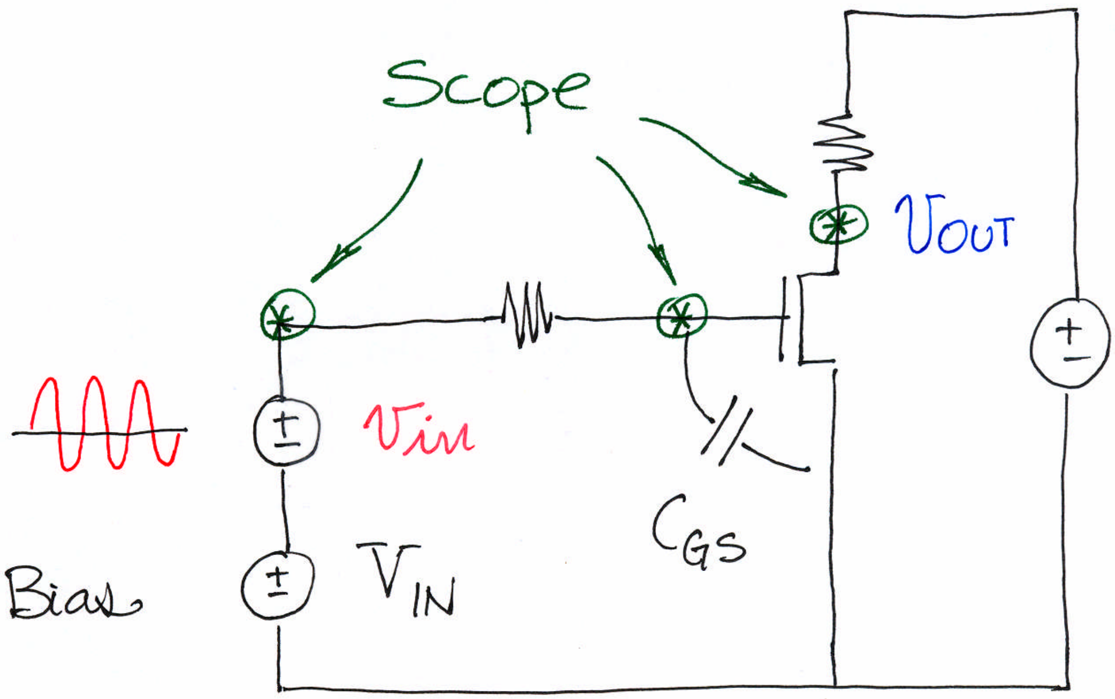
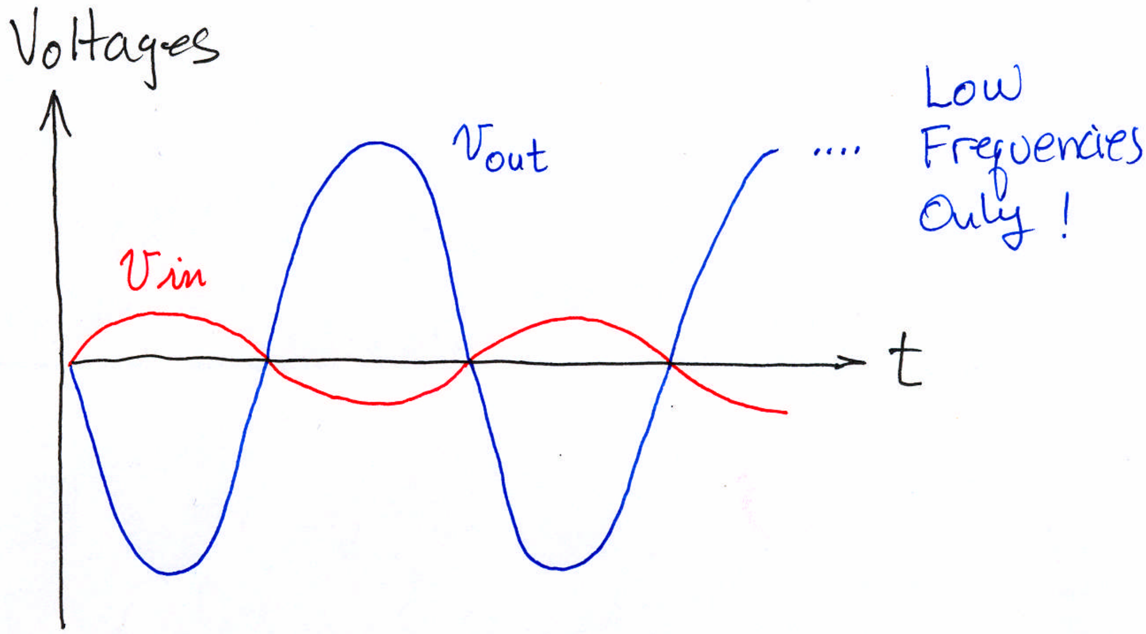
6.002 - Lecture 16

Sinusoidal Steady State

- MOSFET Amplifier ... Again
- Sinusoidal Steady State
- Analysis ... The Direct/Hard Way
- Analysis ... Easier Way Via $e^{j\omega t}$
- Frequency Response

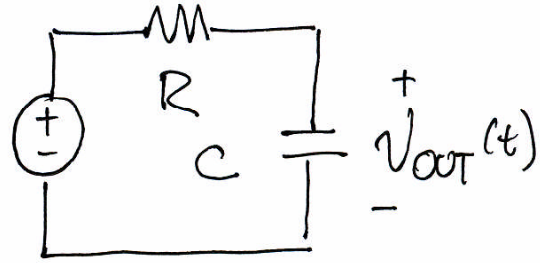
MOSFET Amplifier





SSS Analysis I

$$v_{IN}(t) = V_{in}\cos(\omega t)\,u(t)$$



$$RC\frac{dv_{OUT}}{dt} + v_{OUT} = v_{IN} \quad \& \quad v_{OUT}(0) \equiv 0$$

$$v_{OUT} = v_{\text{Particular}}(t) + v_{\text{Homogeneous}}(t)$$

$$= V_{out}\cos(\omega t + \phi) - \underbrace{V_{out}\cos(\phi)\,e^{-\frac{t}{RC}}}_{\text{Decays Away}}$$

Amplitude Phase

SSS Frequency Response

SSS Frequency Response

- Often used to characterize a linear system.
- Response to a pure tone: amplitude and phase as functions of frequency.
- Frequency domain viewpoint of signals and systems.
- More complex signals can be made by superimposing pure tones. The responses may be superimposed too!

SSS Analysis II

$$RC\frac{dV_{OUT}}{dt} + V_{OUT} = V_{in}\cos(\omega t)$$

$$V_{OUT} = V_{out}\cos(\omega t + \phi)$$

Linear System

Substitute ...

$$-RC\omega V_{out}\sin(\omega t+\phi) + V_{out}\cos(\omega t+\phi) = V_{in}\cos(\omega t)$$

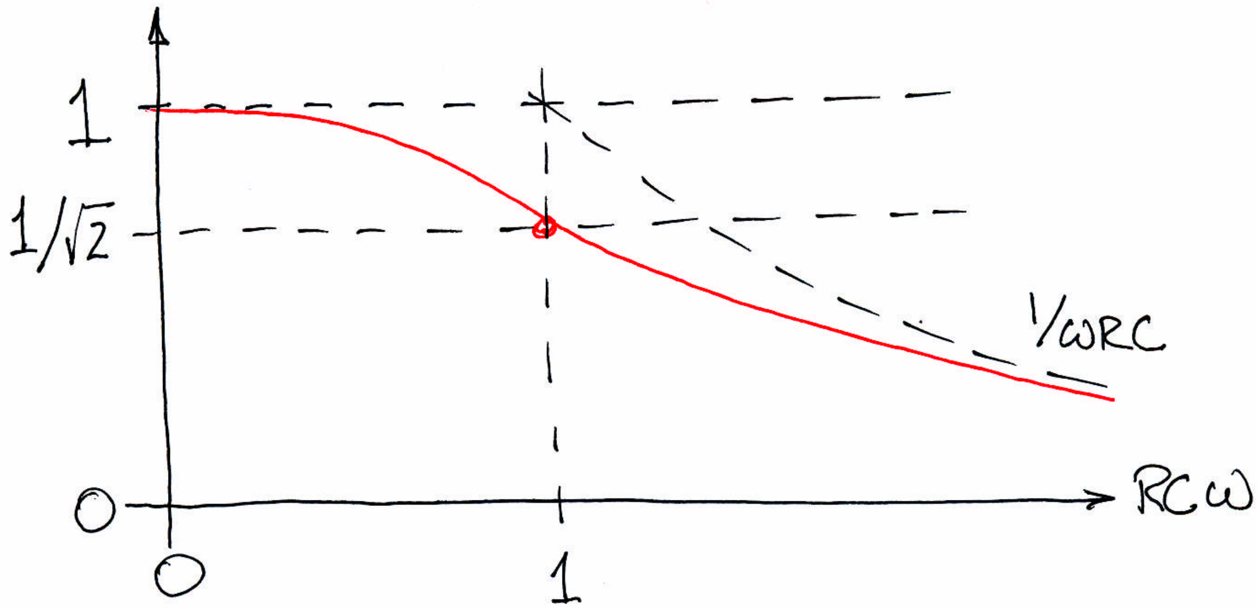
$$V_{out}\sqrt{1+R^2C^2\omega^2}\left[\underbrace{\frac{1}{\sqrt{1+R^2C^2\omega^2}}}_{\cos(\alpha)}\cos(\omega t+\phi) - \underbrace{\frac{1}{\sqrt{1+R^2C^2\omega^2}}}_{\sin(\alpha)}\sin(\omega t+\phi)\right] = V_{in}\cos(\omega t)$$

$$V_{out}\sqrt{1+R^2C^2\omega^2}\cos(\omega t+\phi+\alpha) = V_{in}\cos(\omega t) \quad , \quad \tan(\alpha) \equiv RC\omega$$

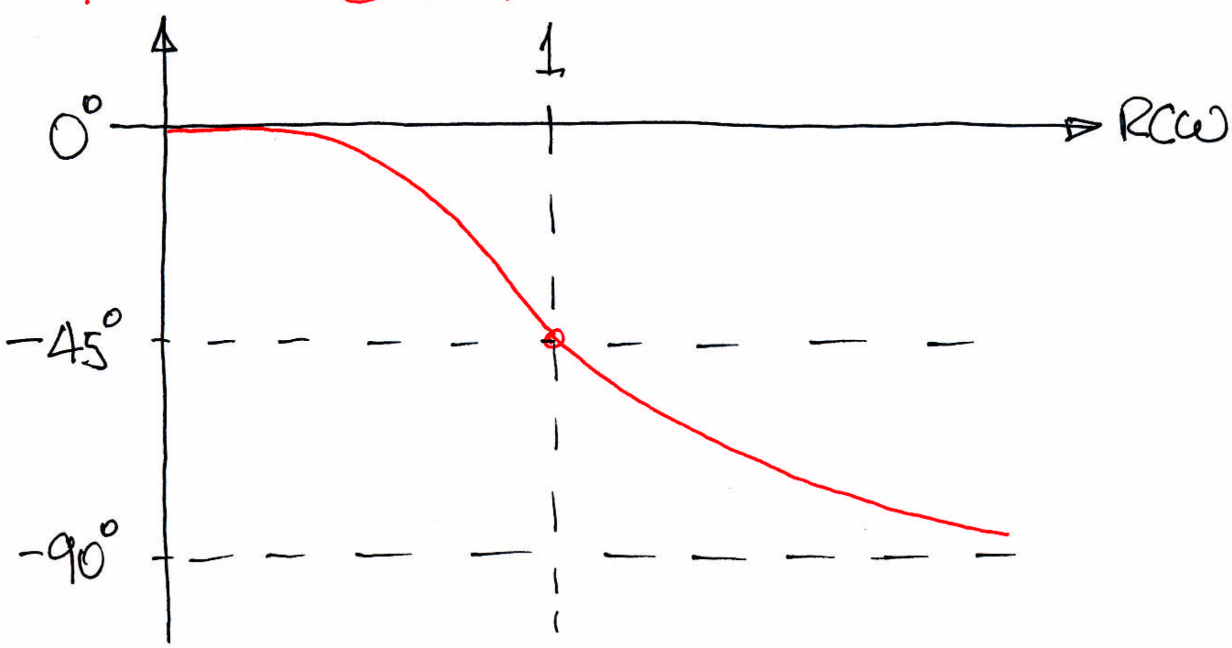
$$V_{out} = \frac{V_{in}}{\sqrt{1+R^2C^2\omega^2}} \qquad \phi = -\alpha = -\tan^{-1}(RC\omega)$$

Frequency Response

$$\left|\frac{V_{out}}{V_{in}}\right| \equiv \text{Gain}$$

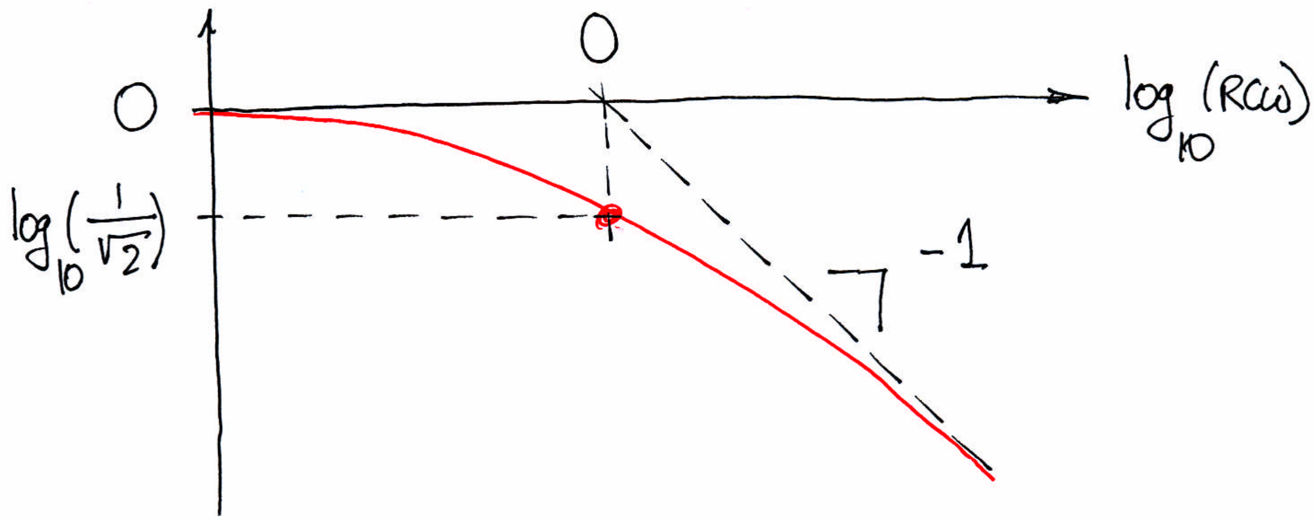


$$\phi \equiv \text{Phase Shift}$$

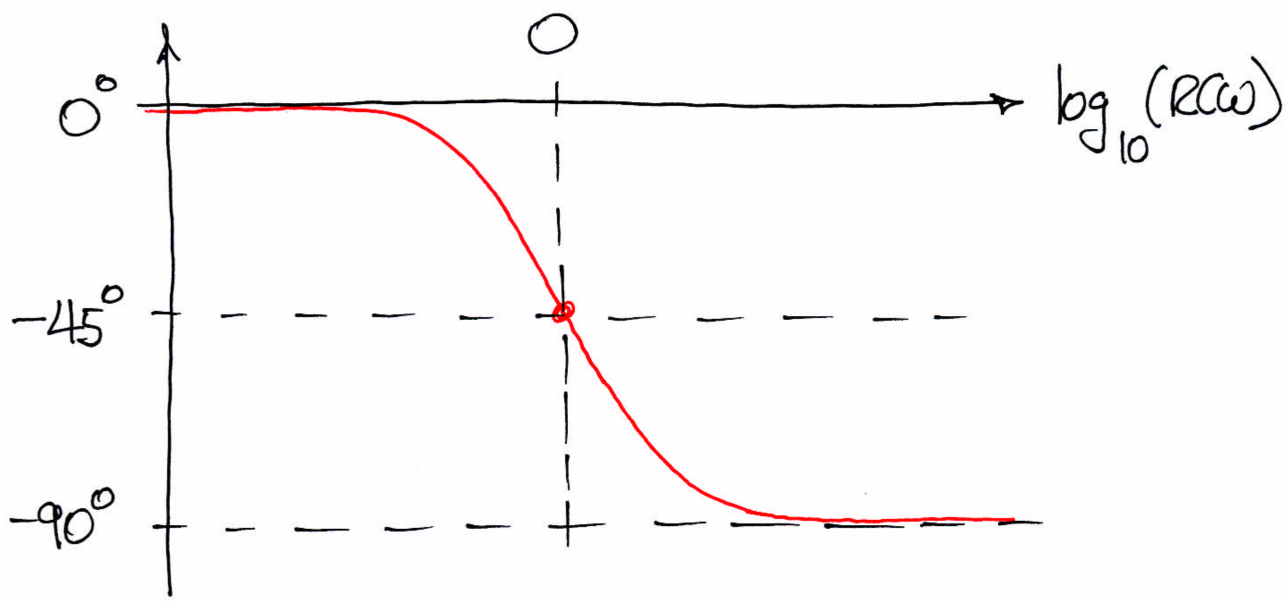


Frequency Response

$$\log_{10}\left|\frac{V_{out}}{V_{in}}\right| \equiv \log_{10}(\text{Gain})$$

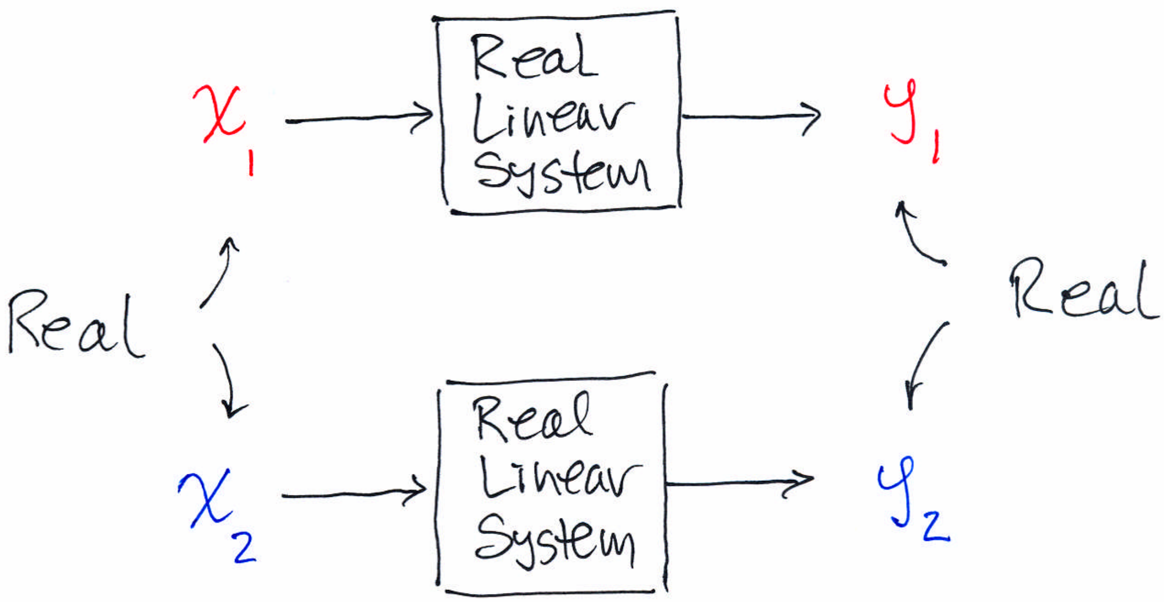


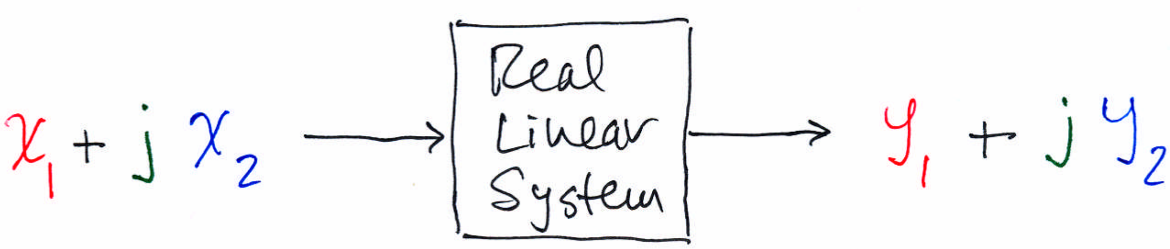
$\Phi \equiv$ Phase Shift



Linearity

Linearity $\Rightarrow$ Homogeneity & Superposition





Complex Exponentials

$$\underbrace{A\cos(\omega t) + jA\sin(\omega t)}_{Ae^{j\omega t}} \longrightarrow \boxed{\begin{array}{l}\text{Real}\\ \text{Linear}\\ \text{System}\end{array}} \longrightarrow Y_1 + jY_2 \equiv Y$$

- $A\cos(\omega t) \longrightarrow Y_1 = \text{Real}\{Y\}$
- $A\sin(\omega t) \longrightarrow Y_2 = \text{Imag}\{Y\}$
- $\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t} \Rightarrow$ No trig identities !
- Sets foundation for impedance

SSS Analysis III

$$RC\frac{dv_{OUT}}{dt} + v_{OUT} = \bar{V}_{in}\left[\cos(\omega t) + j\sin(\omega t)\right] = \bar{V}_{in}e^{j\omega t}$$

$$v_{IN} = \bar{V}_{in}e^{j\omega t} \Rightarrow v_{OUT} = \tilde{\bar{V}}_{out}e^{j\omega t}$$

Substitution $\Rightarrow$ $(j\omega RC + 1)\tilde{\bar{V}}_{out} = \bar{V}_{in}$

$$\tilde{\bar{V}}_{out} = \frac{\bar{V}_{in}}{1 + j\omega RC}$$

$$= \left|\tilde{\bar{V}}_{out}\right| e^{j\measuredangle \tilde{\bar{V}}_{out}}$$

$$v_{OUT} = \left|\tilde{\bar{V}}_{out}\right| e^{j(\omega t + \measuredangle \tilde{\bar{V}}_{out})}$$

SSS Analysis IV

$$V_{OUT} = |\tilde{V}_{out}|\, e^{j(\omega t + \measuredangle \tilde{V}_{out})}$$

$$= |\tilde{V}_{out}| \left(\cos(\omega t + \measuredangle \tilde{V}_{out}) + j \sin(\cdot) \right)$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

Amplitude V_{out} Phase ϕ

$$V_{out} = |\tilde{V}_{out}| = \sqrt{\tilde{V}_{out} \tilde{V}_{out}^{*}} = \frac{V_{in}}{\sqrt{1+\omega^2 R^2 C^2}}$$

$$\phi = \measuredangle \tilde{V}_{out} = -\tan^{-1}(\omega RC)$$

=

$$\measuredangle\, \alpha + j\beta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) \qquad \measuredangle \frac{1}{\alpha + j\beta} = -\tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

SSS Analysis Summary

$$RC\frac{dV_{OUT}}{dt} + V_{OUT} = \bar{V}_{in}e^{j\omega t} \Rightarrow V_{OUT} = \tilde{\bar{V}}_{out}e^{j\omega t}$$

$$RCj\omega\tilde{\bar{V}}_{out} + \tilde{\bar{V}}_{out} = \bar{V}_{in} \Rightarrow \tilde{\bar{V}}_{out} = \frac{\bar{V}_{in}}{1+j\omega RC}$$

Amplitude: $$V_{out} = \left|\tilde{\bar{V}}_{out}\right| = \frac{\bar{V}_{in}}{\sqrt{1+\omega^2R^2C^2}}$$

Phase: $$\phi = \measuredangle\tilde{V}_{out} = -\tan^{-1}(\omega RC)$$

SSS Frequency Response