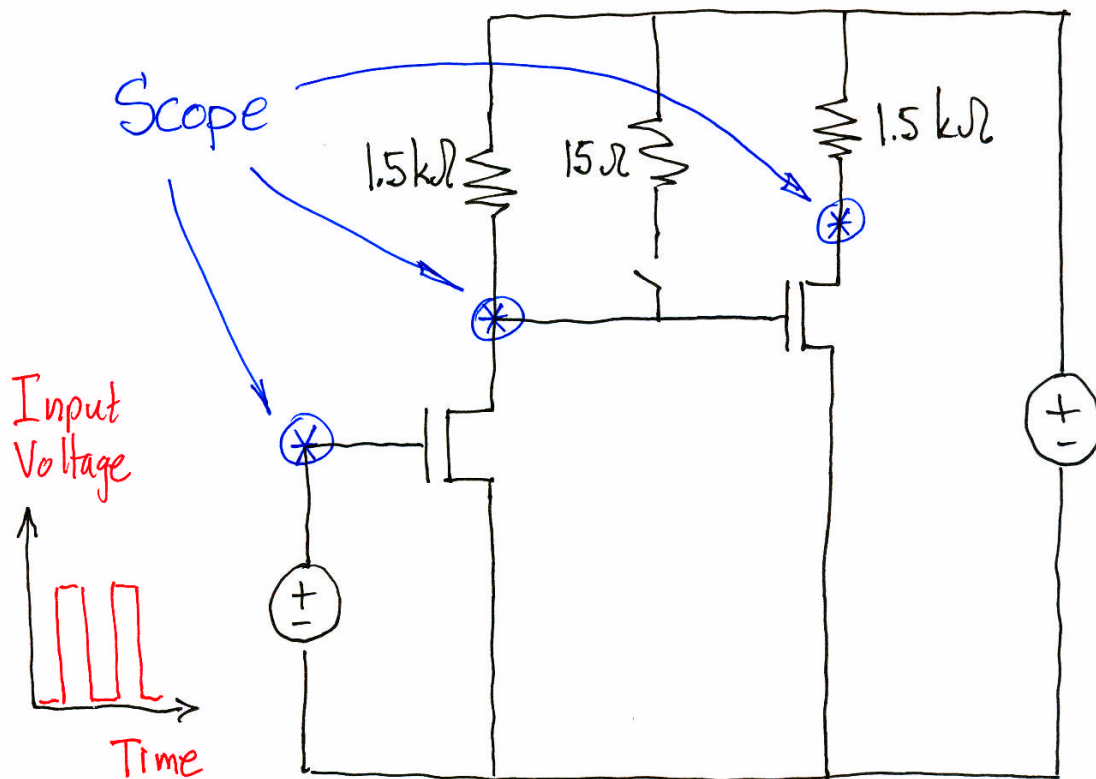


6.002 - Lecture 13

Second-Order (LC) Networks

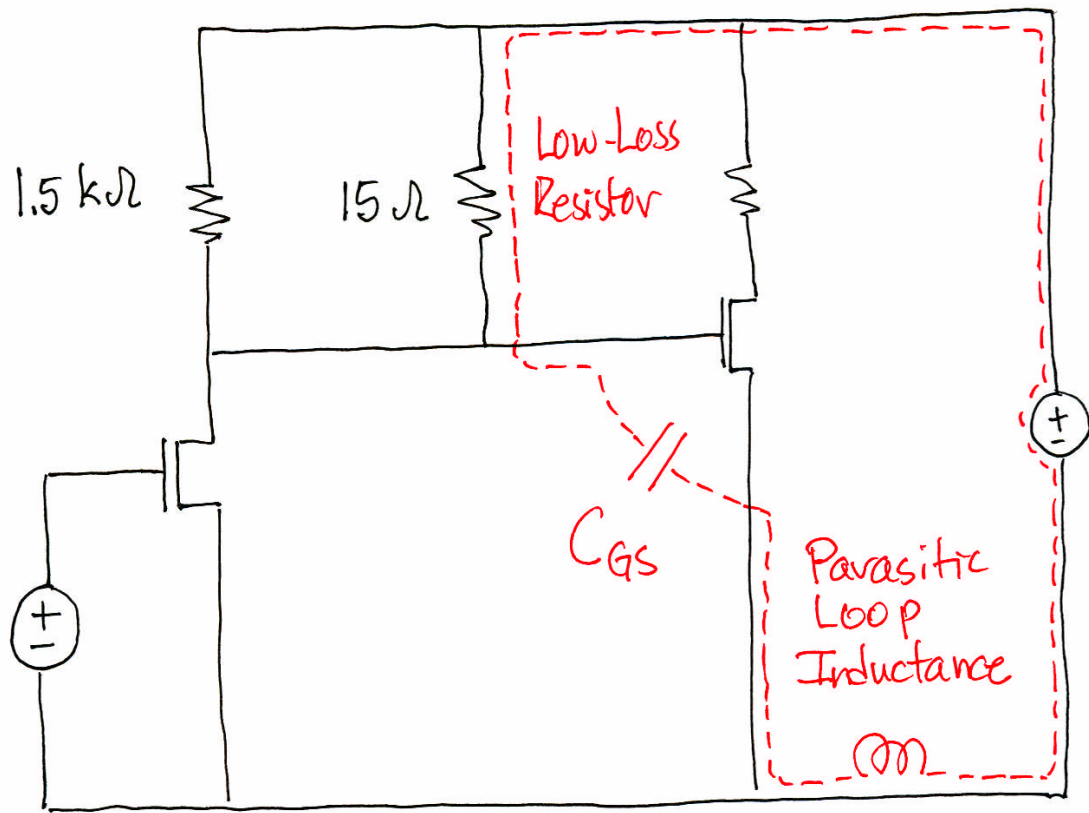
- Node Analysis
- Undamped Oscillations
- Characteristic Frequency
- Characteristic Resistance
- Energy Exchange
- Mechanical Analog

Logic Gate Waveforms



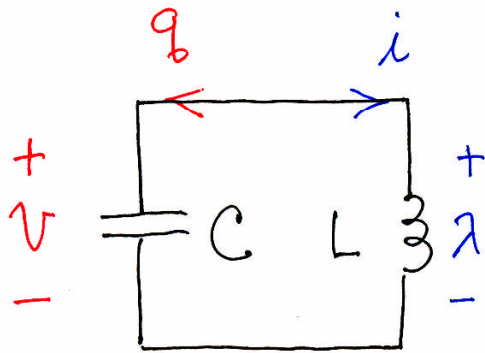
Observe voltages as resistance decreases in an effort to speed up the logic gates.

Oscillations Explained



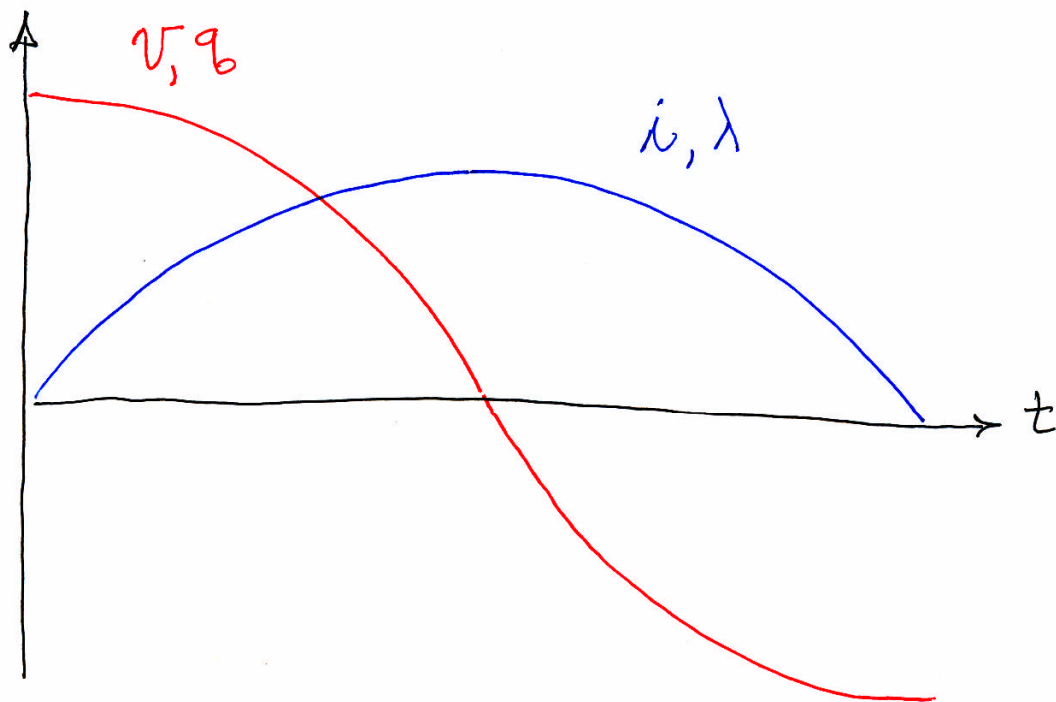
Oscillations occur as energy is exchanged between C_{gs} and the loop inductance.

LC Intuition (Half Cycle)

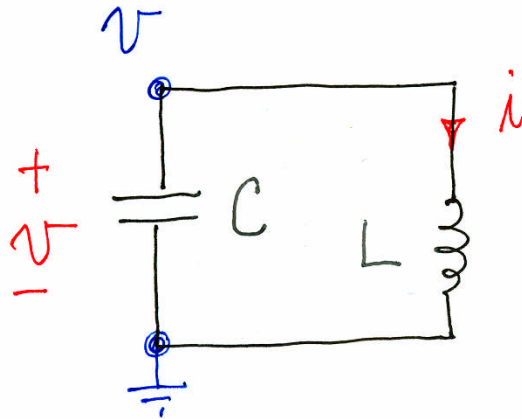


$$v = \frac{d\lambda}{dt} = L \frac{di}{dt}$$

$$i = -\frac{dq}{dt} = -C \frac{dv}{dt}$$



Transient Analysis



$$\underbrace{i}$$

$$\text{Node Method} \Rightarrow C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^t v dt = 0$$

$$\text{Time}^2 \left(LC \frac{d^2 v}{dt^2} + v = 0 \right)$$

Well-Posed Problem $\Rightarrow v(0)$ given

$\frac{dv}{dt}(0)$ given

$$\text{Also } i = -C \frac{dv}{dt}$$

Homogeneous Response I

$$LC \frac{d^2 V}{dt^2} + V = 0$$

$$\text{LCCODE} \Rightarrow V \sim e^{st}$$

$$\Rightarrow LC s^2 + 1 = 0$$

$$\Rightarrow s = \pm j/\sqrt{LC}$$

$$\Rightarrow V = A_+ e^{j t / \sqrt{LC}} + A_- e^{-j t / \sqrt{LC}}$$

$$\Rightarrow V = A_c \cos\left(\frac{t}{\sqrt{LC}}\right) + A_s \sin\left(\frac{t}{\sqrt{LC}}\right)$$

Initial Conditions $\Rightarrow A_c$ & A_s

$$\text{Frequency} = \frac{1}{\sqrt{LC}} \left[\frac{\text{rad}}{\text{s}} \right] = \frac{1}{2\pi\sqrt{LC}} \left[\text{Hz} \right]$$

Homogeneous Response II

$$\text{Initial States} \Rightarrow V(0) = V(0)$$

$$i(0) = -C \frac{dV}{dt}(0)$$

$$\text{Matching} \Rightarrow V(0) = A_c = V(0)$$

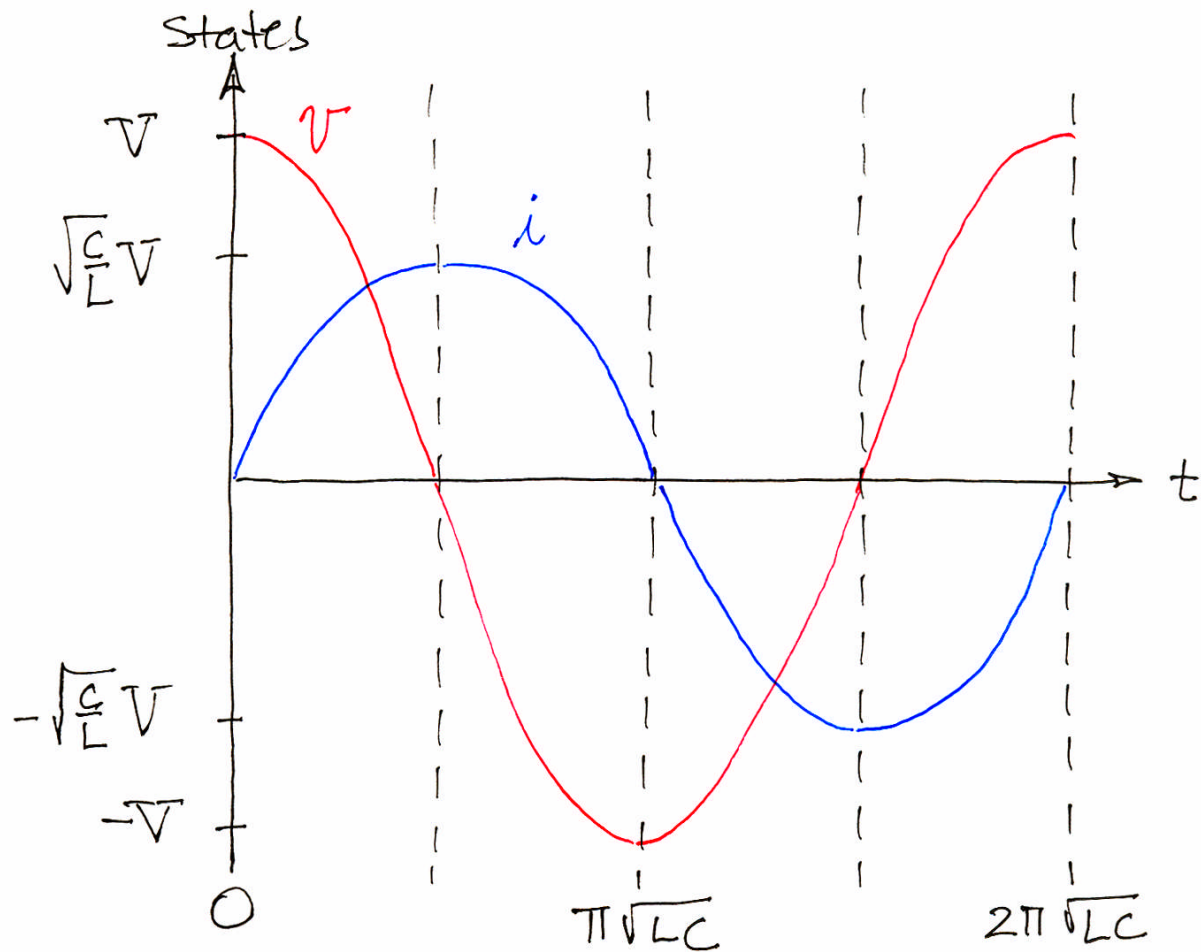
$$\frac{dV}{dt}(0) = \frac{A_s}{\sqrt{LC}} = -\frac{i(0)}{C}$$

$$V(t) = V(0) \cos\left(\frac{t}{\sqrt{LC}}\right) - \underbrace{\sqrt{\frac{L}{C}}}_{\text{Ohms, but not resistance}} i(0) \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$i(t) = -C \frac{dV(t)}{dt} = \sqrt{\frac{C}{L}} V(0) \sin\left(\frac{t}{\sqrt{LC}}\right) + i(0) \cos\left(\frac{t}{\sqrt{LC}}\right)$$

Example: Homogeneous Response

$$V(0) \equiv V \quad \& \quad i(0) \equiv 0$$



$$C = 1 \mu\text{F}$$

$$L = 100 \mu\text{H}$$

$$\sqrt{LC} = 0.32 \frac{\text{ms}}{\text{rad}}$$

$$2\pi\sqrt{LC} = 2 \frac{\text{ms}}{\text{Cycle}}$$

$$\sqrt{\frac{L}{C}} = 316 \Omega$$

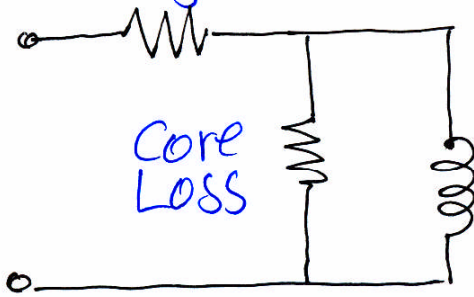
$$V_{pk} = 0.5 \text{ V}$$

$$i_{pk} = 1.6 \text{ mA}$$

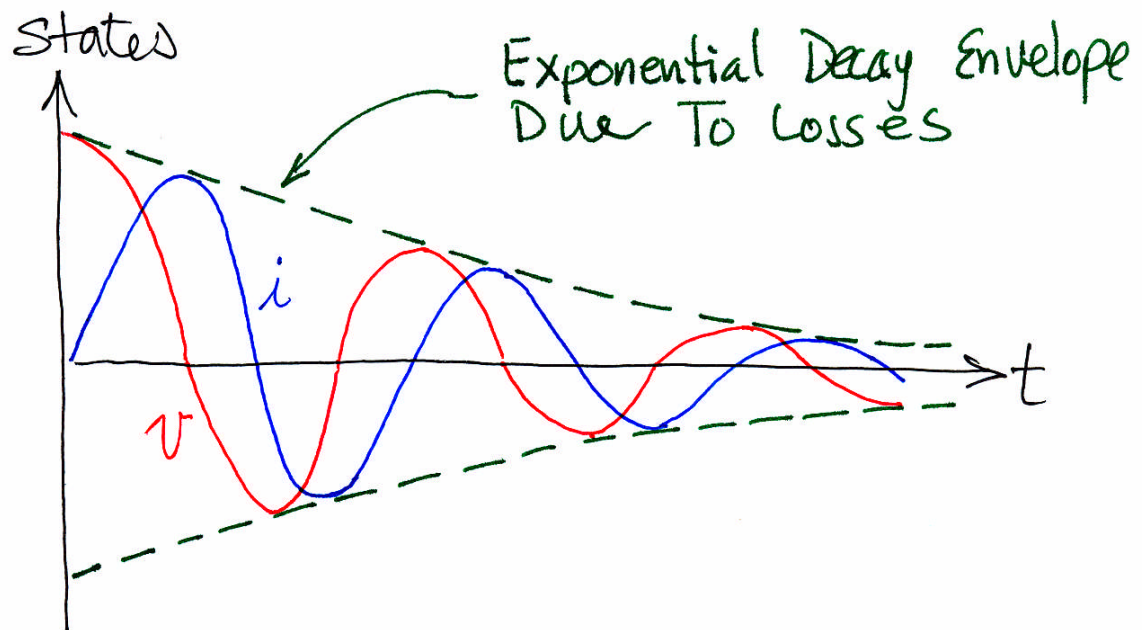
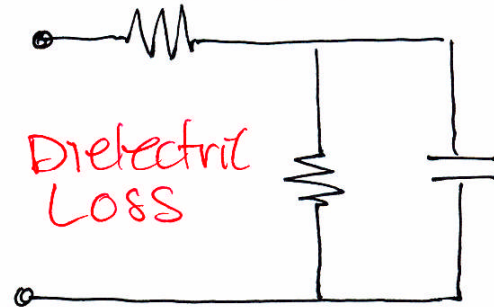
Losses

Real inductors and capacitors have losses. Simple loss models and their consequences follow.

Winding Loss



Winding (Foil) Loss

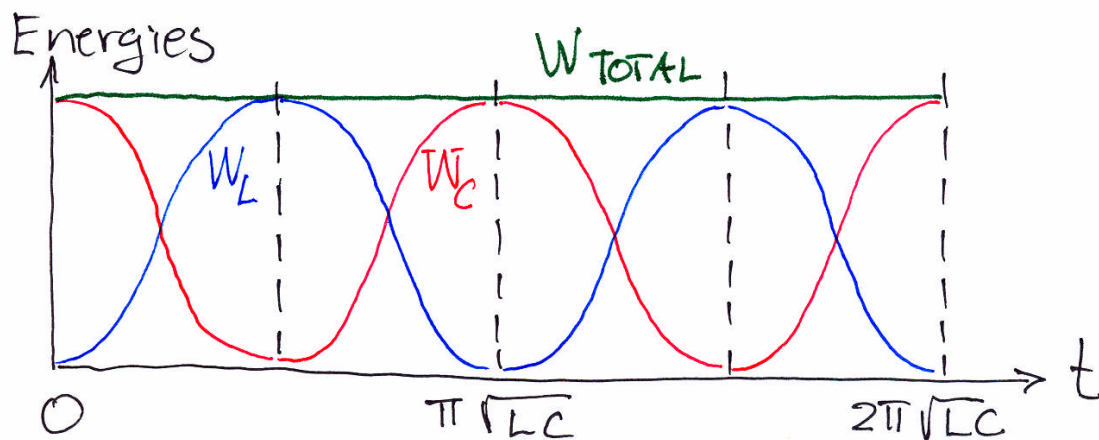
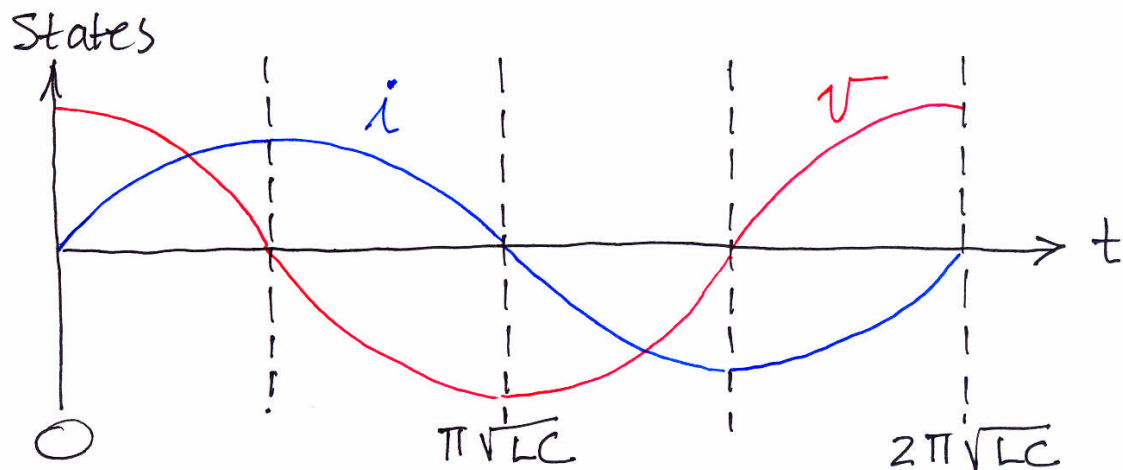


Energy

$$v = V \cos\left(\frac{t}{\sqrt{LC}}\right) \quad i = \sqrt{\frac{C}{L}} V \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$W_C = \frac{1}{2} C v^2 = \frac{C V^2}{2} \cos^2\left(\frac{t}{\sqrt{LC}}\right) \quad W_L = \frac{1}{2} L i^2 = \frac{C V^2}{2} \sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$W_{\text{TOTAL}} = W_C + W_L = \frac{1}{2} C V^2 \dots \text{Constant!}$$



Characteristic Resistance

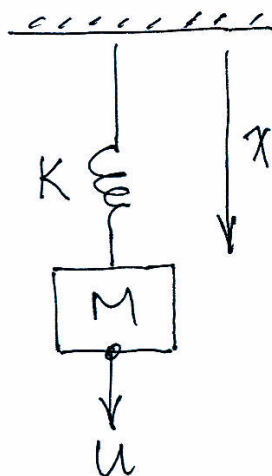
$$W_T = \frac{1}{2} C v_{Pk}^2 \quad \dots \quad \text{When } i = 0$$

$$W_T = \frac{1}{2} L i_{Pk}^2 \quad \dots \quad \text{When } v = 0$$

$$\frac{1}{2} C v_{Pk}^2 = \frac{1}{2} L i_{Pk}^2 \Rightarrow \underbrace{\sqrt{\frac{L}{C}}}_{\text{Characteristic Resistance}} = \frac{v_{Pk}}{i_{Pk}}$$

Characteristic resistance
results from an energy balance.
Not a loss !

Mechanical Example I

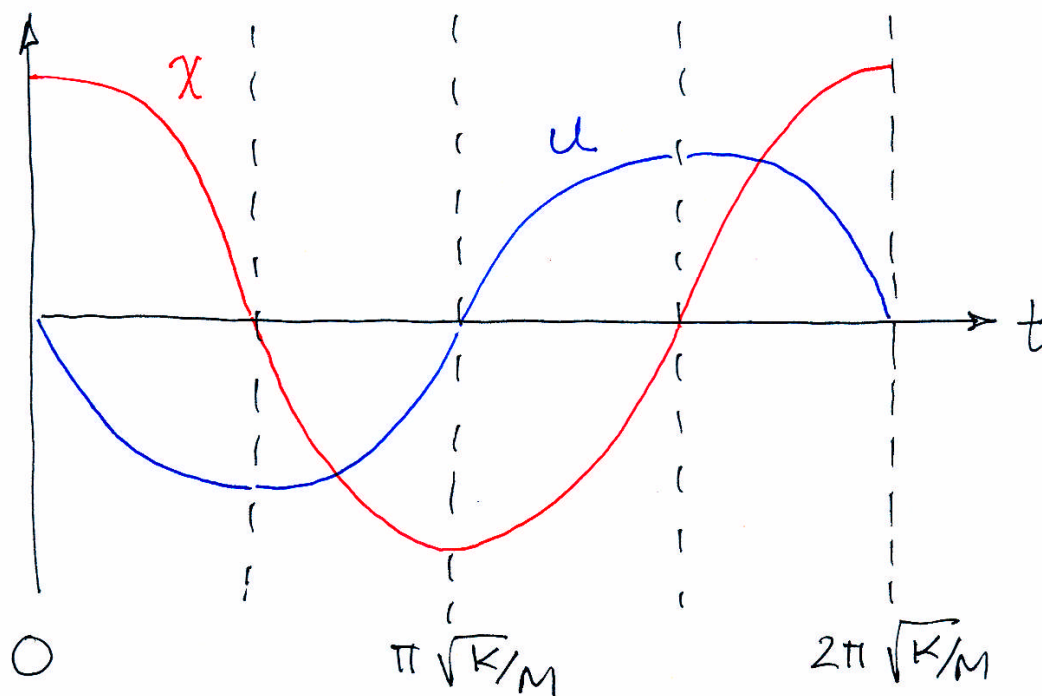


$$m \frac{du}{dt} = -kx$$

$$\frac{dx}{dt} = u$$

$$\frac{d^2x}{dt^2} + \underbrace{\frac{K}{m}}_{\text{Frequency}^2} x = 0$$

States



Mechanical Example II

$$x = X \cos\left(\sqrt{\frac{K}{M}} t\right) \quad u = -\sqrt{\frac{K}{M}} X \sin\left(\sqrt{\frac{K}{M}} t\right)$$

$$\text{Total Energy} = \underbrace{\frac{1}{2} K X^2}_{\text{Potential Energy}} + \underbrace{\frac{1}{2} M u^2}_{\text{Kinetic Energy}}$$

$$= \frac{1}{2} K X^2 \cos^2\left(\sqrt{\frac{K}{M}} t\right) + \frac{1}{2} K X^2 \sin^2\left(\sqrt{\frac{K}{M}} t\right)$$

$$= \frac{1}{2} K X^2 \dots \text{Constant!}$$

$$= \frac{1}{2} K X_{pk}^2 = \frac{1}{2} M u_{pk}^2$$

$$\underbrace{\frac{u_{pk}}{x_{pk}}}_{\text{}} = \sqrt{\frac{K}{M}}$$