Stochastic Programming I
Linear Programming

Berk Ustun
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ESD.862 To Date

\[ V_t(s_t) = \max_{a_t} \{ r_t(s_t, a_t) + E_{w_t}[V_{t+1}(s_{t+1})] \} \]

\[ \pi_t(s_t) = \arg\max_{a_t} \{ r_t(s_t, a_t) + E_{w_t}[V_{t+1}(s_{t+1})] \} \]
Approximate with ADP or Infinite Horizon.

\[
V_t(s_t) = \max_{a_t} \{ r_t(s_t, a_t) + E_{w_t}[V_{t+1}(s_{t+1})] \}
\]

\[
\pi_t(s_t) = \arg\max_{a_t} \{ r_t(s_t, a_t) + E_{w_t}[V_{t+1}(s_{t+1})] \}
\]

Estimate using Monte Carlo Methods.
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Optimization Problems

\[ V_t(s_t) = \max_{a_t} \{ r_t(s_t, a_t) + E_{w_t} [V_{t+1}(s_{t+1})] \} \]

\[ \pi_t(s_t) = \arg\max_{a_t} \{ r_t(s_t, a_t) + E_{w_t} [V_{t+1}(s_{t+1})] \} \]
ESD.862 for the Next 3 Lectures

Decision Making Problem at $t-1$ → Optimal Decision at $t-1$ → Decision Making Problem at $t$ → Optimal Decision at $t$ → Decision Making Problem at $t+1$
ESD.862 for the Next 3 Lectures

Model these as **Constrained Optimization Problems**

**Sequentiality:** dependence on previous stage decision  
**Uncertainty:** parameters are uncertain with a probability distribution
Decision Making Problem at $t-1$

Optimal Decision at $t-1$

Decision Making Problem at $t$

Optimal Decision at $t$

Decision Making Problem at $t+1$

Decision at stage $t$: $x_t$

Uncertainty at stage $t$: $\omega_t$

Cost at stage $t$: $z_t$

Decision making problem at stage $t$:

$$\min_{x_t} z_t = f_t(x_t, \omega_t, x_{t-1})$$
Upcoming Lectures

**Linear Programming**
Formulating and solving static decision making problems

**Two Stage Stochastic Programming**
Formulating and solving sequential decision making problems under uncertainty

**Multistage Stochastic Programming**
Formulating and solving multistage decision making problems under uncertainty using sampling
Mathematical Programming

Formulating and Solving Constrained Optimization Problems

$$\min_{x} \quad f(x)$$
$$\text{st.} \quad g(x) \geq 0$$
$$x \in X$$
Mathematical Programming

Formulating and Solving Constrained Optimization Problems

Find a decision $x$

$$
\min_{x} f(x) \\
\text{st.} \
g(x) \geq 0 \\
x \in X
$$
Mathematical Programming

Formulating and Solving Constrained Optimization Problems

Find a decision $x$

$$\min_{x} f(x)$$

$$\text{st. } g(x) \geq 0$$

$x \in X$

Minimizes a cost $f(x)$
Mathematical Programming

Formulating and Solving Constrained Optimization Problems

\[
\begin{align*}
\text{min} & \quad f(x) \\
\text{st.} & \quad g(x) \geq 0 \\
& \quad x \in X
\end{align*}
\]

Find a decision \( x \)

Minimizes a cost \( f(x) \)

Such that constraints \( g(x) \geq 0 \) are valid
Linear Programming

$f(x)$ and $g(x)$ are linear functions of $x$

$X$ is convex

\[
\begin{align*}
\min_{x} & \quad f(x) \\
\text{st.} & \quad g(x) \geq 0 \\
& \quad x \in X
\end{align*}
\]
Linear Programming Framework

Standard Form for \( n \) variables and \( m \) constraints

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{st.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]
Linear Programming Framework

Standard Form for $n$ variables and $m$ constraints

$$\min_{x} \quad c^T x$$
$$st. \quad Ax \geq b$$
$$x \geq 0$$

$(n \times 1)$
Linear Programming Framework

Standard Form for $n$ variables and $m$ constraints

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{st.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]
Linear Programming Framework

Standard Form for \( n \) variables and \( m \) constraints

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{st.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]
LP Example: Advertising Problem

- A company has started an advertising campaign and has to decide how many commercials spots to purchase during comedy shows and football games.

- Each comedy show is watched by 7M women and 2M men.
- Each football game is watched by 2M women and 12M men.

- Spots during comedy shows cost $50’000
- Spots during football games cost $100’000

- Ads need to be seen by at least 28M women and 24M men.
Advertising Problem LP

\[ \min_{x_1, x_2} \quad z = 50x_1 + 100x_2 \]

s.t. \[ 7x_1 + 2x_2 \geq 28 \]
\[ 2x_1 + 12x_2 \geq 24 \]
\[ x_1, \quad x_2 \geq 0 \]

\( x_1 \): # spots purchased during comedy shows
\( x_2 \): # spots purchased during football games
Advertising Problem LP

\[ \min_{x} \; z = \; c^{T} \; x \]

\[ A \; x \; \geq \; b \]

\[ x \; \geq \; 0 \]
Advertising Problem LP

\[
\begin{align*}
\min_{x} z &= \quad c^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\geq b \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\geq 0
\end{align*}
\]
Advertising Problem LP

$$\min_{x} z = \begin{bmatrix} c^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 28 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0$$
Advertising Problem LP

\[
\min_{x_1,x_2} z = \begin{bmatrix} 50 & 100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\begin{bmatrix} 7 & 2 \\ 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 28 \\ 24 \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
Advertising Problem Solution

• $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 3.6 \\ 1.4 \end{bmatrix}$

• $z^* = c^T x^*$
  
  $= [50 \ 100] \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$
  
  $= 320$

  $= 320'000$
Transportation Problem

• $i = 1 \ldots M$ Factories
• $j = 1 \ldots N$ Stores
• $s_i$ units produced at each factory $i$
• $d_j$ units demanded at each store $j$
• $c_{ij} =$ cost to ship from factory $i$ to store $j$
• Minimize shipping costs required to meet the demand at each store
Transportation Problem LP

\[
\min_{x_{ij}} \quad z = \sum_i \sum_j c_{ij} x_{ij}
\]

subject to

\[
\sum_i x_{ij} \geq d_j
\]

\[
\sum_j x_{ij} \leq s_i
\]

\[
x_{ij} \geq 0
\]
Transportation Problem LP

4M

1

$2

2

$1

$2

3

$3

10M

25M

25M

Transportation Problem LP

4M

1

$2

2

$1

$2

3

$3

10M

25M

Transportation Problem LP

4M

1

$2

2

$1

$2

3

$3

10M

25M
Transportation Problem LP

\[
\begin{align*}
\min_{x_{ij}} \quad & z = 2x_{11} + x_{21} + 2x_{22} + 3x_{32} \\
\text{st.} \quad & x_{11} + x_{21} \quad \geq 10 \\
& x_{11} \quad \leq 4 \\
& x_{21} + x_{22} \quad \leq 20 \\
& x_{32} \quad \leq 50 \\
& x_{ij} \quad \geq 0
\end{align*}
\]
Transportation Problem LP

\[
\min_{x_{ij}} \quad z = \quad 2x_{11} + x_{21} + 2x_{22} + 3x_{32} \\
\text{st.} \quad x_{11} + x_{21} \quad \geq \quad 10 \quad \\
x_{22} + x_{32} \quad \geq \quad 25 \quad \\
x_{11} \quad \leq \quad 4 \quad \\
x_{21} + x_{22} \quad \leq \quad 20 \quad \\
x_{32} \quad \leq \quad 50 \quad \\
x_{ij} \quad \geq \quad 0
\]
Transportation Problem LP

\[
\begin{align*}
\min_{x_{ij}} & \quad z = 2x_{11} + x_{21} + 2x_{22} + 3x_{32} \\
\text{st.} & \quad x_{11} + x_{21} \geq 10 \\
& \quad x_{22} + x_{32} \geq 25 \\
& \quad -x_{11} \geq -4 \\
& \quad -x_{21} - x_{22} \geq -20 \\
& \quad -x_{32} \geq -50 \\
& \quad x_{ij} \geq 0
\end{align*}
\]
Transportation Problem LP

\[
\min_{x_{ij}} \quad z = \begin{bmatrix} 2 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{22} \\ x_{32} \end{bmatrix} \leq \begin{bmatrix} 10 \\ 25 \\ -4 \\ -20 \\ -50 \end{bmatrix}
\]

\[
x_{ij} \geq 0
\]
Transportation Problem LP

\[ x_{11}^* = 4M \]
\[ x_{21}^* = 6M \]
\[ x_{22}^* = 14M \]
\[ x_{32}^* = 4M \]
\[ z^* = 75M \]
Q: When will an LP have a solution?

Q: Why can we solve LPs so quickly
Advertising Problem LP

\[ x_1: \text{# spots purchased during comedy shows} \]
\[ x_2: \text{# spots purchased during football games} \]

\[
\min_{x_1,x_2} \quad z = 50x_1 + 100x_2 \\
\text{st.} \quad 7x_1 + 2x_2 \geq 28 \\
2x_1 + 12x_2 \geq 24 \\
x_1, x_2 \geq 0
\]
Geometry of Advertising Problem

\[
\min_{x_1, x_2} \quad z = \quad 50x_1 + 100x_2 \\
\text{st.} \quad 7x_1 + 2x_2 \quad \geq \quad 28 \\
2x_1 + 12x_2 \quad \geq \quad 24 \\
x_1, \quad x_2 \geq 0
\]
Geometry of Advertising Problem

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
st. & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Geometry of Advertising Problem

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\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
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& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
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Geometry of Advertising Problem

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\min_{x_1, x_2} \quad & z = 50x_1 + 100x_2 \\
\text{st.} \quad & 7x_1 + 2x_2 \geq 28 \\
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& x_1, \ x_2 \geq 0
\end{align*} \]
Geometry of Advertising Problem

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\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
st. & \quad 7x_1 + 2x_2 \geq 28 \\
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Geometry of Advertising Problem

\[ \min_{x_1, x_2} \quad z = 50x_1 + 100x_2 \]

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Geometry of Advertising Problem

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Geometry of Advertising Problem

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& \quad x_1, x_2 \geq 0
\end{align*}
\]
Geometry of Advertising Problem

\[
\min_{x_1, x_2} \quad z = 50x_1 + 100x_2
\]
\[
\text{st.} \quad 7x_1 + 2x_2 \geq 28
\]
\[
2x_1 + 12x_2 \geq 24
\]
\[
x_1, \ x_2 \geq 0
\]
Geometry of Advertising Problem

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\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
st. & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Geometry of Advertising Problem

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\begin{align*}
\min_{x_1, x_2} \quad & z = 50x_1 + 100x_2 \\
st. \quad & 7x_1 + 2x_2 \geq 28 \\
& 2x_1 + 12x_2 \geq 24 \\
& x_1, x_2 \geq 0
\end{align*}
\]
Geometry of Advertising Problem

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\begin{align*}
\min_{x_1, x_2} \quad & z = 50x_1 + 100x_2 \\
st. \quad & 7x_1 + 2x_2 \geq 28 \\
& 2x_1 + 12x_2 \geq 24 \\
& x_1, x_2 \geq 0
\end{align*}
\]
LP Geometry 101

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{st.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]

- Each constraint in \( Ax \geq b \) marks a half-space
- All constraints result in a polyhedron \( P = \{ x \mid Ax \geq b, x \geq 0 \} \)
- Every point within \( P \) is feasible
- \( x^* \) is at one of the vertices of \( P \)
Will an LP always have a Solution?

- LP will have a solution if $P$ is closed and nonempty.
- LP will be infeasible if $P$ is empty.
- LP will be unbounded if $P$ is not closed.
Infeasibility

When $\not\exists \ x \ st. Ax \geq b, \ x \geq 0$

\[
\min_{x_1, x_2} \quad z = \quad 50x_1 + 100x_2 \\
\text{st.} \quad 7x_1 + 2x_2 \quad \geq \quad 28 \\
2x_1 + 12x_2 \quad \geq \quad 24 \\
x_1, \ x_2 \geq 0
\]
Infeasibility

When $\nexists \; x \; st. \; Ax \geq b, \; x \geq 0$

\[ \begin{align*}
\min_{x_1, x_2} \quad & z = 50x_1 + 100x_2 \\
\text{st.} \quad & 7x_1 + 2x_2 \geq 28 \\
\quad & 2x_1 + 12x_2 \geq 24 \\
\quad & x_1 + x_2 \leq 2 \\
\quad & x_1, \; x_2 \geq 0
\end{align*} \]
Infeasibility

When \( \not\exists \ x \text{ st. } Ax \geq b, \ x \geq 0 \)

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
\text{st.} & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Infeasibility

When \( \not\exists \; x \; st. \; Ax \geq b, \; x \geq 0 \)

\[
\min_{x_1, x_2} \quad z = \quad 50x_1 + 100x_2 \\
\text{st.} \quad 7x_1 + 2x_2 \quad \geq \quad 28 \\
2x_1 + 12x_2 \quad \geq \quad 24 \\
x_1 + x_2 \quad \leq \quad 2 \\
x_1, \; x_2 \geq 0
\]
Unboundedness

When $\exists x \text{ st. } Ax \geq b, x \geq 0$ and $c^T x = -\infty$

$$\begin{align*}
\min_{x_1, x_2} \quad & z = 50x_1 + 100x_2 \\
\text{st.} \quad & 7x_1 + 2x_2 \geq 28 \\
\quad & 2x_1 + 12x_2 \geq 24 \\
\quad & x_1, x_2 \geq 0
\end{align*}$$
Unboundedness

When $\exists \ x \geq 0 \ s.t. \ Ax \geq b$ and $c^T x = -\infty$

\[
\begin{align*}
\min_{x_1,x_2} \quad & z = 50x_1 + 100x_2 \\
\text{st.} \quad & 7x_1 + 2x_2 \geq 28 \\
\quad & 2x_1 + 12x_2 \geq 24 \\
\quad & x_1, \ x_2 \geq 0
\end{align*}
\]
Unboundedness

When $\exists \ x \geq 0 \ st. \ Ax \geq b$ and $c^T x = -\infty$

$$\min_{x_1, x_2} \ z = -50x_1 - 100x_2$$

$$st. \quad 7x_1 + 2x_2 \geq 28$$
$$\quad 2x_1 + 12x_2 \geq 24$$
$$\quad x_1, \ x_2 \geq 0$$
Sensitivity Analysis

Existing LP

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
\text{st.} & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[z^* = 320\]

New LP

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
\text{st.} & \quad 7x_1 + 2x_2 \geq 30 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[z^* = ?\]

**Q:** Can we get an estimate of \(z^*\) without having to solve the LP once again?
Sensitivity Analysis

**Existing LP**

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
\text{st.} & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[z^* = 320\]

**New LP**

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
\text{st.} & \quad 7x_1 + 2x_2 \geq 30 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[z^* = ?\]

**Q:** Can we get an estimate of \(z^*\) without having to solve the LP once again?

**A:** Yes but we will need to know how \(z^*\) changes with respect to \(b\).
Dual Variables

- Every LP has dual variables $\pi$ that describe how $z^*$ changes wrt $b$

- For an LP has $m$ constraints, then $\pi$ is an $m \times 1$ vector where

$$\pi_i = \frac{\delta z^*}{\delta b_i}$$

$$\pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_m \end{bmatrix} = \begin{bmatrix} \frac{\delta z^*}{\delta b_1} \\ \vdots \\ \frac{\delta z^*}{\delta b_m} \end{bmatrix}$$
Sensitivity Analysis

Existing LP

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
st. & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[
\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \frac{\delta z^*}{\delta b_1} \\ \frac{\delta z^*}{\delta b_2} \end{bmatrix} = \begin{bmatrix} 5 \\ 7.5 \end{bmatrix}
\]

\[
z^* = 320
\]

New LP

\[
\begin{align*}
\min_{x_1, x_2} & \quad z = 50x_1 + 100x_2 \\
st. & \quad 7x_1 + 2x_2 \geq 30 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[
z^* = z + \frac{\delta z^*}{\delta b_1} (\delta b_1)
\]

\[
= z + \pi_1 (\delta b_1)
\]

\[
= 320 + 5(2) = 330
\]
More about Dual Variables

$\pi$ only gives **local** information with respect to **one** component
- Estimate of $z^*$ only accurate for **small** changes in **one** $b_j$

More about dual variables:

- **Strong Duality**: if an LP has a Solution: $\pi b = cx^*$

- Every LP can be formulated using **only** its Dual Variables
  - Explains many infeasibility and unboundedness results
  - Motivates numerous algorithms to solve LP
Wrap Up

Linear Programming is only one of the ways to formulate static decision making problems

Other ways to formulate and solve static problems

- Integer Programming
  - linear cost + linear constraints + integer variables

- Quadratic Programming
  - quadratic cost + linear constraints + real variables

- Nonlinear Programming
  - nonlinear cost + nonlinear constraints + real variables