Poisson Statistics

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Prior scientific knowledge has shown that the radioactive decay of nuclei can be modeled as a series of independent, random events. [1] The probability for the occurence of such events can thus be modeled by Poisson statistics. In this experiment, we studied the radioactive decay of ^{137}Cs by using a NaI scintillator. We recorded the counts per second for 100 consecutive one-second long intervals at countrates of approximately 1, 5, 10, and 100 counts per second and attempted to model the data using the Poisson probability distribution. It was found that the mean countrates converged to values of $0.44 \pm 0.07 s^{-1}, 4.54 \pm 0.21 s^{-1}; 12.74 \pm 0.36 s^{-1}$, and $96.05 \pm 0.98 s^{-1}$. The second part of this experiment involved using Monte Carlo calculations to simulate random data and creating Poisson distributions that were comapared to our experimental data. It was found that the experimental data correlated with the simulated data. We concluded that the Poisson probability distribution can be observed in the decay of ^{137}Cs.

1. INTRODUCTION

Historically, it has been predicted and demonstrated that the decay of a single nucleus in a radioactive sample is a completely random event. This phenomenon was first postulated by E. Schweidler in 1905, then proven by Rutherford and Geiger in 1910. [1] In our experiment, we set out to confirm these results by using experimental and computer simulated data sets and comparing them to theoretical models.

First, we studied the pulses from a scintillation detector exposed to gamma rays from a radioactive source. A ^{137}Cs source was used in this experiment, and measurements were taken at countrates of approximately 1, 4, 10, and 100 counts per second. After recording data, we compared our measurements to theoretical Poisson and Gaussian curves, two probability distributions often observed in nature.

The second part of this experiment involved comparing the experimental distributions to those generated by a computer simulation of a random Poisson process. Using a computer simulation allowed for a greater numbers of runs and also provided a demonstration of typical statistical fluctuations. For each of our measured decay rates, we used Monte Carlo techniques to construct Poisson distributions of the same average countrates and compared these to our experimental data sets.

Monte Carlo calculations are statistical methods based on the use of random numbers. They derive their name from a city of the same name that is home to a famous casino–the random numbers generated by Monte Carlo calculations are similar to those involved in casino games of chance. [2]

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2. THEORY

2.1. Derivation of the Poisson Probability Distribution

Random, independent events in physical measurements are often modeled by Poisson distributions. The Poisson distribution is used to describe data in counting experiments where the number of items or event observed per unit interval is recorded.

Three assumptions must be made for the Poisson distribution to be used as an accurate model:

1. The probability of observing one event in a very small time interval $\Delta\tau$ is proportional to $\Delta\tau$.

2. The probability that more than one event will be observed in $\Delta\tau$ is negligible.

3. The number of events observed in a certain $\Delta\tau$ does not depend on what happens in any other interval of time, so long as the intervals do not overlap.

Using only these three assumptions, one can derive the Poisson distribution by first calculating the probablity that zero events occur in a certain time interval τ, then calculating the probability that a certain number of events k occur in the time interval $\tau + \Delta\tau$. [3], [4]

A quicker derivation, however, can be performed by modeling the Poisson probability distribution as an approximation of the binomial probability distribution. The binomial probability distribution describes the probability for something to occur when there are a limited number of possible outcomes. An example would be counting the number of heads and tails when flipping a coin repeatedly. The probability for observing x of n items to be in the state with probability p is given by the following statement of the binomial distribution:

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-x)^{n+x}$$

The equation can be factored as below:

$$P_B(x;n,p) = \frac{1}{x!}\frac{n!}{(n-x)!}p^x(1-p)^{-x}(1-p)^n$$

Then the limit at $n \to \infty$ while the mean rate $\mu = np$ remains constant can be taken to obtain the Poisson equation:

$$\lim_{n\to\infty} P_B(x;n,p) = \lim_{n\to\infty}\frac{1}{x!}\frac{n!}{(n-x)!}p^x(1-p)^{-x}(1-p)^n$$

$$= \frac{1}{x!}(np)^x(1+px)e^{-\mu} = \frac{\mu^x}{x!}e^{-\mu}$$

The Poisson distribution is not only used to describe experiments where data is recorded as counts in unit time intervals–whenever data is grouped into bins to form a histogram or frequency plot, the number of events in each bin will obey Poisson statistics. [2]

2.2. Properties of the Poisson Distribution

The Poisson distribution is applied to experiments where the data is strictly bounded on one side; the curve of the graph is highly assymetric. A way of looking at the properties of the Poisson is to compare the Poisson distribution to the Gaussian (or Normal) probability distribution. Both the Gaussian and the Poisson distributions are approximations of the binomial probability distributions. The Gaussian distribution is also an approximation for when n, the total number of possible events, is large, but it differs from the Poisson in that p, the probability that an event will be observed, must also be large. When $\mu \to \infty$, the Poisson distribution becomes more symmetric and approaches the Gaussian distribution. Because of this, the Gaussian distribution is often preferred in calculating probabilities, because it is usually easier to calculate. It should be noted that the Gaussian distribution is continuous, while the Poisson and binomial distributions are both discrete, which means that they are only defined at integral values of the variable x. The parameter μ is always a positive, real number, however. [4]

2.3. Monte Carlo methods

When the calculation of experimental results becomes tedious and lengthy, the Monte Carlo method provides us with a way of evaluating multidimensional integrals. The need for this method might arise in a scattering experiment where the goal is to measure the angular distribution of particles scattered from protons in a fixed target. Many variables come into play and can be described with probability distributions: the magnitude and direction of the momentum vector of incident particles, the probability that a particle will collide with a photon in the target, etc. A multiple integration will have to be performed over the pertinent probability distribution fuctions, and this integration may not be possible to obtain through analytical methods.

Monte Carlo methods are a way of calculating these multiple integrals via random sampling. It provides a method of simulating experiments and creating models of experimental data. The process we model here with a Monte Carlo calculation is relatively simple, as it is described by a single probability distribution, but most experiments, like the scattering one described above, involve a combination of many different probability distributions, making it more unwieldy to extract useful parameters. It is in these experiments that the full power of the Monte Carlo method can be seen. Even in our simple case, the convenience of being able to model the experiment and quickly obtain data sets is evident. [4]

3. EXPERIMENT

The experimental setup for recording $\gamma - ray$ counts from the ^{137}Cs source is shown below. The discriminator settings on the scintillation counter were adjusted to obtain four mean countrates of approximately 1, 5, 10, and 100 counts/second. For each countrate, data was collected for 100 consecutive one-second intervals. The actual average countrate was determined, as well as the standard deviation of the mean. These were compared with the means and standard deviations of the theoretical Poisson distributions.

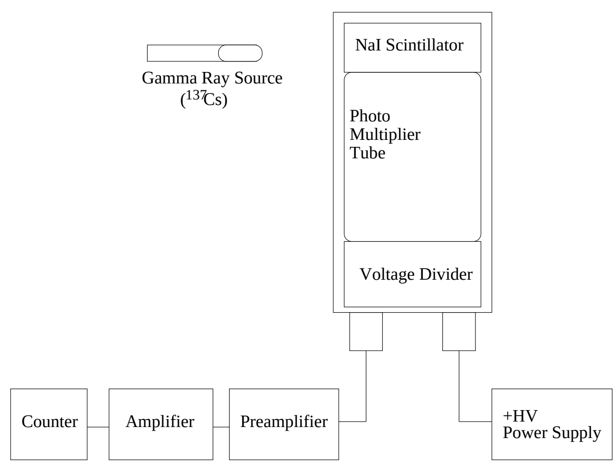

FIG. 1: This is a schematic diagram of the setup for Poisson Statistics. The ^{137}Cs source was placed approximately 10cm from the detector. Diagram adapted from [5].

In the second part of the experiment, Poisson distributions were generated via Monte Carlo simulations and compared to the distributions observed with the counting measurements.

4. DATA AND ANALYSIS

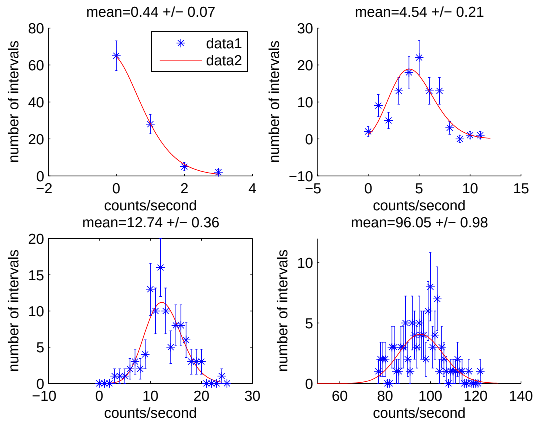

FIG. 2: Here, data1 represents the experimental data, while data2 represents the theoretical curve for the Poisson distribution, modeled by using the experimental mean.

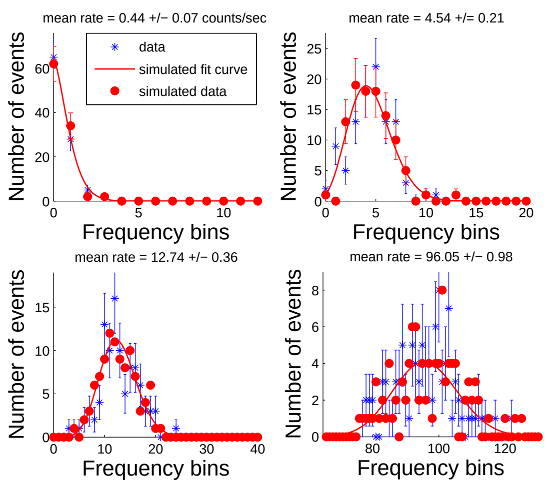

FIG. 3: Comparison of the experimental data to the Monte Carlo calculations which generated Poisson distributions.

The actual mean countrates obtained and their respective discriminator settings were as follows: $0.44 \pm 0.07s^{-1}, 9.50volts; 4.54 \pm 0.21s^{-1}, 9.20V; 12.74 \pm 0.36s^{-1}, 7.80V; 96.05 \pm 0.98s^{-1}$.

The cumulative mean for the 100 runs was plotted with errorbars against the index number of the runs to determine if the mean converged over the course of 100 runs. Errorbars were obtained by taking the standard deviation of the mean:

$$\sigma_\mu = \sqrt{\frac{\mu}{N}}.$$

This represented a 68% probability for the mean to fall into a certain range if we were to take many sets of data consisting of 100 consecutive one-second intervals, then average the counts per second for each data set. It was observed that the 68% confidence interval decreased and the values of the mean gradually became constant over more trials. This showed that the probability of events occurring in a given $\Delta\tau$ was indeed proportional to $\Delta\tau$.

The counts for each second were collected into one-second bins and plotted as histograms with errorbars, again of 68% confidence, obtained by taking the standard deviation of the number of counts per bin, which followed Poisson statistics and was equal to the square root of the number of counts. The histograms were then compared to theoretical models. It was observed that the experimental data corresponded more closely to the theoretical curves at the lower mean countrates.

In the graph for the $12.74 \pm 0.36s^{-1}$ mean countrate, for instance, it is observed that all but two of the experimental data points fall within one standard deviation of the theoretical curve, and those that do not are within two standard deviations. This is roughly consistent with the expected probabilities for 12 histogram bins.

A Matlab script that employed a Monte Carlo calculation was then used to generate Poisson distributions. The data in the figure below represent the four computer-simulated 100 consecutive one-second-long trial runs. The mean countrates that were experimentally obtained were also used in these Monte Carlo calculations.

5. CONCLUSIONS

It was found that the mean of the counts per second did converge over time. This indicates that the probability of obtaining a certain number of counts in an interval $\Delta\tau$ is proportional to $\Delta\tau$, as is the case in a Poisson probability distribution. Also, in comparing the experimental data to the theoretical Poisson curves, it was found that the data corresponded well. We conclude that the decay of ^{137}Cs can be modeled by the Poisson probability distribution.

Improvements that could be made on this experiment might be to conduct more consecutive runs. With more data, better Poisson curves might be observed. Another possible route of exploration would be to choose different radioactive elements to study and see if their decay could also be modeled by Poisson statistics.

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[2] P. Bevington and D. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, 2003).
[3] *chap4-repeated trials*, Internet (2002), http://www.cis.rit.edu/class/simg713/notes/chap4-Repeated%20Trials.pdf.
[4] A. Melissinos and J. Napolitano, *Experiments in Modern Physics* (Academic Press, 2003), 2nd ed.
[5] J. L. Staff, *Junior Lab Reader* (2004).

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