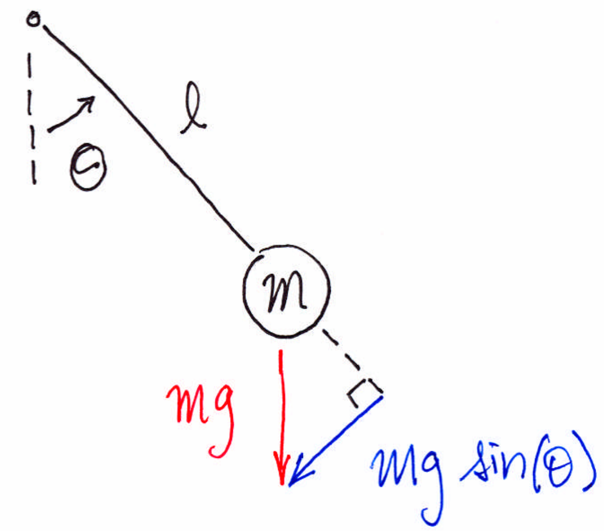
6.002 - Lecture 15

Clocks

- Mechanical clocks
- Phase - Plane Analysis
- Electrical Clocks
- LC Oscillations Can Be Good

Pendulum



Damping:

Bearing Loss

Windage

$$m\ell^2 \frac{d^2\theta}{dt^2} = -mg\ell\sin(\theta) - b\frac{d\theta}{dt}$$

$\theta \ll \frac{\pi}{2}$

$$\frac{d^2\theta}{dt^2} + \overbrace{\boxed{\frac{b}{m\ell^2}}}^{2\alpha}\frac{d\theta}{dt} + \overbrace{\boxed{\frac{g}{\ell}}}^{\omega_0^2}\theta \approx 0$$

$$\theta \sim e^{st} \Rightarrow s = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} \approx -\alpha \pm j\omega_0$$

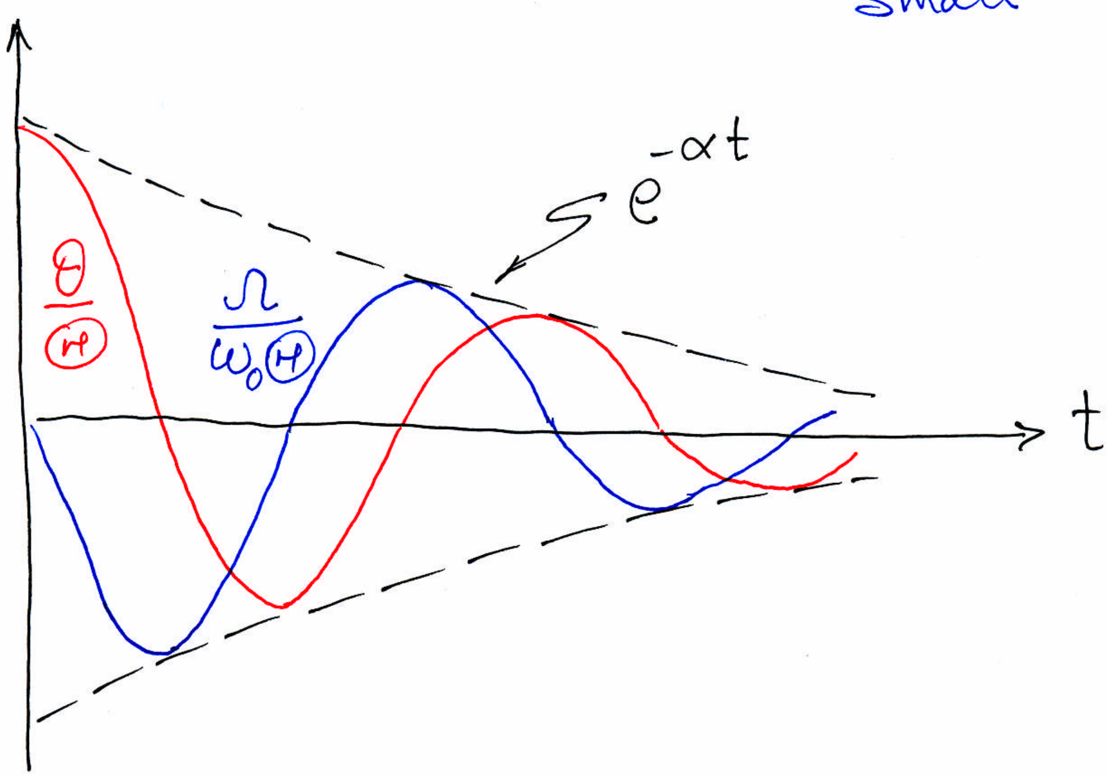
Low Loss $\Rightarrow \alpha \ll \omega_0$

Trajectories

$$\Theta(t) \approx \Theta \cos(\omega_0 t)\, e^{-\alpha t}$$

$$\begin{aligned}
\Omega(t) &\equiv \frac{d\theta(t)}{dt} \\
&\approx -\omega_0 \Theta \sin(\omega_0 t) e^{-\alpha t} - \underbrace{\alpha \Theta \cos(\omega_0 t)\, e^{-\alpha t}}_{\substack{\alpha \ll \omega_0 \\ \Downarrow \\ \text{small}}} \\
&\approx -\omega_0 \Theta \sin(\omega_0 t) e^{-\alpha t}
\end{aligned}$$

States

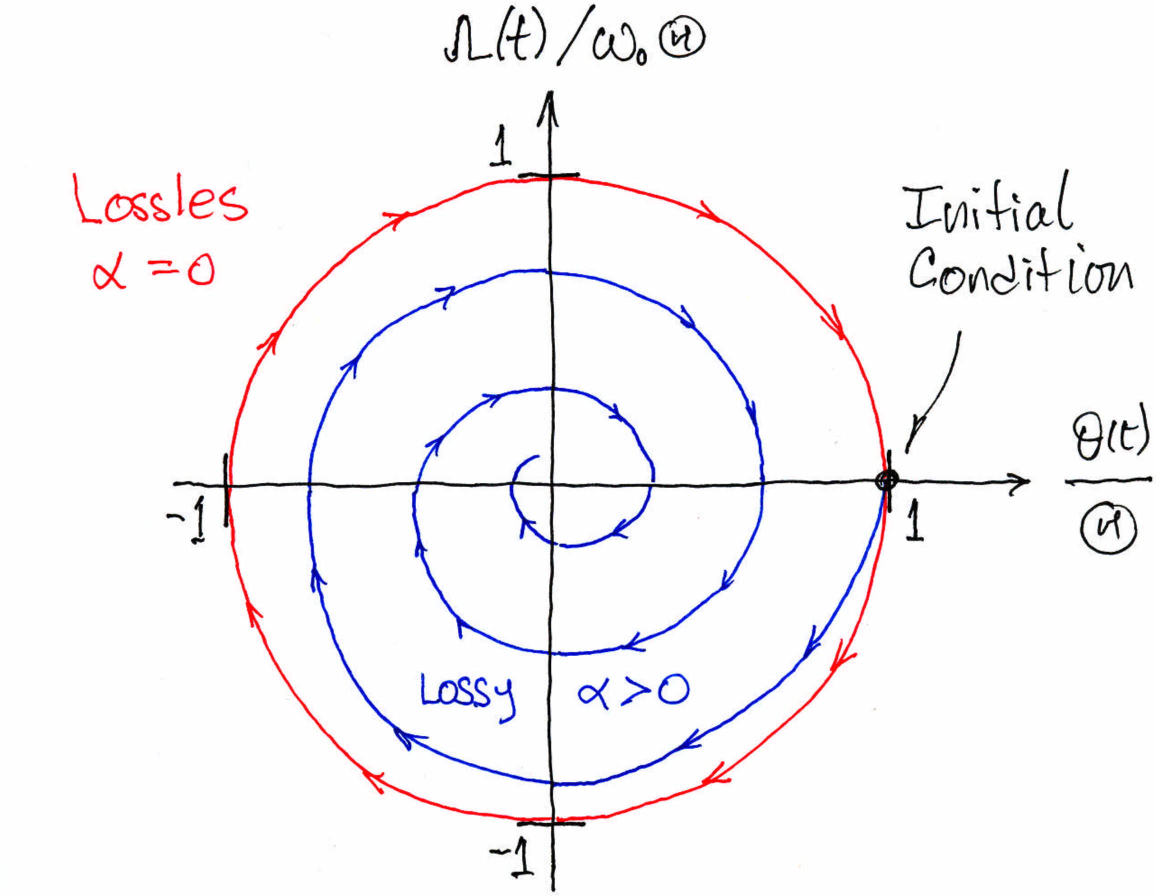


Θ-Ω Phase Plane

Phase-Plane Coordinates: $\dfrac{\theta(t)}{\textcircled{M}}$, $\dfrac{\Omega(t)}{\omega_0 \textcircled{M}}$

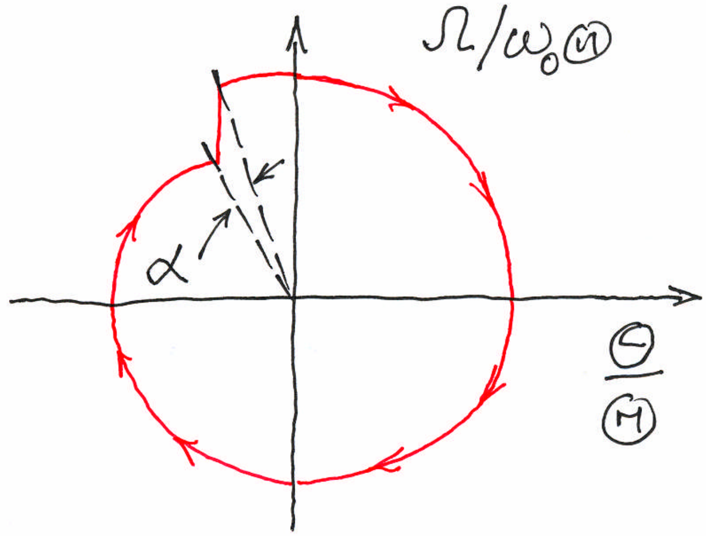
Phase-Plane Angle: $\dfrac{\Omega(t)/\omega_0 \textcircled{M}}{\theta(t)/\textcircled{M}} = \operatorname{Tan}^{-1}(-\omega t)$

Phase-Plane Radius: $\left(\dfrac{\theta(t)}{\textcircled{M}}\right)^2 + \left(\dfrac{\Omega(t)}{\omega_0 \textcircled{M}}\right)^2 = e^{-2\alpha t}$

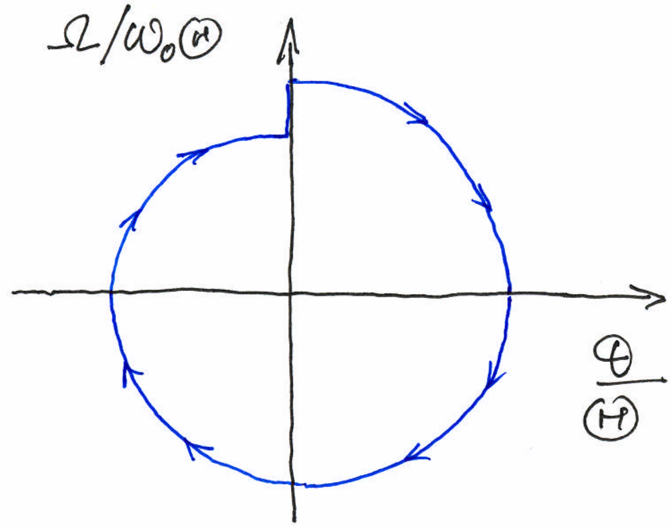


Loss Mitigatcin

Add kinetic energy (via a force impulse and a momentum step) when the pendulum velocity is at a peak.

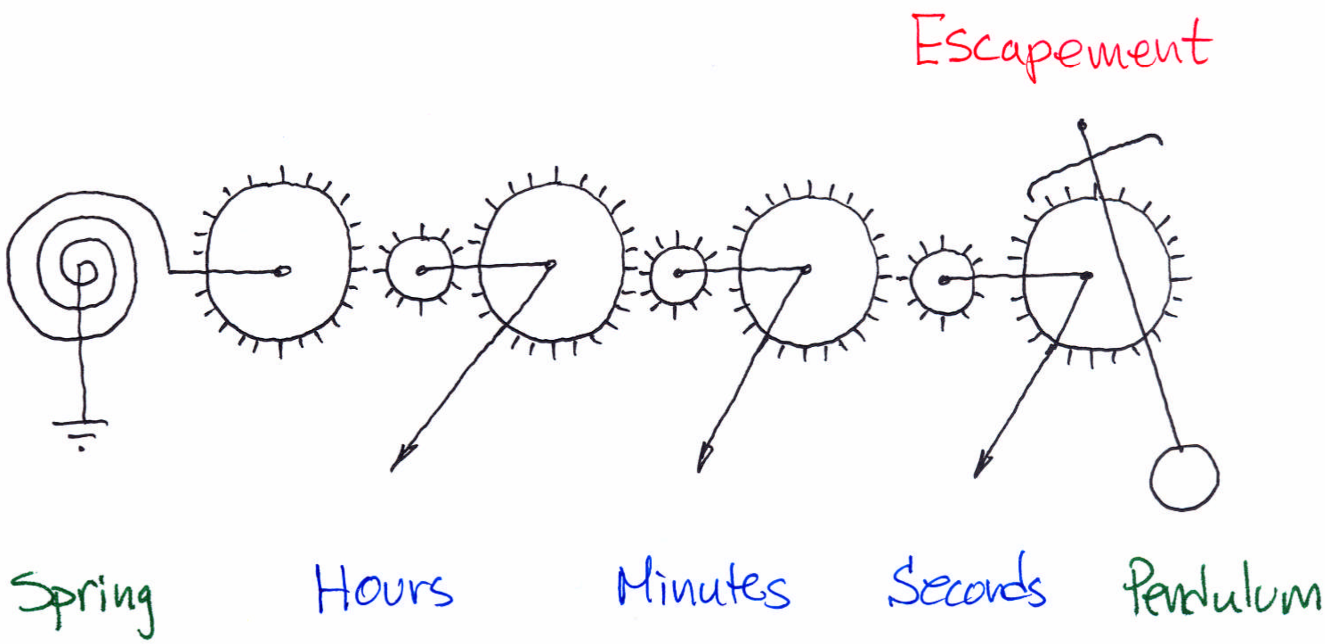


This period differs from 2π by α, which depends on the velocity step. Not good!



This period is 2π, independent of the velocity step. Good!

Escapement Mechanism

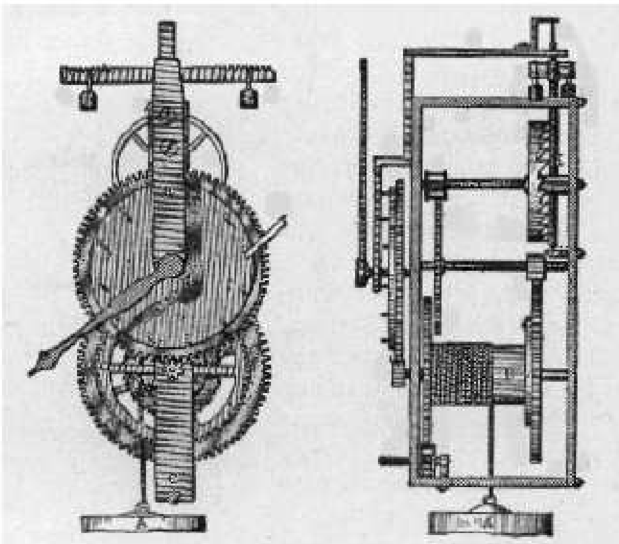


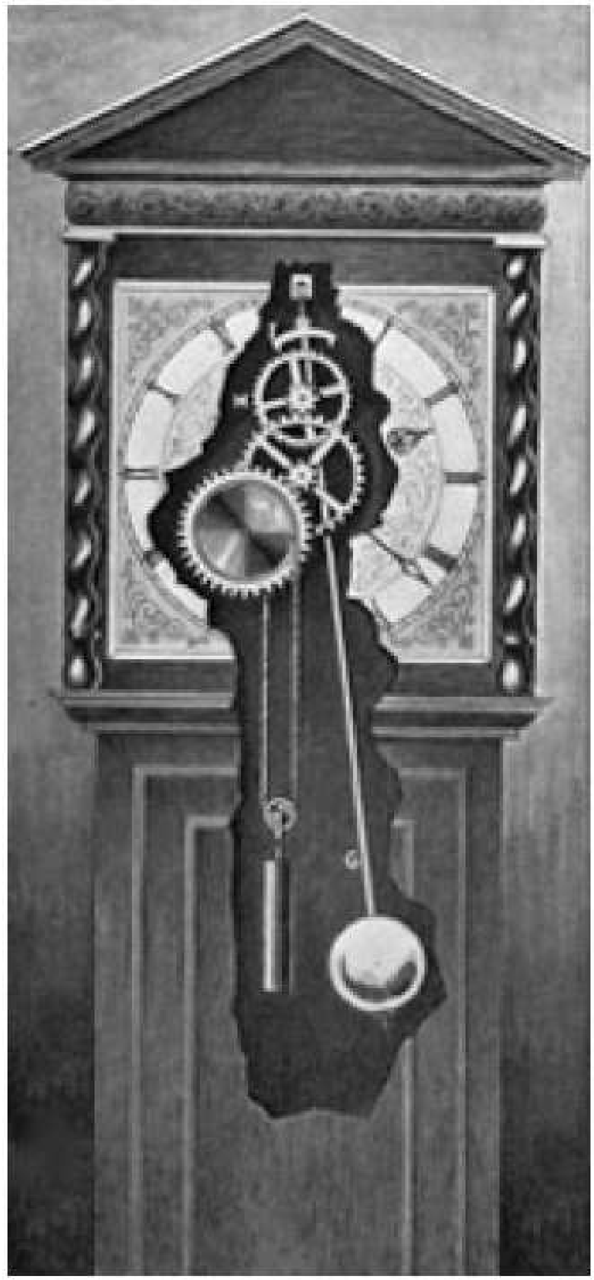
Escapement mechanism:

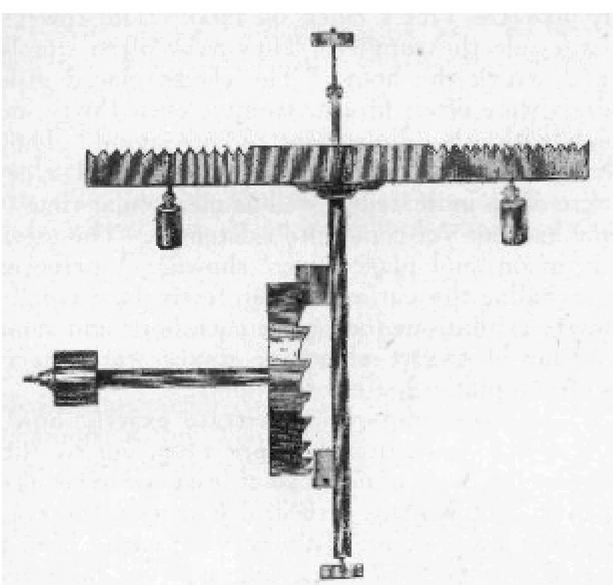
(1) synchronizes gear motion to the pendulum;

(2) delivers a force impulse to the pendulum when $\theta = 0$.

Pendulum Glocks



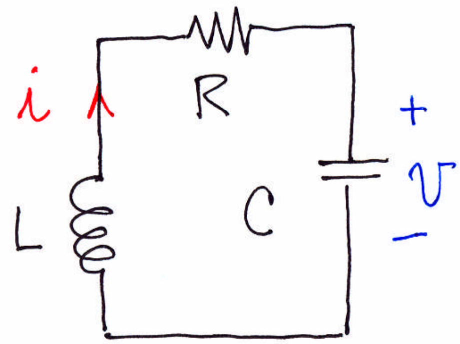




Christiaan Huygens (1629 - 1695)

- Mathematician
- Astronomer
- Engineer
- Inventor Of Pendulum Clock

LC Clock I



$$L\frac{di}{dt} + Ri + v = 0$$

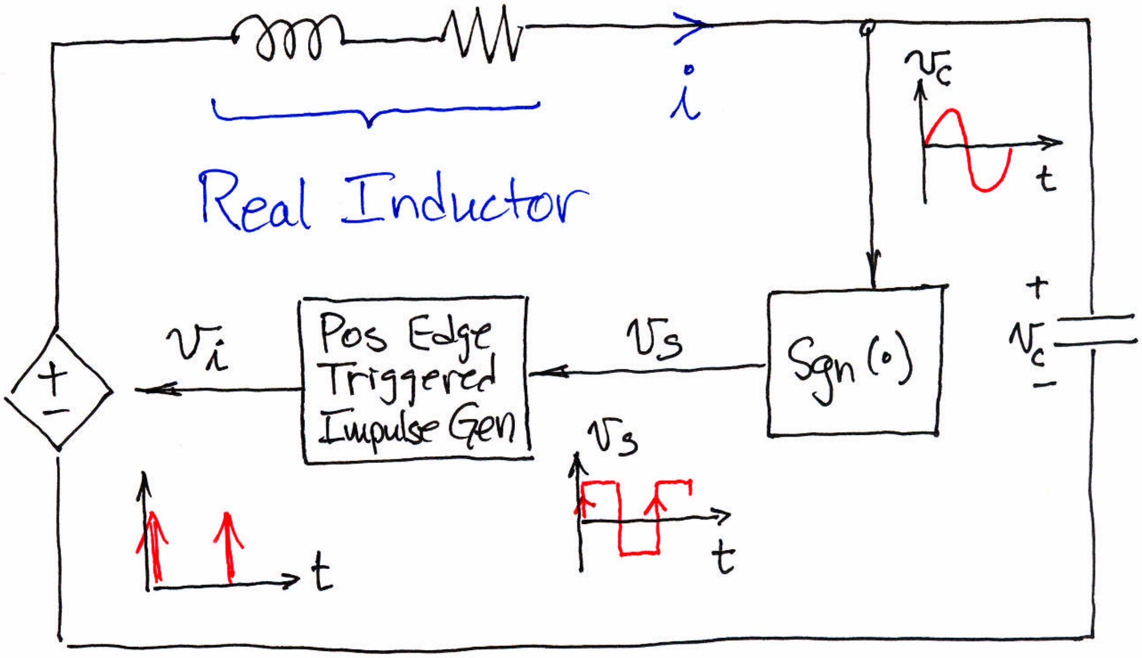
$$i = C\frac{dv}{dt}$$

$$\frac{d^2 v(t)}{dt^2} + \underbrace{\boxed{\frac{R}{L}}}_{2\alpha}\frac{dv(t)}{dt} + \underbrace{\boxed{\frac{1}{LC}}}_{\omega_0^2} v(t) = 0$$

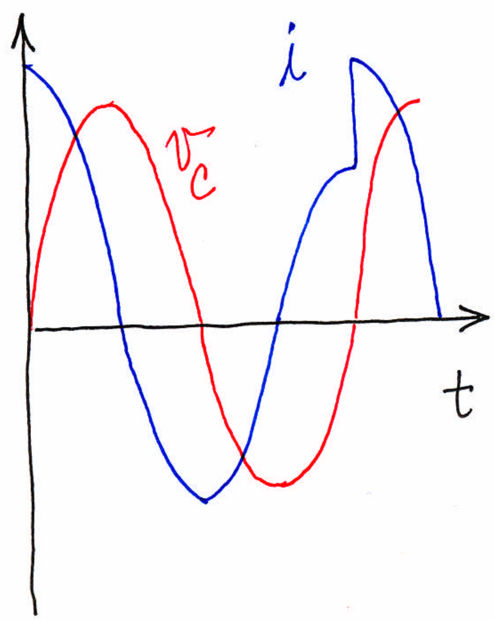
$$R \ll \sqrt{\frac{L}{C}} \Rightarrow \alpha \ll \omega_0 \Rightarrow e^{st},\ s \approx \alpha \pm j\omega_0$$

$$\left.\begin{aligned} v(t) &\approx V_0 \sin(\omega_0 t)\, e^{-\alpha t} \\ i(t) &\approx \frac{V_0}{\sqrt{LC}} \cos(\omega_0 t)\, e^{-\alpha t} \end{aligned}\right\} \text{Similar to the pendulum!}$$

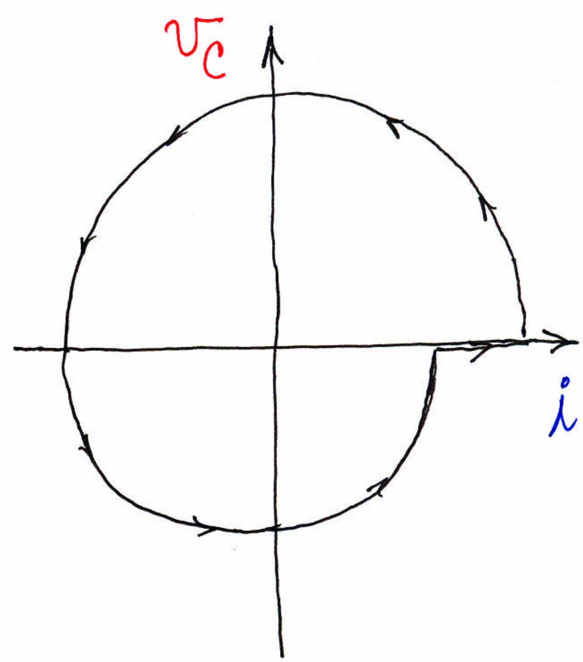
LC Clock II



i
Real Inductor
v_c
t
$+$
v_c
$-$
Sgn (·)
v_s
Pos Edge Triggered Impulse Gen
v_i
t



i
v_c
t



v_c
i

LC Clock III

Suppose voltage impulse has area Λ.

$$i(t) \approx I \cos(\omega_0 t)\, e^{-\alpha t} \qquad 0 \le t < \frac{2\pi}{\omega}$$

$$i\left(\frac{2\pi}{\omega_0}^{-}\right) = I\, e^{-\frac{2\pi\alpha}{\omega_0}}$$

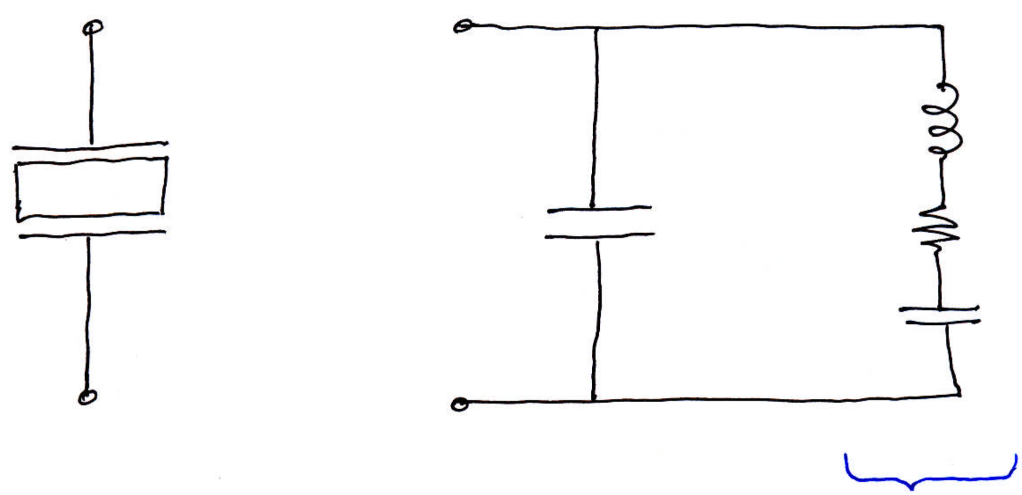
$$i\left(\frac{2\pi}{\omega_0}^{+}\right) = i\left(\frac{2\pi}{\omega_0}^{-}\right) + \frac{\Lambda}{L}$$

$$I = \frac{\Lambda / L}{1 - e^{-\frac{2\pi\alpha}{\omega_0}}}$$

The amplitude I falls with Λ, perhaps due to decaying power supply, but the frequency remains $\approx \omega_0$.

LC Clock IV

Use a quartz crystal as a very-low-loss resonator.



Mechanical resonator with piezoelectric electromechanical coupling.