Lecture 1: Random Walks and 1-D Diffusion

- A localized high concentration of substance will spread out to make the concentration spatially homogeneous (2nd law of thermo)
- Molecules "move" from high to low density
- Individual particles are randomly battered by solvent molecules \( \Rightarrow \) causes a random walk
- Many random walks made by many molecules is diffusion

Diffusion is dominant at \( \mu m \) length scales

Show Applet Movie of Random Walk

Beach ball at concert example

1st observation: pollen particles Robert Brown 1828
the concept of molecules was controversial
'Brownian motion' was mistaken for a life process

Simple molecule measurements: Random walk
Bulk measurements: Diffusion

1D \hspace{1cm} \text{transcription factor sliding on DNA growth of actin filaments}

2D \hspace{1cm} \text{protein traveling in lipid membrane}

3D \hspace{1cm} \text{protein traveling in cytoplasm cell motility}
One-Dimensional Random Walk

- Flip a coin to determine whether you go left or right
- How far are you likely to go in a random walk?
- Cannot solve for one walk, but we can determine the average displacement

Enumeration of the possibilities for N = 3 steps:

<table>
<thead>
<tr>
<th>x = 0</th>
<th>Distance (x^2)</th>
<th>x^2/3L^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3L</td>
<td>+3L</td>
<td>+9L^2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-3L</td>
<td>-9L^2</td>
</tr>
</tbody>
</table>

\[ \langle x_3 \rangle = 0 \quad \langle x_3^2 \rangle = 3L^2 \]

\( \langle x_n \rangle = 0 \quad \langle x_n^2 \rangle = NL^2 \)

The mean displacement of a random walk is zero.

- That does not mean that you don't go anywhere!

**Add time to the process**

\[ N = \frac{t}{\Delta t} \text{ total time} \]

\[ \Delta t = \text{amount of time per step} \]

- Define diffusion constant \[ D = \frac{\langle x_n^2 \rangle}{2\Delta t} = \frac{NL^2}{2\Delta t} \]

\[ \langle x_n^2 \rangle = 2D\Delta t \]

\[ \langle x_n \rangle \text{ (mean-square displacement increases linearly with time)} \]

Completely arbitrary/convenient later

Any individual random walk will not conform to the diffusion law, even approximately!
we will observe excursions of any size $X$ as long as we are
willing to wait a time $v_0 \frac{x^2}{2Dt}$

can easily experimentally measure $D$: wait time $t$, measure
displacement, and repeat

Higher-dimensions

1D

$\langle r^2 \rangle = 2Dt$

2D

$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle = 4Dt$

3D

$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 6Dt$

Example

About how long does it take for a sudden supply of sugar molecules to spread uniformly?

$\langle r^2 \rangle = 6Dt$

$r \sim 500 \text{ nm}$

$\frac{r^2}{6D} = \frac{500^2 \text{ nm}^2}{6}$

$\epsilon_{cell} = \frac{(1 \mu m)^2}{6(500 \text{ nm})} = 0.0003 \text{ s}$

$\epsilon_{bact} = \frac{(10^2 \mu m)^2}{6(500 \text{ nm})} = 0.03 \text{ s}$

$\epsilon_{neu} = \frac{(1000 \mu m)^2}{6(500 \text{ nm})} = 300 \text{ s}$

Compare with $\mathcal{D} \approx 10 - 5 \text{ m} \text{ s}^{-1}$ for an average-sized
protein in the cytoplasm.
Explicit calculation of the probability distribution

10 random walk

\[ P(m,N) \]

- each step is like flipping a coin \( P_+ = P_- = \frac{1}{2} \)
- How many heads in 100 flips?

Binomial probability distribution: \( P(m,N) = \left( \frac{1}{2} \right)^N \frac{N!}{m!(N-m)!} \)

- factorials are mathematically intractable
- want to approximate function \( P \)

Calculate a Taylor series around the most probable end point \( m^* \)

Reminder: purpose of a Taylor Series is to approximate a function with a simpler mathematical expression

\[ f(x) \approx f(a) + \left( \frac{df}{dx} \right)_{x=a} (x-a) + \frac{1}{2} \left( \frac{d^2f}{dx^2} \right)_{x=a} (x-a)^2 + \ldots \]

- Approximate \( P(m,N) \) - work w/ log is easier

\[ \ln \phi P(m) = \ln P(m^*) + \left( \frac{d \ln P}{dm} \right)_{m^*} (m-m^*) + \frac{1}{2} \left( \frac{d^2 \ln P}{dm^2} \right)_{m^*} (m-m^*)^2 + \ldots \]
Need to evaluate the derivatives \( \frac{d\ln P}{dm} \) and \( \frac{d^2\ln P}{dm^2} \).

Take log of 2 and use Stirling's Approximation \((\ln N! \approx N\ln N)\)

\[
\ln P = N \ln N - m \ln m - (N-m) \ln (N-m) - N \ln (\frac{1}{2})
\]

\( \frac{d\ln P}{dm} \bigg|_{m^*} = 0 = -1 - \ln m^* + \ln (N-m^*) + 1 \)

Solve for \( m^* \), \( m^* = \frac{N}{Z} \) (makes sense, right?)

\( \frac{d^2\ln P}{dm^2} \bigg|_{m^*} = (\frac{1}{m} - \frac{1}{N-m}) \bigg|_{m^*} = \frac{1}{N} \)

Put 4 and 5 into Taylor Series

\[
\ln P = \ln P^* + \frac{1}{2} \left( -\frac{1}{N} \right) (m-m^*)^2
\]

\( P = P^* e^{-\frac{2(m-m^*)^2}{N}} \) - form of Gaussian Distribution

Now, convert from \( m \) to \( x \)

Displacement forward-progression is forward-reverse steps

\[
x = m - (N-m) = 2m - N
\]

\[
m = \frac{x+N}{2}
\]

\[
P(x) = P^* e^{-\frac{x^2}{2N}}
\]

at \( x = 0 \)
Now, calculate $P^*$

- this is a probability distribution that needs to be normalized

$$\int_{-\infty}^{\infty} P(x) \, dx = 1 = \int_{-\infty}^{\infty} P^* e^{-\frac{x^2}{2N}} \, dx$$

$$P^* = \left( \int_{-\infty}^{\infty} e^{-\frac{x^2}{2N}} \, dx \right)^{-1}$$

$$= \frac{1}{\sqrt{2\pi N}} = (2\pi N)^{-\frac{1}{2}}$$

$$P(x) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{x^2}{2N}}$$

a Gaussian of this form $\langle x^2 \rangle = N$

from the diffusion law

$$\langle x^2 \rangle = 2D\tau = N$$

$$P(x) = \frac{1}{\sqrt{4\pi D\tau}} e^{-\frac{x^2}{4D\tau}}$$

- can calculate $D$ if you have a distribution $P(x)$ of displacements at time $\tau$