The Dark Side of Circuit Breakers

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Abstract

As an important part of overall market architecture, market-wide trading halts, also called circuit breakers, have been widely adopted as a measure to stabilize the stock market when experiencing large price movements. We develop an intertemporal equilibrium model to examine how circuit breakers impact the market when investors trade for risk sharing. We show that a downside circuit breaker tends to lower the stock price and increase its volatility, both conditional and realized. Due to this increase in volatility, the circuit breaker's own presence actually raises the likelihood of reaching the triggering price. In addition, the circuit breaker also increases the chance of hitting the triggering price as the stock price approaches it – the so-called "magnet effect." Our model highlights the fact that changes in market liquidity can endogenously trigger leverage constraints and in turn affect trading and price dynamics. This mechanism also applies to other forms of market interventions.

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1 Introduction

Large stock market swings in the absence of significant macroeconomic shocks often raise questions about the confidence in the financial market from market participants, policy makers and general public alike. While the cause of these swings are still not well understood, various measures have been adopted to intervene in the normal trading process during these extreme times in the hope to stabilize prices and maintain proper functioning of the market. These measures, sometimes referred to as throwing sand in the gears, range from market-wide trading halts, price limits on the whole market or individual assets, to limits on order flows, positions and margins, even transaction taxes, just to name a few.[1] They have grown to be an important part of the broad market architecture. Yet, the merits of these measures, either from a theoretical or an empirical perspective, remain largely unclear (see, for example, Grossman, 1990).

Probably one of the most prominent of these measures is the market-wide circuit breaker in the U.S., which was advocated by the Brady Commission (Presidential Task Force on Market Mechanisms, 1988) following the Black Monday of 1987 and subsequently implemented in 1988. It temporarily halts trading in all stocks and related derivatives when a designated market index drops by a significant amount. Following this lead, circuit breakers of various forms have been widely adopted by equity and derivative exchanges around the globe.[2] Table 1 shows the adoption of market-wide circuit breakers among the leading stock markets in both the developed and developing economies.

Since its introduction, the U.S. circuit breaker was triggered only once on October 27, 1997 (see, e.g., Figure 1, left panel). At that time, the threshold was based on points movement of the DJIA index. At 2:36 p.m., a 350-point (4.54%) decline in the DJIA led to a 30-minute trading halt on stocks, equity options, and index futures. After trading

[1] It is worth noting that contingent trading halts and price limits are part of the normal trading process for individual stocks and futures contracts. However, their presence there have quite different motivations. For example, the trading halt of an individual stock prior to major corporate announcements is motivated by the desire for fair information disclosure, and daily price limits on futures are motivated by the desire to guarantee the proper implementation of the mark-to-market mechanism as well as to deter market manipulation. In this paper, we focus on market-wide trading interventions in the underlying markets such as stocks as well as their derivatives, which have very different motivations and implications.

[2] According to a 2016 report, "Global Circuit Breaker Guide" by ITG, over 30 countries around the world have rules of trading halts in the form of circuit breakers, price limits and volatility auctions.

Table 1: Adoption of market-wide circuit breakers among leading stock markets. The table also reports markets with price limits on individual stocks. Markets with market-wide circuit breakers as well as individual stock price limits are denoted by Y and N otherwise. Y/N for China denotes its adaptation of circuit breakers and then the abandonment.

	2018 Market Cap (Tn $)	Rank	Circuit Breaker	Price Limit
Developed markets				
United States	30.4	1	Y	Y
Japan	5.3	3	N	Y
Hong Kong	3.8	4	N	N
France	2.4	5	Y	N
Canada	1.9	7	Y	N
Germany	1.8	8	Y	N
Developing markets				
China	6.3	2	Y/N	Y
India	2.1	6	Y	Y
Brazil	0.9	12	Y	N
Russia	0.6	16	Y	N

resumed at 3:06 p.m., prices fell rapidly to reach the second-level 550-point circuit breaker at 3:30 p.m., leading to the early market closure for the day.[3] But the market stabilized the next day. This event led to the redesign of the circuit breaker rules, moving from change in the level of DJIA to percentage drop of S&P 500, with a considerably wider bandwidth.[4]

After the Chinese stock market experienced extreme price declines in 2015, a market-wide circuit breaker was introduced in January 2016, with a 15-minute trading halt when the CSI 300 Index falls by 5% (Level 1) from previous day's close, and market closure for the day after a 7% decline (Level 2).[5] On January 4, 2016, the first trading day after the circuit breaker was put in place, both thresholds were reached (Figure 1, middle panel), and it took only 7 minutes from the re-opening of the markets following the 15-minute halt for the index to reach the 7% threshold. Three days later, on January 7, both circuit breakers were triggered again (Figure 1, right panel), and the entire trading session lasted just 30 minutes.

[3]For a detailed review of this event, see Securities and Exchange Commission (1998).

[4]In its current form, the market-wide circuit breaker can be triggered at three thresholds: 7% (Level 1), 13% (Level 2), both of which will halt market-wide trading for 15 minutes when the decline occurs between 9:30 a.m. and 3:25 p.m. Eastern time, and 20% (Level 3), which halts market-wide trading for the remainder of the trading day; these triggers are based on the prior day's closing price of the S&P 500 Index.

[5]The CSI 300 index is a market-cap weighted index of 300 major stocks listed on the Shanghai Stock Exchange and the Shenzhen Stock Exchange, compiled by the China Securities Index Company, Ltd.

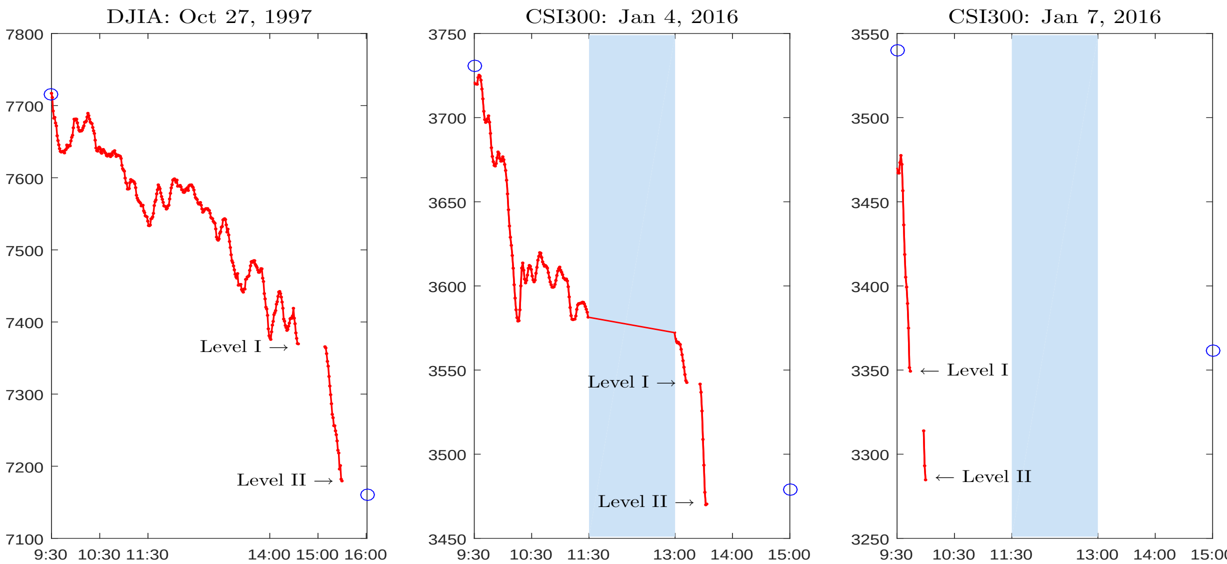

Figure 1: **Circuit breakers in the U.S. and Chinese stock market.** The left panel plots the DJIA index on Oct 27, 1997, when the market-wide circuit breaker was triggered, first at 2:36 p.m., and then at 3:30 p.m. The middle and right panels plot the CSI300 index on January 4 and January 7 of 2016. Trading hours for the Chinese stock market are 9:30-11:30 and 13:00-15:00 (the shaded interval in the panels marks the lunch break). Level 1 (2) circuit breaker is triggered after a 5% (7%) drop in price from the previous day's close. The blue circles on the left (right) vertical axes mark the price on the previous day's close (following day's open).

On the same day, the circuit breaker was suspended indefinitely.

These events have revived debates about circuit breakers and market interventions in general. What are the theoretical and empirical basis for introducing circuit breakers? What are their goals? How do they actually impact the market? How to assess their success or failure? How may their effectiveness depend on the particular market, the actual design, and the specific market conditions? More broadly, these questions can be raised about any form of interventions in the trading process.

In this paper, we develop an intertemporal equilibrium model to capture investors' most fundamental trading needs, namely to share risk. We then examine how the introduction of a downside circuit breaker affects investors' trading behavior and the equilibrium price dynamics. In addition to welfare loss by reduced risk sharing, which can be substantial, we show that a circuit breaker also lowers price levels, increases conditional and realized volatility, and increases the likelihood of triggering trading halts. These consequences run contrary to the often stated goal of circuit breakers, which is to calm the markets. Our model not only demonstrates the potential cost of circuit breakers, but also provides a basic setting to further incorporate market imperfections to fully examine their costs and benefits.

In our model, two (classes of) investors have log preferences over terminal wealth and have heterogeneous beliefs about the dividend growth rate, which generates trading. For simplicity, one investor's belief is set to be the same as the objective belief while another investor's belief is different, who will also be referred to as the irrational investor.[6] Without the circuit breaker, the stock price is a weighted average of the prices under the two investors' beliefs, with the weights being their respective shares of total wealth.

The introduction of a downside circuit breaker in the market, however, makes the equilibrium stock price disproportionately reflect the beliefs of the relatively pessimistic investor, especially when the stock price approaches the circuit breaker limit. To understand this result, first consider the scenario when the stock price has just reached the circuit breaker threshold. Immediate market closure is an extreme form of illiquidity, which forces the relatively optimistic investor to refrain from taking on leverage due to the inability to rebalance his portfolio during closure and the risk of default it may entail. As the optimistic investor faces binding leverage constraints, the pessimistic investor becomes the marginal investor, and the equilibrium stock price upon market closure entirely reflects his belief, regardless of his wealth share.

Next, the threat of market closure also affects trading and prices before the circuit breaker is triggered. Compared to the case without circuit breaker, the relatively optimistic investor will preemptively reduce his leverage as the price approaches the circuit breaker limit. For a downside circuit breaker, the price-dividend ratios become lower throughout the trading interval. Thus, a downside circuit breaker tends to drive down the overall asset price levels.

In addition, in the presence of a downside circuit breaker, the conditional volatilities of stock returns can become significantly higher. These effects are stronger when the price is closer to the circuit breaker threshold, when it is earlier during a trading session. Surprisingly,

[6]Interpreting the investor heterogeneity as difference in beliefs is solely for convenience. We can also interpret it as difference in preferences or endowments. See the discussion on the model in Section 2.

the volatility amplification effect of downside circuit breakers is stronger when the initial wealth share for the irrational investor (who tends to be pessimistic at the triggering point) is smaller, because the gap between the wealth-weighted belief of the representative investor and the belief of the pessimist is larger in such cases.

Our model shows that circuit breakers have multifaceted effects on intra-day price variability. On the one hand, almost mechanically, a (tighter) downside circuit breaker eliminates a possibility of very large downward price movements. Such effects could be beneficial, for example, in reducing inefficient liquidations due to intra-day mark-to-market. On the other hand, a (tighter) downside circuit breaker tends to raise the probabilities of intermediate and large price ranges, and can significantly increase the median of daily realized volatilities as well as the probabilities of very large conditional and realized volatilities. These effects could exacerbate market instability in the presence of imperfections.

Furthermore, our model demonstrates a "magnet effect." The very presence of downside circuit breakers makes it more likely for the stock price to reach the threshold in a given amount of time than when there are no circuit breakers (the opposite is true for upside circuit breakers). The difference between the probabilities is negligible when the stock price is sufficiently far away from the threshold, but it gets bigger as the stock price gets closer to the threshold. Eventually, when the price is sufficiently close to the threshold, the gap converges to zero as both probabilities converge to one.

This "magnet effect" is important for the design of circuit breakers. It suggests that using the historical data from a period when circuit breakers were not implemented can substantially underestimate the likelihood of future circuit breaker triggers, which might result in picking a downside circuit breaker threshold that is excessively tight.

Prior theoretical work on circuit breakers focuses on their role in reducing excess volatility and restore orderly trading in the presence of market imperfections such as limited participation, information asymmetry and market power. For example, Greenwald and Stein (1991) argue that, in a market with limited participation and the resulting execution risk, circuit breakers can help to better synchronize trading for market participants and improve the efficiency of allocations (see also Greenwald and Stein, 1988).[7] On the other

[7]In Greenwald and Stein (1991), limited participation takes several forms. In particular, value traders,

hand, Subrahmanyam (1994) shows that in the presence of asymmetric information, circuit breakers can increase price volatility by causing investors to shift their trades to earlier periods with lower liquidity supply (see also Subrahmanyam, 1995).

By developing a model to reflect investors' first-order trading needs, our work complements the studies above in three important dimensions. First, it properly captures the cost of circuit breakers, in welfare, price distortion, and excess volatility, in a benchmark setting without imperfections. For instance, we show that the excess volatility effect that Subrahmanyam (1994) demonstrates in his model is more generic and is present even in the absence of asymmetric information. This is relevant in analyzing market-wide circuit breakers because various forms of market imperfections such as information asymmetry and strategic behavior could be less important for broad markets, such as the aggregate stock market, than for narrow markets, such as markets for individual stocks.

Second, our model provides a basis to further include different forms of market imperfections, if suitable, such as asymmetric information, strategic behavior, cost of participation and failure of coordination, which are needed to justify and quantify the benefits of circuit breakers. Such imperfections are undoubtedly important in providing the basis for interventions. However, focusing solely on their influence may understate the fundamental merits of the market mechanism itself.

Third, our model sheds light on the importance of properly capturing the most fundamental trading needs of investors, i.e., risk sharing or liquidity, in analyzing, understanding and managing financial markets. In models of information asymmetry, these trading needs are represented by the liquidity demands of "noise traders," which are treated as exogenous. Our results suggest that the behavior of these liquidity traders can be significantly affected by circuit breakers, which should be carefully taken into account.

In this spirit, our paper is closely related to Hong and Wang (2000), who study the effects of periodic market closures in the presence of asymmetric information. The liquidity effect caused by market closures as we see here is qualitatively similar to what they find. By

who act as price stabilizers, enter the market at different times with uncertainty. This uncertainty in their participation, which is assumed to be exogenous, gives rise to the additional risk in execution prices. Also, these value traders can only rely on market orders or simple limit orders, rather than limit order schedules, in their trading.

modeling the stochastic nature of a circuit breaker, we are able to fully capture its impact on market dynamics, such as volatility and conditional distributions.

While our model focuses on circuit breakers, our theoretical results about the dynamic impact of disappearing market liquidity in the presence of levered investors is more broadly applicable. Besides exchange-implemented trading halts, other types of market interventions such as price limits, short-sale bans, trading speed restrictions (e.g., penalties for HFT), and other forms of market freezes could also have similar effects on the willingness of highly-levered investors to continue to hold their risky positions. As these investors preemptively de-lever, it will depress prices, amplify volatility, and further raise the chances of market freeze. In fact, the setup we have developed here can be extended to examine the impact of these interventions.

The empirical work on the impact of market-wide circuit breakers is scarce due to the fact that their likelihood to be approached, not to mention triggered, is very small by design. Goldstein and Kavajecz (2004) provide a detailed analysis on the behavior of market participants in the period around October 27, 1997, the only time the circuit breaker has been triggered in the U.S. since its introduction. They find that leading up to the trading halt, market participants accelerated their trades, which is consistent with the magnet effect. In addition, they show that sellers' behavior is less influenced when approaching circuit breaker than the buyers', who are withdrawing from the market by canceling their buy limit orders. This is consistent with what our model predicts: sellers are becoming marginal traders when approaching a circuit breaker.[8]

There is, however, more extensive empirical work on the impact of conditional trading restrictions on individual securities including futures. For example, Bertero and Mayer (1990) and Lauterbach and Ben-Zion (1993) study the effects of trading halts based on price limits imposed on individual stocks around the 1987 stock market crash. Bertero and Mayer (1990) find that stock indices of countries with price limits imposed on individual stocks experienced declines in magnitude of up to 9% lower compared to aggregate indices in countries without circuit breakers. Lauterbach and Ben-Zion (1993) find that stocks

[8] Ackert, Church, and Jayaraman (2001) study the impact of market-wide circuit breakers through experiments. They find that circuit breakers do not impact prices significantly but alter market participants' trading behavior substantially by accelerating trading when the breakers are approaching.

with trading halts experienced smaller decline on the day of the crash, but trading halts did not have any effect on long run performance. Lee, Ready, and Seguin (1994) explore behavior of individual stock prices traded at NYSE around times of trading halts. They find that individual stock volatility and trading volume both increase on the days following a trading halt. On the other hand, Christie, Corwin, and Harris (2002) find that for news related trading halts of individual NASDAQ stocks, longer halts tend to reduce post-halt uncertainty.[9] Although the focus of our paper is on market-wide circuit breakers, the results we obtain are broadly compatible with the empirical findings on the impact of trading halts for individual assets. But the results from individual assets are in general richer and less robust. In many ways, this is expected given the relative importance of various type of imperfections in these markets.

The rest of the paper is organized as follows. Section 2 describes the basic model for our analysis. Section 3 provides the solution to the model. In Section 4, we examine the impact of a downside circuit breaker on investor behavior and equilibrium prices. Section 5 discusses the robustness of our results with respect to some of our modeling choices such as continuous trading and no default. In Section 6, we consider extensions of the model to different types of trading halts. Section 7 concludes. All proofs are given in the appendix.

2 The Model

We consider a continuous-time endowment economy over the finite time interval $[0, T]$. Uncertainty is described by a one-dimensional standard Brownian motion Z, defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the augmented filtration generated by Z.

There is a single share of an aggregate stock, which pays a terminal dividend of D_T at

[9] Chen (1993), Santoni and Liu (1993), Kim and Rhee (1997), Corwin and Lipson (2000), Jiang, McInish, and Upson (2009), Gomber, Haferkorn, Lutat, and Zimmermann (2012), Brugler and Linton (2014), among others, study the effects of trading halts and price limits on the market behavior individual stocks. Chen, Gao, He, Jiang, and Xiong (2017) examine the impact of daily price limits on trading patterns and price dynamics in the Chinese stock market. Kuserk, Locke, and Sayers (1992), Berkman, Steenbeek, et al. (1998), Coursey and Dyl (1990), Ma, Rao, and Sears (1989b), Ma, Rao, and Sears (1989a), Chen and Jeng (1996) study effects of trading restrictions related to price fluctuations in futures markets.

time T. The process for D is exogenous and publicly observable, given by:

$$dD_t = \mu D_t dt + \sigma D_t dZ_t, \quad D_0 = 1, \tag{1}$$

where μ and $\sigma > 0$ are the expected growth rate and volatility of D_t, respectively.[10] Besides the stock, there is also a riskless bond with total net supply $\Delta \geq 0$. Each unit of the bond yields a terminal pays off of one at time T.

There are two competitive agents A and B, who are initially endowed with ω and $1-\omega$ shares of the aggregate stock and $\omega\Delta$ and $(1-\omega)\Delta$ units of the riskless bond, respectively, with $0 \leq \omega \leq 1$ determining the initial wealth distribution between the two agents. Both agents have logarithmic preferences over their terminal wealth at time T:

$$u_i(W_T^i) = \ln(W_T^i), \quad i = \{A, B\}. \tag{2}$$

There is no intermediate consumption.

The two agents have heterogeneous beliefs about the terminal dividend, and they "agree to disagree" (i.e., they do not learn from each other or from prices).[11] Agent A has the objective beliefs in the sense that his probability measure is consistent with $\mathbb{P}$ (in particular, $\mu^A = \mu$). Agent B's probability measure, denoted by $\mathbb{P}^B$, is different from but equivalent to $\mathbb{P}$.[12] In particular, he believes that the dividend growth rate at time t is:

$$\mu_t^B = \mu + \delta_t, \tag{3}$$

where the difference in beliefs δ_t follows an Ornstein-Uhlenbeck process:

$$d\delta_t = -\kappa(\delta_t - \bar{\delta})dt + \nu dZ_t, \tag{4}$$

[10] For brevity, throughout the paper we will refer to D_t as "dividend" and S_t/D_t as the "price-dividend ratio," even though dividend will only be realized at time T.

[11] The formulation here follows earlier work of Detemple and Murthy (1994) and Zapatero (1998), among others.

[12] More precisely, $\mathbb{P}$ and $\mathbb{P}^B$ are equivalent when restricted to any σ-field $\mathcal{F}_T = \sigma(\{D_t\}_{0 \leq t \leq T})$. Two probability measures are equivalent if they agree on zero probability events. Agents beliefs should be equivalent to prevent seemingly arbitrage opportunities under any agents' beliefs.

with $\kappa \geq 0$ and $\nu \geq 0$. Equation (4) describes the dynamics of the gap in beliefs under physical probability measure, which is the same as from the perspective of agent A.[13] Notice that δ_t is driven by the same Brownian motion as the aggregate dividend. With $\nu > 0$, agent B becomes more optimistic (pessimistic) following positive (negative) shocks to the aggregate dividend, and the impact of these shocks on his belief decays exponentially at the rate κ. Thus, the parameter ν controls how sensitive B's conditional belief is to realized dividend shocks, while κ determines the relative importance of shocks from recent past vs. distant past. The average long-run disagreement between the two agents is $\bar{\delta}$. In the special case with $\nu = 0$ and $\delta_0 = \bar{\delta}$, the disagreement between the two agents remains constant over time. In another special case where $\kappa = 0$, δ_t follows a random walk.

Heterogeneous beliefs are a simple way to introduce heterogeneity among agents, which is necessary to generate trading. The heterogeneity in beliefs can easily be interpreted as heterogeneity in utility, which can be state dependent. For example, time-varying beliefs could represent behavioral biases ("representativeness") or path-dependent utility that makes agent B effectively more (less) risk averse following negative (positive) shocks to fundamentals (e.g., "catching up with the Joneses" utility as in Abel, 1990). Alternatively, we could introduce heterogeneous endowment shocks to generate trading (see, e.g., Wang, 1994). In all these cases, trading allows agents to share risk.

Let the Radon-Nikodym derivative of the probability measure $\mathbb{P}^B$ with respect to $\mathbb{P}$ be η. Then from Girsanov's theorem, we have:

$$\eta_t = \exp\left(\frac{1}{\sigma}\int_0^t \delta_s dZ_s - \frac{1}{2}\frac{1}{\sigma^2}\int_0^t \delta_s^2 ds\right). \tag{5}$$

Intuitively, since agent B will be more optimistic than A when $\delta_t > 0$, those paths with high realized values for $\int_0^t \delta_s dZ_s$ will be assigned higher probabilities under $\mathbb{P}^B$ than under $\mathbb{P}$.

Because there is no intermediate consumption, we use the riskless bond as the numeraire. Thus, the price of the bond is always 1. Let S_t denote the price of the stock at t.

[13]Under agent B's beliefs, δ_t will follow a different O-U process (see Eq. (A.4b) in Appendix A). In other words, the two agents not only disagree about future dividend growth, but also about how their disagreement will evolve in the future.

Circuit Breaker. To capture the essence of a circuit breaker rule, we assume that the stock market will be closed whenever the stock price S_t hits a threshold $(1-\alpha)S_0$, where S_0 is the endogenous initial price of the stock, and $\alpha \in [0,1]$ is a constant parameter determining the floor of downside price fluctuations during the interval $[0,T]$. Later in Section 6, we extend the model to allow for market closures for both downside and upside price movements, which represent price limit rules. The closing price for the stock is determined such that both the stock market and bond market clear when the circuit breaker is triggered. After that, the stock market will remain closed until time T.

In practice, the circuit breaker threshold is often based on the closing price from the previous trading session instead of the opening price of the current trading session. For example, in the U.S., a cross-market trading halt can be triggered at three circuit breaker thresholds (7%, 13%, and 20%) based on the prior day's closing price of the S&P 500 Index. However, the distinction between today's opening price and the prior day's closing price is not crucial for our analysis. The circuit breaker not only depends on but also endogenously affects the initial stock price, just like it does for prior day's closing price in practice.[14]

Finally, we impose usual restrictions on trading strategies to rule out arbitrage.

3 The Equilibrium

3.1 Benchmark Case: No Circuit Breaker

In this section, we solve for the equilibrium when there is no circuit breaker. To distinguish from the case with circuit breakers, we use the symbol "$\widehat{\ }$" to denote variables in the case without circuit breakers.

In the absence of circuit breakers, markets are dynamically complete. The equilibrium

[14] Other realistic features of the circuit breaker in practice is to close the market for m minutes and reopen (Level 1 and 2), or close the market until the end of the day (Level 3). In our model, we can think of T as one day. The fact that the price of the stock reverts back to the fundamental value D_T at T resembles the rationale of CB to "restore order" in the market.

allocation in this case can be characterized as the solution to the following planner's problem:

$$\max_{\widehat{W}_T^A, \widehat{W}_T^B} \mathbb{E}_0 \left[\lambda \ln \left(\widehat{W}_T^A \right) + (1-\lambda)\eta_T \ln \left(\widehat{W}_T^B \right) \right], \tag{6}$$

subject to the resource constraint:

$$\widehat{W}_T^A + \widehat{W}_T^B = D_T + \Delta. \tag{7}$$

From the agents' first-order conditions and the budget constraints, we obtain $\lambda = \omega$, and

$$\widehat{W}_T^A = \frac{\omega}{\omega + (1-\omega)\eta_T}(D_T + \Delta), \tag{8a}$$

$$\widehat{W}_T^B = \frac{(1-\omega)\eta_T}{\omega + (1-\omega)\eta_T}(D_T + \Delta). \tag{8b}$$

As it follows from the equations above agent B will be allocated a bigger share of the aggregate dividend when realized value of the Radon-Nikodym derivative η_T is higher, i.e., under those paths that agent B considers to be more likely.

The state price density under agent A's beliefs, which corresponds the objective probability measure $\mathbb{P}$, is given by:

$$\widehat{\pi}_t^A = \mathbb{E}_t \left[\xi u'(\widehat{W}_T^A) \right] = \mathbb{E}_t \left[\xi (\widehat{W}_T^A)^{-1} \right], \quad 0 \le t \le T \tag{9}$$

for some constant ξ. Then, from the budget constraint for agent A we see that the planner's weights are equal to the shares of endowment, $\lambda = \omega$. Using the state price density, one can then derive the price of the stock and individual investors' portfolio holdings.

In the limiting case with bond supply $\Delta \to 0$, the complete markets equilibrium can be characterized in closed form. We focus on this limiting case in the rest of this section. First, the following proposition summarizes the pricing results.

Proposition 1. *When there are no circuit breakers, the price of the stock in the limiting* *case with bond supply* $\Delta \to 0$ *is:*

$$\widehat{S}_t = \frac{\omega + (1-\omega)\eta_t}{\omega + (1-\omega)\,\eta_t\, e^{a(t,T)+b(t,T)\delta_t}} D_t e^{\left(\mu-\sigma^2\right)(T-t)}, \tag{10}$$

where

$$\begin{aligned} a(t,T) = {} & \left[\frac{\kappa\bar{\delta}-\sigma\nu}{\frac{\nu}{\sigma}-\kappa} + \frac{1}{2}\frac{\nu^2}{\left(\frac{\nu}{\sigma}-\kappa\right)^2}\right](T-t) - \frac{1|}{4}\frac{\nu^2}{\left(\frac{\nu}{\sigma}-\kappa\right)^3}\left[1-e^{2\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right] \\ & + \left[\frac{\kappa\bar{\delta}-\sigma\nu}{\left(\frac{\nu}{\sigma}-\kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma}-\kappa\right)^3}\right]\left[1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right], \end{aligned} \tag{11a}$$

$$b(t,T) = \frac{1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}}{\frac{\nu}{\sigma}-\kappa}. \tag{11b}$$

From Equation (10), we can derive the conditional volatility of the stock $\widehat{\sigma}_{S,t}$ in closed form, which is available in the appendix.

Next, we turn to the wealth distribution and portfolio holdings of individual agents. At time $t \le T$, the shares of total wealth of the two agents are:

$$\widehat{\omega}_t^A = \frac{\omega}{\omega + (1-\omega)\eta_t}, \quad \widehat{\omega}_t^B = 1 - \widehat{\omega}_t^A. \tag{12}$$

The number of shares of stock $\widehat{\theta}_t^A$ and units of riskless bonds $\widehat{\phi}_t^A$ held by agent A are:

$$\widehat{\theta}_t^A = \frac{\omega}{\omega + (1-\omega)\eta_t} - \frac{\omega(1-\omega)\eta_t}{\left[\omega + (1-\omega)\eta_t\right]^2}\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}} = \widehat{\omega}_t^A\left(1 - \widehat{\omega}_t^B\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}}\right), \tag{13}$$

$$\widehat{\phi}_t^A = \widehat{\omega}_t^A\widehat{\omega}_t^B\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}}\widehat{S}_t, \tag{14}$$

and the corresponding values for agent B are $\theta_t^B = 1 - \theta_t^A$ and $\phi_t^B = -\phi_t^A$.

As Equation (13) shows, there are several forces affecting the agents' portfolio positions. First, all else equal, agent A owns fewer shares of the stock when B has more optimistic beliefs (larger δ_t). This effect becomes weaker when the volatility of stock return $\widehat{\sigma}_{S,t}$ is high. Second, changes in the wealth distribution (as indicated by (12)) also affect the portfolio holdings, as the richer agent will tend to hold more shares of the stock.

We can gain more intuition on the stock price by rewriting Equation (10) as follows:

$$\widehat{S}_t = \frac{1}{\frac{\omega}{\omega+(1-\omega)\eta_t}\mathbb{E}_t\left[D_T^{-1}\right] + \frac{(1-\omega)\eta_t}{\omega+(1-\omega)\eta_t}\mathbb{E}_t^B\left[D_T^{-1}\right]} = \left(\frac{\widehat{\omega}_t^A}{\widehat{S}_t^A} + \frac{\widehat{\omega}_t^B}{\widehat{S}_t^B}\right)^{-1}, \tag{15}$$

which states that the stock price is a weighted harmonic average of the prices of the stock in two single-agent economies with agent A and B being the representative agent, respectively, denoted by $\widehat{S}_t^A$ and $\widehat{S}_t^B$, where

$$\widehat{S}_t^A = e^{\left(\mu-\sigma^2\right)(T-t)} D_t, \tag{16a}$$

$$\widehat{S}_t^B = e^{\left(\mu-\sigma^2\right)(T-t)-a(t,T)-b(t,T)\delta_t} D_t, \tag{16b}$$

and the weights $(\widehat{\omega}_t^A, \widehat{\omega}_t^B)$ are the two agents' shares of total wealth. Controlling for the wealth distribution, the equilibrium stock price is higher when agent B has more optimistic beliefs (larger δ_t).

One special case of the above result is when the amount of disagreement between the two agents is the zero, i.e., $\delta_t = 0$ for all $t \in [0, T]$. The stock price then becomes:

$$\widehat{S}_t = \widehat{S}_t^A = \frac{1}{\mathbb{E}_t[D_T^{-1}]} = e^{\left(\mu-\sigma^2\right)(T-t)} D_t, \tag{17}$$

which is a version of the Gordon growth formula, with σ^2 being the risk premium for the stock. The instantaneous volatility of stock returns becomes the same as the volatility of dividend growth, $\widehat{\sigma}_{S,t} = \sigma$. The shares of the stock held by the two agents will remain constant and be equal to the their endowments, $\widehat{\theta}_t^A = \omega$, $\widehat{\theta}_t^B = 1 - \omega$.

Another special case is when the amount of disagreement is constant over time ($\delta_t = \delta$ for all t). The results for this case are obtained by setting $\nu = 0$ and $\delta_0 = \bar{\delta} = \delta$ in Proposition 1. Equation (10) then simplifies to:

$$\widehat{S}_t = \frac{\omega + (1-\omega)\eta_t}{\omega + (1-\omega)\eta_t e^{-\delta(T-t)}} e^{\left(\mu-\sigma^2\right)(T-t)} D_t. \tag{18}$$

As expected, $\widehat{S}_t$ increases with δ, which reflects agent B's optimism on dividend growth.

3.2 Circuit Breaker

We start this section by introducing some notation. By θ_t^i, ϕ_t^i, and W_t^i we denote stock holdings, bond holdings, and wealth of agent i at time t, respectively, in the market with a circuit breaker. Let τ denote the time when the circuit breaker is triggered. It follows from the definition of the circuit breaker and the continuity of stock prices that τ satisfies

$$\tau = \inf\{t \geq 0 : S_t = (1-\alpha)S_0\}. \tag{19}$$

We use the expression $\tau \wedge T$ to denote $\min\{\tau, T\}$. Next, we define the equilibrium with a circuit breaker.

Definition 1. *The equilibrium with circuit breaker is defined by an $\mathcal{F}_t$-stopping time τ, trading strategies $\{\theta_t^i, \phi_t^i\}$ $(i = A, B)$, and a continuous stock price process S defined on the interval $[0, \tau \wedge T]$ such that:*

1. *Taking stock price process S as given, the two agents' trading strategies maximize their expected utilities under their respective beliefs and budget constraints.*
2. *For any $t \in [0, T]$, both the stock and bond markets clear,*

$$\theta_t^A + \theta_t^B = 1, \quad \phi_t^A + \phi_t^B = \Delta. \tag{20}$$

3. *The stopping time τ is consistent with the circuit breaker rule in (19).*

One crucial feature of the model is that markets remain dynamically complete until the circuit breaker is triggered. Hence, we solve for the equilibrium with the following three steps. First, consider an economy in which trading stops when the stock price reaches any given triggering price $\underline{S} \geq 0$. By examining the equilibrium conditions upon market closure, we can characterize the $\mathcal{F}_t$-stopping time τ that is consistent with $S_\tau = \underline{S}$. Next, we solve for the optimal allocation at $\tau \wedge T$ through the planner's problem as a function of $\underline{S}$, as well as the stock price prior to $\tau \wedge T$, also as a function of $\underline{S}$. Finally, the equilibrium is the fixed point whereby the triggering price $\underline{S}$ is consistent with the initial price, $\underline{S} = (1-\alpha)S_0$. We describe these steps in detail below.

Suppose the circuit breaker is triggered before the end of the trading session, i.e., $\tau < T$. We start by deriving the agents' indirect utility functions at the time of market closure. Agent i has wealth W^i_τ at time τ. Since the two agents behave competitively, they take the stock price S_τ as given and choose the shares of stock θ^i_τ and bonds ϕ^i_τ to maximize their expected utility over terminal wealth, subject to the budget constraint:

$$\begin{aligned} V^i(W^i_\tau, \tau) \quad = \quad & \max_{\theta^i_\tau, \phi^i_\tau} \quad \mathbb{E}^i_\tau \left[\ln(\theta^i_\tau D_T + \phi^i_\tau) \right], \\ & \quad s.t. \quad \theta^i_\tau S_\tau + \phi^i_\tau = W^i_\tau, \end{aligned} \tag{21}$$

where $V^i(W^i_\tau, \tau)$ is the indirect utility function for agent i at time $\tau < T$.

The market clearing conditions at time τ are:

$$\theta^A_\tau + \theta^B_\tau = 1, \quad \phi^A_\tau + \phi^B_\tau = \Delta. \tag{22}$$

For any $\tau < T$, the Inada condition implies that terminal wealth for both agents needs to stay non-negative, which implies $\theta^i_\tau \geq 0$ and $\phi^i_\tau \geq 0$. That is, neither agent will take short or levered positions in the stock. This is a direct result of the inability to rebalance one's portfolio after market closure, which is an extreme version of illiquidity.

Solving the problem (21) – (22) gives us the indirect utility functions $V^i(W^i_\tau, \tau)$. It also gives us the stock price at the time of market closure, S_τ, as a function of the dividend D_τ, the gap in beliefs δ_τ, and the wealth distribution at time τ (which is determined by the Radon-Nikodym derivative η_τ). Thus, the condition $S_\tau = \underline{S}$ translates into a condition on D_τ, δ_τ, and η_τ, which in turn characterizes the stopping time τ as a function of exogenous state variables. As we will see later, in the limiting case with bond supply $\Delta \to 0$, the stopping rule satisfying this condition can be expressed in closed form. When $\Delta > 0$, the solution can be obtained numerically.

Next, the indirect utility for agent i at $\tau \wedge T$ is given by:

$$V^i(W^i_{\tau\wedge T}, \tau \wedge T) = \begin{cases} \ln(W^i_T), & \text{if} \quad \tau \geq T \\ V^i(W^i_\tau, \tau), & \text{if} \quad \tau < T. \end{cases} \tag{23}$$

These indirect utility functions make it convenient to solve for the equilibrium wealth allocations in the economy at time $\tau \wedge T$ through the following planner problem:

$$\max_{W^A_{\tau\wedge T}, W^B_{\tau\wedge T}} \mathbb{E}_0 \left[\lambda V^A(W^A_{\tau\wedge T}, \tau \wedge T) + (1-\lambda)\eta_{\tau\wedge T} V^B(W^B_{\tau\wedge T}, \tau \wedge T)\right], \tag{24}$$

subject to the resource constraint:

$$W^A_{\tau\wedge T} + W^B_{\tau\wedge T} = S_{\tau\wedge T} + \Delta, \tag{25}$$

where

$$S_{\tau\wedge T} = \begin{cases} D_T, & \text{if} \quad \tau \geq T \\ \underline{S}, & \text{if} \quad \tau < T. \end{cases} \tag{26}$$

Taking the equilibrium allocation $W^A_{\tau\wedge T}$ from the planner's problem, the state price density for agent A at time $\tau \wedge T$ can be expressed as his marginal utility of wealth times a constant ξ:

$$\pi^A_{\tau\wedge T} = \xi \, \frac{\partial V^A(W, \tau \wedge T)}{\partial W}\Big|_{W=W^A_{\tau\wedge T}}. \tag{27}$$

The price of the stock at any time $t \leq \tau \wedge T$ is then given by:

$$S_t = \mathbb{E}_t \left[\frac{\pi^A_{\tau\wedge T}}{\pi^A_t} S_{\tau\wedge T}\right], \tag{28}$$

where like in Equation (9),

$$\pi^A_t = \mathbb{E}_t \left[\pi^A_{\tau\wedge T}\right]. \tag{29}$$

The expectations above are straightforward to evaluate, at least numerically.

Having obtained the solution for S_t as a function of $\underline{S}$, we can finally solve for the equilibrium triggering price $\underline{S}$ through the following fixed point problem,

$$\underline{S} = (1-\alpha) S_0. \tag{30}$$

Proposition 2. *There exists a solution to the fixed-point problem in (30) for any $\alpha \in [0, 1]$.*

To see why Proposition 2 holds, consider S_0 as a function of $\underline{S}$, $S_0 = f(\underline{S})$. First notice that when $\underline{S} = 0$, there is essentially no circuit breaker, and $f(0)$ will be the same as the initial stock price in the complete markets case. Next, there exists $s^* > 0$ such that $s^* = f(s^*)$, which is the initial price when the market closes immediately after opening. The fact that f is continuous ensures that there exists at least one crossing between the function $f(s)$ and $s/(1-\alpha)$, which will be a solution for (30).

Below we will show how these steps can be neatly solved in the special case when riskless bonds are in zero net supply.

Because neither agent will take levered or short positions during market closure, there cannot be any lending or borrowing in that period. Thus, in the limiting case with net bond supply $\Delta \to 0$, all the wealth of the two agents will be invested in the stock upon market closure. Consequently the leverage constraint will always bind for the relatively optimistic investor in the presence of heterogeneous beliefs. The result is that the relatively pessimistic investor becomes the marginal investor, as summarized in the following proposition.

Proposition 3. *Suppose the stock market closes at time $\tau < T$. In the limiting case with bond supply $\Delta \to 0$, at τ both agents will hold all of their wealth in the stock, $\theta^i_\tau = \frac{W^i_\tau}{S_\tau}$, and hold no bonds, $\phi^i_\tau = 0$. The market clearing price is:*

$$S_\tau = \min\left\{\widehat{S}^A_\tau, \widehat{S}^B_\tau\right\} = \begin{cases} e^{\left(\mu-\sigma^2\right)(T-\tau)} D_\tau, & \text{if} \quad \delta_\tau > \underline{\delta}(\tau) \\ e^{\left(\mu-\sigma^2\right)(T-\tau)-a(\tau,T)-b(\tau,T)\delta_\tau} D_\tau, & \text{if} \quad \delta_\tau \leq \underline{\delta}(\tau) \end{cases} \tag{31}$$

where $\widehat{S}^i_\tau$ denotes the stock price in a single-agent economy populated by agent i, as given in (16a)-(16b)*:*

$$\underline{\delta}(t) = -\frac{a(t,T)}{b(t,T)}, \tag{32}$$

and $a(t,T)$, $b(t,T)$ are given in Proposition 1.

Clearly, the market clearing price S_τ only depends on the belief of the relatively pessimistic agent. This result is qualitatively different from the complete markets case, where the stock price is a wealth-weighted average of the prices under the two agents' beliefs. It is a crucial result: the lower stock valuation upon market closure affects both the stock price level and dynamics before market closure, which we analyze in Section 4. Notice that having the lower expectation of the growth rate at the current instant is not sufficient to make the agent marginal. One also needs to take into account the agents' future beliefs and the risk premium associated with future fluctuations in the beliefs, which are summarized by $\underline{\delta}(t)$.[15]

Equation (31) implies that we can characterize the stopping time τ using a stochastic threshold for dividend D_t, as summarized below.

Lemma 1. *Take the triggering price $\underline{S}$ as given. Define a stopping time:*

$$\tau = \inf\{t \geq 0 : D_t = \underline{D}(t, \delta_t)\}, \tag{33}$$

where

$$\underline{D}(t, \delta_t) = \begin{cases} \underline{S} e^{-\left(\mu - \sigma^2\right)(T-t)}, & \textit{if} \quad \delta_t > \underline{\delta}(t) \\ \underline{S} e^{-\left(\mu - \sigma^2\right)(T-t) + a(t,T) + b(t,T)\delta_t}, & \textit{if} \quad \delta_t \leq \underline{\delta}(t). \end{cases} \tag{34}$$

Then, in the limiting case with bond supply $\Delta \to 0$, the circuit breaker is triggered at time τ whenever $\tau < T$.

Having characterized the equilibrium at time $\tau < T$, we plug the equilibrium portfolio holdings into (21) to derive the indirect utility of the two agents at τ:

$$V^i(W^i_\tau, \tau) = \mathbb{E}^i_\tau \left[\ln \left(\frac{W^i_\tau}{S_\tau} D_T \right) \right] = \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)]. \tag{35}$$

The indirect utility for agent i at $\tau \wedge T$ is then given by:

$$V^i(W^i_{\tau \wedge T}, \tau \wedge T) = \begin{cases} \ln(W^i_T), & \text{if} \quad \tau \geq T \\ \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)], & \text{if} \quad \tau < T. \end{cases} \tag{36}$$

Substituting these indirect utility functions into the planner's problem (24) and taking

[15] Technically, there is a difference between the limiting case with $\Delta \to 0$ and the case with $\Delta = 0$. When $\Delta = 0$, any price equal or below S_τ in (31) will clear the market. At such prices, both agents would prefer to invest more than 100% of their wealth in the stock, but both will face binding leverage constraints, which is why the stock market clears at these prices. However, these alternative equilibria are ruled out by considering a sequence of economies with bond supply $\Delta \to 0$. In each of these economies where $\Delta > 0$, the relatively pessimistic agent needs to hold the bond in equilibrium, which means his leverage constraint cannot not be binding.

the first order condition, we get the wealth of agent A at time $\tau \wedge T$:

$$W^A_{\tau\wedge T} = \frac{\omega S_{\tau\wedge T}}{\omega + (1-\omega)\,\eta_{\tau\wedge T}}, \tag{37}$$

where $S_{\tau\wedge T}$ is given in (26). Then, we obtain the state price density for agent A and the price of the stock at time $t \le \tau \wedge T$ as in (27) and (28), respectively. In particular,

$$S_t = \left(\omega^A_t \mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \omega^B_t \mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right]\right)^{-1}. \tag{38}$$

Here, ω^i_t is the share of total wealth owned by agent i, which, in the limiting case with $\Delta \to 0$, is identical to $\widehat{\omega}^i_t$ in (12) before market closure. Equation (38) is reminiscent of its complete markets counterpart (15). Unlike in the case of complete markets, the expectations in (38) are no longer the inverse of the stock prices from the respective representative agent economies.

From the equilibrium stock price, we can then compute the conditional mean $\mu_{S,t}$ and volatility $\sigma_{S,t}$ of stock returns, which are given by:

$$dS_t = \mu_{S,t} S_t dt + \sigma_{S,t} S_t dZ_t. \tag{39}$$

In Appendix A.3, we provide the closed-form solution for S_t in the special case with constant disagreements ($\delta_t \equiv \delta$).

Finally, by evaluating S_t at time $t = 0$, we can solve for $\underline{S} = (1-\alpha)S_0$ from the fixed point problem (30). Beyond the existence result of Proposition 2, one can further show that the fixed point is unique when the riskless bond is in zero net supply.[16]

The case of positive bond supply. When the riskless bond is in positive net supply, there are four possible scenarios upon market closure: the relatively optimistic agent faces binding leverage constraint, while the relatively pessimistic agent is either unconstrained (Scenario i) or faces binding short-sale constraint (Scenario ii); the relatively optimistic agent

[16]The uniqueness is due to the fact that S_0 will be monotonically decreasing in the triggering price $\underline{S}$ in the limiting case when $\Delta \to 0$, which is not necessarily true when $\Delta > 0$.

is unconstrained, while the relatively pessimistic agent is either unconstrained (Scenario iii) or faces binding short-sale constraint (Scenario iv). In contrast, only Scenario (i) is possible when the riskless bond is in zero net supply. The three new scenarios originate from the fact that when $\Delta > 0$ the two agents can hold different portfolios without borrowing and lending; furthermore, when the relatively optimistic agent is sufficiently wealthy, he could potentially hold the entire stock market without having to take on any leverage.

In particular, Scenario (iv) is the opposite of Scenario (i) in that the relatively optimistic agent, instead of the pessimistic one, becomes the marginal investor. As a result, the price level can become higher and volatility lower in the economy with a circuit breaker. Under Scenarios (ii) and (iii), the equilibrium stock price upon market closure is somewhere in between the two agents' valuations.

In Section 5.1, we examine the conditions (wealth distribution, size of bond supply, and amount of disagreement upon market closure) that determine which of the four scenarios occur in equilibrium. As we show later, which of the scenarios is realized has important implications for the equilibrium price process.

Circuit breaker and wealth distribution. We conclude this section by examining the impact of circuit breakers on the wealth distribution. As explained earlier, the wealth shares of the two agents before market closure (at time $t \leq \tau \wedge T$) will be the same as in the economy without circuit breakers, and take the form in (12) when the riskless bonds are in zero net supply.

However, the wealth shares at the end of the trading day (time T) will be affected by the presence of the circuit breaker. This is because if the circuit breaker is triggered at $\tau < T$, the wealth distribution after τ will remain fixed due to the absence of trading. Since irrational traders on average lose money over time, market closure at $\tau < T$ will raise their average wealth share at time T. This "mean effect" implies that circuit breakers will help "protecting" the irrational investors in this model. How strong this effect is depends on the amount of disagreement and the distribution of τ. In addition, circuit breakers will also make the tail of the wealth share distribution thinner as they put a limit on the amount of wealth that the relatively optimistic investor can lose over time along those paths with low realizations of D_t.

4 Impact of Circuit Breakers on Market Dynamics

We now turn to the quantitative implications of the model. In Section 4.1 we examine the special case of constant disagreement, $\delta_t \equiv \delta$. This case helps to demonstrate the main mechanism through which circuit breakers affect trading and asset prices. Then, in Section 4.2, we examine the general case with time-varying disagreements. Throughout this section we focus on the case where riskless bonds are in zero net supply ($\Delta \to 0$). We examine the robustness of these results in Section 5.

4.1 Constant Disagreement

For calibration, we normalize $T = 1$ to denote one trading day. We set the expected value of the dividend growth $\mu = 10\%/250 = 0.04\%$ (implying an annual dividend growth rate of 10%) and its (daily) volatility $\sigma = 3\%$. The downside circuit breaker threshold is set at $\alpha = 5\%$. For the initial wealth distribution, we assume agent A (with rational beliefs) owns 90% of total wealth ($\omega = 0.9$) at $t = 0$. For the amount of disagreement, we set $\delta = -2\%$. This means agent B is relatively pessimistic about dividend growth, and his valuation of the stock at $t = 0$, $\widehat{S}_0^B$, will be 2% lower than that of agent A, $\widehat{S}_0^A$, which is fairly modest.

In Figure 2, we plot the equilibrium price-dividend ratio S_t/D_t (left column), the conditional volatility of returns (middle column), and the stock holding for agent A (right column). The stock holding for agent B can be inferred from that of agent A, as $\theta_t^B = 1 - \theta_t^A$. In each panel, the solid line denotes the solution for the case with circuit breaker, while the dotted line denotes the case without circuit breaker. To examine the time-of-the-day effect, we plot the solutions at two different points in time, $t = 0.25$ and 0.75, respectively.

Let's start with the price-dividend ratio. As discussed in Section 3.1, the price of the stock in the case without circuit breaker is the weighted (harmonic) average of the prices of the stock from the two representative-agent economies populated by agent A and B, respectively, with the weights given by the two agents' shares of total wealth (see equation

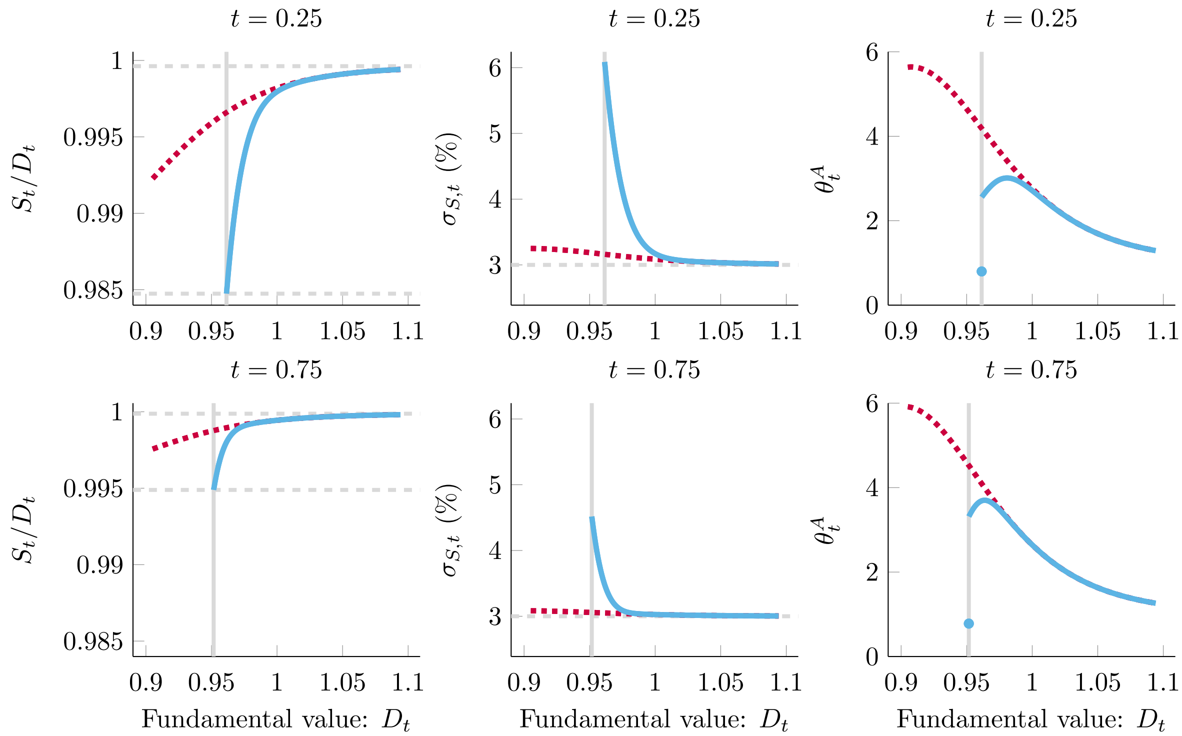

Figure 2: **Price-dividend ratio, conditional return volatility, and agent A's (rational optimist) portfolio holding.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

(15)). Under our calibration, the price-dividend ratio is close to one for any $t \in [0, T]$ under agent A's beliefs ($\widehat{S}_t^A/D_t$), and it is approximately equal to $e^{\delta(T-t)} \leq 1$ under agent B's beliefs ($\widehat{S}_t^B/D_t$). These two values are denoted by the upper and lower horizontal dash lines in the left column of Figure 2.

The price-dividend ratio in the economy without circuit breaker (red dotted line) indeed lies between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$. Since agent A is relatively more optimistic, he will hold levered position in the stock (see the red dotted line in the middle column), and his share of total wealth will become higher following positive shocks to the dividend. Thus, as dividend value D_t rises (falls), the share of total wealth owned by agent A increases (decreases), which makes the equilibrium price-dividend ratio approach the value $\widehat{S}_t^A/D_t$ ($\widehat{S}_t^B/D_t$).

In the case with circuit breaker, the price-dividend ratio (blue solid line) still lies between the price-dividend ratios from the two representative agent economies, but it is always below the price-dividend ratio without circuit breaker for any given level of dividend. The gap between the two price-dividend ratios is negligible when D_t is sufficiently high, but it widens as D_t approaches the circuit breaker threshold $\underline{D}(t)$.

The reason that stock price declines more rapidly with dividend in the presence of a circuit breaker can be traced to how the stock price is determined upon market closure. As explained in Section 3.2, at the instant when the circuit breaker is triggered, neither agent will be willing to take on levered position in the stock due to the inability to rebalance the portfolio. With bonds in zero net supply, the leverage constraint always binds for the relatively optimistic agent (agent A), and the market clearing stock price has to be such that agent B is willing to hold all of his wealth in the stock, *regardless of his share of total wealth*. Indeed, we see the price-dividend ratio with circuit breaker converging to $\widehat{S}_t^B/D_t$ when D_t approaches $\underline{D}(t)$, instead of the wealth-weighted average of $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$. The lower stock price at the circuit breaker threshold also drives the stock price lower before market closure, with the effect becoming stronger as D_t moves closer to the threshold $\underline{D}(t)$. This explains the accelerated decline in stock price as D_t drops.

The higher sensitivity of the price-dividend ratio to dividend shocks due to the circuit breaker manifests itself in elevated conditional return volatility, as shown in the middle column of Figure 2. Quantitatively, the impact of the circuit breaker on the conditional volatility of stock returns can be quite sizable. Without circuit breaker, the conditional volatility of returns (red dotted lines) peaks at about 3.2%, only slightly higher than the fundamental volatility of $\sigma = 3\%$. This small amount of excess volatility comes from the time variation in the wealth distribution between the two agents. With circuit breaker, the conditional volatility (blue solid lines) becomes substantially higher as D_t approaches $\underline{D}(t)$. For example, when $t = 0.25$, the conditional volatility reaches 6% at the circuit breaker threshold, almost twice as high as the return volatility without circuit breaker.

We can also analyze the impact of the circuit breaker on the equilibrium stock price by connecting it to how the circuit breaker influences the equilibrium portfolio holdings of the two agents. Let us again start with the case without circuit breaker (red dotted lines in right column of Figure 2). The stock holding of agent A, $\widehat{\theta}_t^A$, continues to rise as D_t falls to $\underline{D}(t)$ and beyond. This is the result of two effects: (i) with lower D_t, the stock price is lower, implying higher expected return under agent A's beliefs; (ii) lower D_t also makes agent B (who is shorting the stock) wealthier and thus more capable of lending to agent A, who then takes on a more levered position.

With circuit breaker, while the stock holding θ_t^A takes on similar values as $\widehat{\theta}_t^A$, its counterpart in the case without circuit breaker, for large values of D_t, it becomes visibly lower than $\widehat{\theta}_t^A$ as D_t approaches the circuit breaker threshold, and it eventually starts to decrease as D_t continues to drop.

This is because agent A becomes increasingly concerned with the rising return volatility at lower D_t, which eventually dominates the effect of higher expected stock return. Finally, θ_t^A takes a discrete drop when $D_t = \underline{D}(t)$. With the leverage constraint binding, agent A will hold all of his wealth in the stock, which means θ_t^A will be equal to his wealth share ω_t^A. The preemptive deleveraging by agent A can be interpreted as a form of "self-predatory" trading. The stock price in equilibrium has to fall enough such that agent A has no incentive to sell more of his stock holding.

Time-of-the-day effect. Comparing the cases with $t = 0.25$ and $t = 0.75$, we see that the impact of circuit breaker on the price-dividend ratio and return volatility weakens as t approaches T. For example, at $t = 0.25$, the price-dividend ratio with circuit breaker can be as much as 1.2% lower than the level without circuit breaker, and the conditional return volatility peaks 6%. In contrast, at $t = 0.75$, the gap in price-dividend ratio is at most 0.3%, and the peak return volatility is 4.5%.

The reason behind this result is quite straightforward: A shorter remaining horizon reduces the potential impact of agent B's pessimistic beliefs on the equilibrium stock price, as reflected in the shrinking gap between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ (the two horizontal dash lines) from the top left panel to the bottom left panel in Figure 2. Thus, this "time-of-the-day" effect really reflects the fact that the potential impact of circuit breaker is larger when there is more disagreement.

Notice also that the circuit breaker threshold $\underline{D}(t)$ becomes lower as t increases. That is, the dividend needs to drop more to trigger the circuit breaker later in the day. This is because the price-dividend ratio for any given D_t becomes higher as t increases.

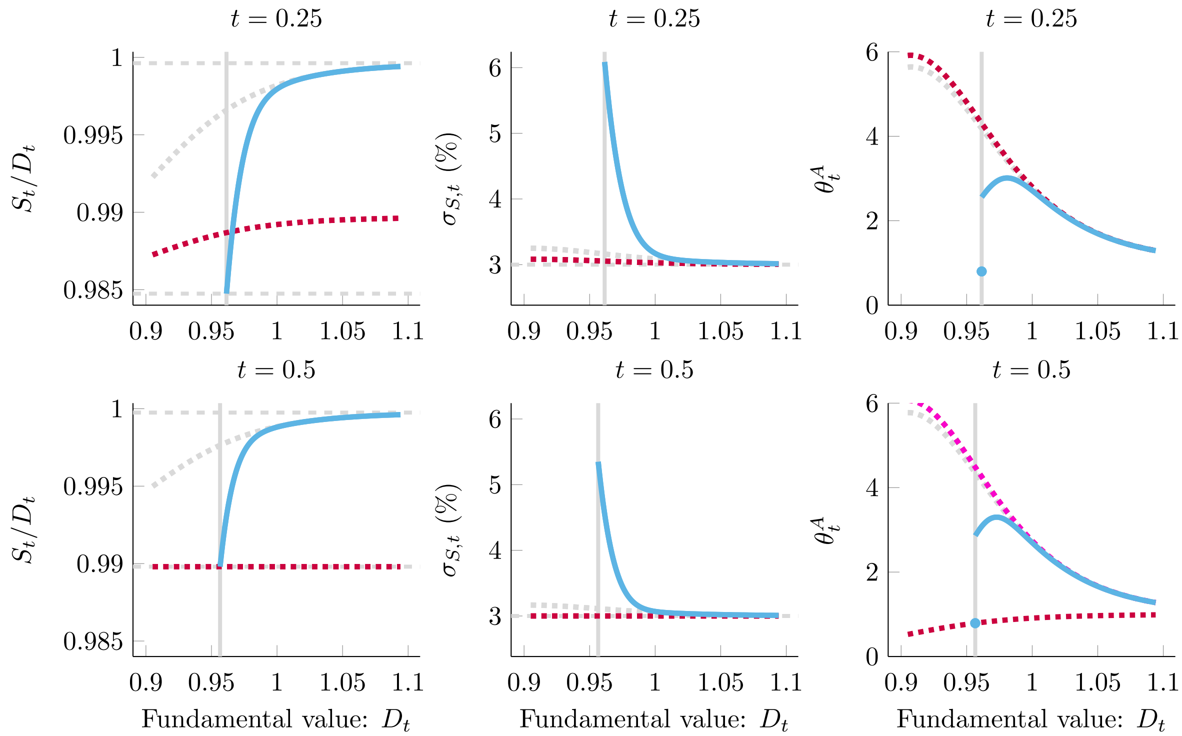

Figure 3: **Circuit breaker vs. pre-scheduled trading halt.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case with pre-scheduled trading halt at $\mathcal{T} = 0.5$. Grey dotted lines are for the case without trading halts. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$. The purple dotted line in the last panel denotes θ_t^A for $t = 0.49$.

Circuit breaker vs. pre-scheduled trading halt. Like the price-based circuit breaker, pre-scheduled trading halts, such as daily market closures, will also prevent investors from rebalancing their portfolios for an extended period of time. However, the implications of such pre-scheduled trading halts on trading behavior and price dynamics are quite different from those of circuit breakers. The key difference is that, in the case of a circuit breaker, the trigger of trading halt endogenously depends on the dividend. A negative shock to fundamentals not only reduces the price-dividend ratio through its impact on the wealth distribution (as in the case without circuit breakers), but also drives the price-dividend ratio closer to the level based on the pessimist's beliefs by moving the markets closer to the trading halt threshold.

This second effect is absent in the case of a pre-scheduled trading halt. As t approaches the pre-scheduled time of market closure $\mathcal{T}$, the price-dividend ratio converges to the pessimist valuation for all levels of dividend (which only occurs when D_t approaches $\underline{D}(t)$ in the case with circuit breaker). Away from $\mathcal{T}$, the price level is lower for all levels of dividend due to the expectation of trading halt, but there is no additional sensitivity of the price-dividend ratio to fundamental shocks, hence no volatility amplification.

To illustrate these differences, Figure 3 plots the price-dividend ratio, the conditional return volatility, and agent A's portfolio holding in the case when the market is scheduled to close at $\mathcal{T} = 0.5$ and remains closed until $T = 1$ (red dotted lines). We then compare these results against the case with a 5% circuit breaker (blue solid lines) as well as the case without any trading halts (grey dotted lines).

Among the three cases, the price-dividend ratio has the most sensitivity to changes in dividend in the case of a circuit breaker; consequently, the conditional return volatility is the highest in that case. Interestingly, the conditional return volatility is the lowest with the pre-scheduled trading halt, and it almost does not change with the dividend. Moreover, unlike in the circuit breaker case, there is no preemptive deleveraging with pre-scheduled trading halt – agent A continues to take levered positions in the stock market as t approaches $\mathcal{T}$, and only delevers at the instant of market closure (see the purple dotted line – agent A's stock holding at $t = 0.49$ – and the red dotted line – stock holding at $t = 0.5$ – in the bottom right panel).

4.2 Time-varying Disagreement

In the previous section, we use the special case of constant disagreement to illustrate the impact of circuit breakers on trading and price dynamics. We now turn to the full model with time-varying disagreement, where the difference in beliefs δ_t follows a random walk. We do so by setting $\kappa = 0$, $\nu = \sigma$, and $\delta_0 = \bar{\delta} = 0$. Thus, there is neither initial nor long-term bias in agent B's belief.

Under this specification, Agent B's beliefs resemble the "representativeness" bias in behavioral economics. As a form of non-Bayesian updating, he extrapolates his belief about future dividend growth from the realized path of dividend.[17] As a result, he becomes overly

[17]Specifically, $\delta_t = \ln\left[\frac{D_t}{D_0} e^{-(\mu-\sigma^2/2)t}\right]$, which is a mean-adjusted nonannualized realized growth rate.

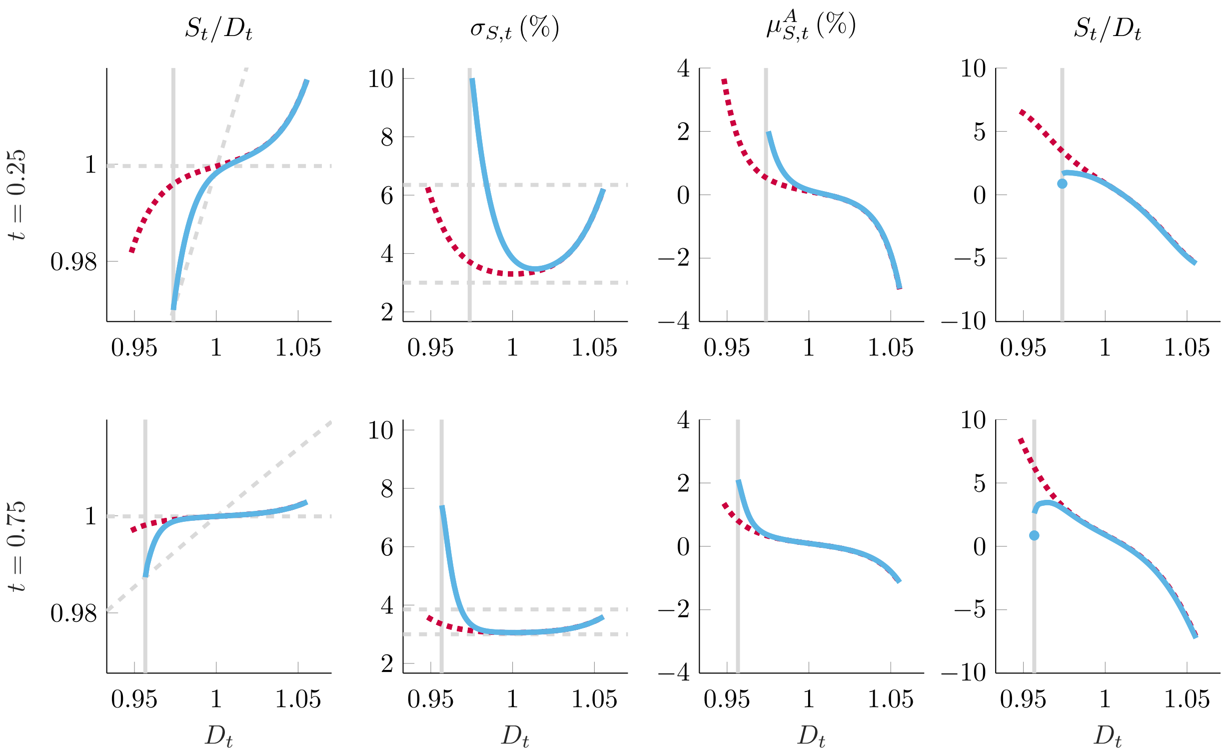

Figure 4: **Price-dividend ratio and agent A's portfolio in the case of time-varying disagreements.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

optimistic following large positive dividend shocks and overly pessimistic following large negative dividend shocks. An alternative interpretation of such beliefs is that they capture in reduced form the behavior of constrained investors, who effectively become more (less) pessimistic or risk averse as the constraint tightens (loosens).

In Figure 4, we plot the price-dividend ratio, conditional return volatility, conditional expected returns under the objective probability measure, and agent A's stock holding. Unlike the constant disagreement case, dividend D_t and time of the day t are no longer sufficient to determine the state of the economy. Thus, we plot the average values of the variables conditional on t and D_t.[18]

[18] Given our calibration of δ_t process as a random walk, the one additional state variable besides t and D_t is the Radon-Nikodym derivative η_t, or equivalently, $\int_0^t Z_s^2 ds$ (which together with D_t determines η_t). There is no need to keep track of δ_t separately because of the one-to-one mapping between δ_t and D_t. Thus, we plot the variables of interest while setting the integral $\int_0^t Z_s^2 ds$ equal to its expected value conditional on D_t.

Let's start with the price-dividend ratio, shown in the first column of Figure 4. Since agent A's belief about the dividend growth rate is constant over time, the price-dividend ratio under his beliefs is constant over different values of D_t (the horizontal grey dash line). However, due to the variation in δ_t which is perfectly correlated with D_t, the price-dividend ratio under agent B's beliefs now increases with D_t (the upward-sloping grey dash line). The price-dividend ratio in the equilibrium without circuit breaker (red dotted line) is still a wealth-weighted average of the price-dividend ratios under the two agents' beliefs. In the presence of a circuit breaker, for any given level of dividend D_t above the circuit breaker threshold, the price-dividend ratio is lower than the value without circuit breaker, and the difference becomes more pronounced as D_t approaches the threshold $\underline{D}(t)$.[19] These properties are qualitatively the same as in the case of constant disagreement.

The circuit breaker does rule out extreme low values for the price-dividend ratio during the trading session, which could have occurred at extreme low dividend values had trading continued. This could be one of the benefits of circuit breakers. When there are intra-day mark-to-market requirements for some of the market participants, a narrower range for the price-dividend ratio can help reduce the chances of inefficient liquidations that could further destabilize the market. Formally modeling such frictions will be an interesting direction for future research.

However, the circuit breaker generates significant volatility amplification when D_t is close to $\underline{D}(t)$ (see Figure 4, second column). The conditional return volatility with time-varying disagreement can reach as high as 10% at $t = 0.25$, compared to the peak volatility of 6% in the constant disagreement case and the fundamental volatility of 3%. Like in the constant disagreement case, the volatility amplification effect weakens as t approaches T ("time-of-the-day effect"), which is because the effective amount of disagreement between the two agents is falling with t.

The third column of Figure 4 plot the conditional expected returns under the agent A's (objective) beliefs. Even when there is no circuit breaker, the conditional expected return rises as dividend falls. This is because the irrational agent B is both gaining wealth

[19] In general cases, the threshold $\underline{D}(t, \delta_t)$ depends on both t and δ_t. Since our calibration of the δ process implies a one-to-one mapping between δ_t and D_t, the threshold becomes unique for any t.

share and becoming more pessimistic as D_t falls, driving prices lower and expected returns higher for agent A higher. The presence of the circuit breaker accelerates the increase in the conditional expected return as D_t approaches the threshold $\underline{D}(t)$. Despite the higher expected returns, agent A still becomes more and more conservative when investing in the stock (see Figure 4, last column) due to the concern of market closure. In fact, the preemptive deleveraging by agent A is again evident as D_t approaches $\underline{D}(t)$.

Unconditional distributions of price and volatility. So far we have been analyzing the conditional effects of the circuit breaker on prices, volatility, and portfolio holdings. Next, in Figure 5, we examine the impact of circuit breakers on the distribution of daily average price-dividend ratios, daily price ranges, and daily return volatility. Daily price range is defined as daily high minus low prices, while daily return volatility is defined as the square root of the quadratic variation of $\log(S_t)$ over the period $[0, \tau \wedge T]$ and scaled back to daily value.

The top panel of Figure 5 shows that the distribution of daily average price-dividend ratio is shifted to the left in the presence of a circuit breaker, and the left tail of the distribution becomes fatter. The magnitude of the price distortion is small on average, because the large price distortions (when D_t approaches $\underline{D}(t)$) occur infrequently.

The middle and bottom panels illustrate the impact of the circuit breaker on volatility. Theoretically, when it comes to the daily price range, a commonly used measure of volatility in market microstructure studies, the circuit breaker, by limiting stock price from below, has a potential to reduce daily price ranges for certain paths of the dividend process. As the middle panel shows, however, statistically this effect is dominated by an increased price ranges for other realizations of the dividend process. So the presence of the circuit breaker shifts the whole distribution of the daily price range to the right. One gets a similar message when it comes to daily realized volatility. The presence of a circuit breaker generates a significantly fatter right tail for the distribution of daily realized volatility.

The "magnet effect". The "magnet effect" is a popular term among practitioners that refers to the changes in price dynamics as the price moves closer to the limit. While there is

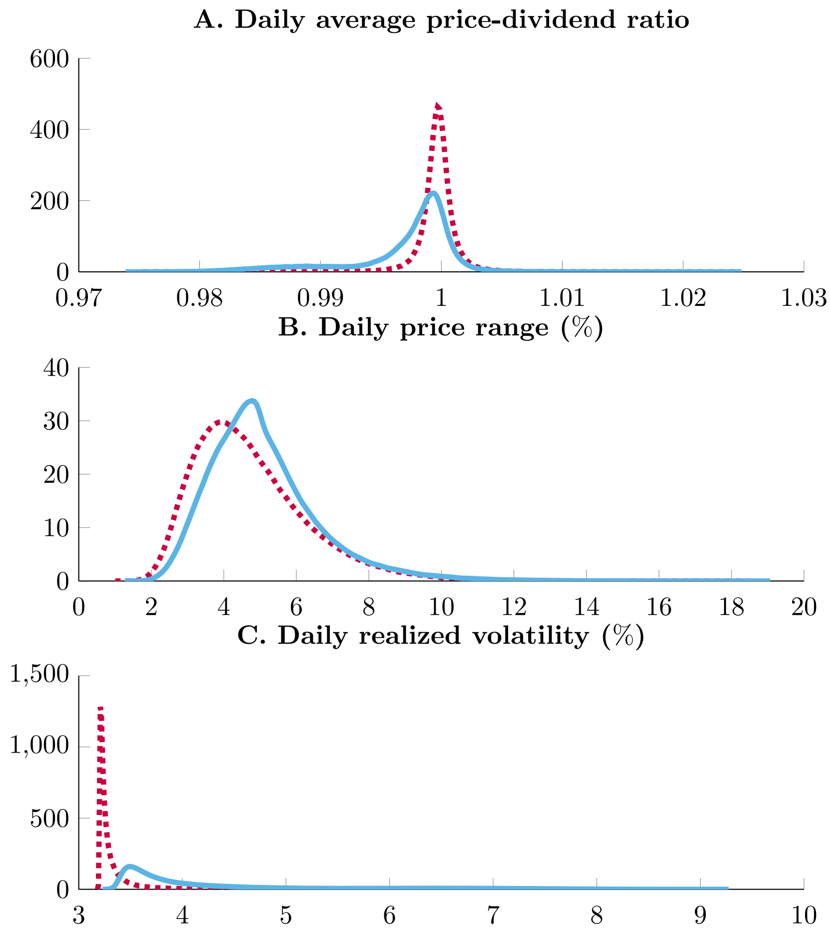

Figure 5: **Distributions of price-dividend ratio, daily price range, and realized volatility.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker.

no formal definition of this effect, we try to formalize this notion in our model by computing the conditional probability that the stock price, currently at S_t, will reach the circuit breaker threshold $(1-\alpha)S_0$ within a given period of time h, which we refer to as conditional hitting probability, and comparing these probabilities to their counterparts in absence of the circuit breaker.

In Figure 6, we plot the conditional hitting probabilities for the horizon of $h = 10$ minutes. When S_t is sufficiently far from $(1-\alpha)S_0$ (say $S_t > 0.98$), the conditional hitting probabilities with and without circuit breaker are both essentially zero. The gap between the two hitting probabilities quickly widens as the stock price moves closer to the threshold. By the time S_t reaches 0.96, the conditional hitting probability with circuit breaker has

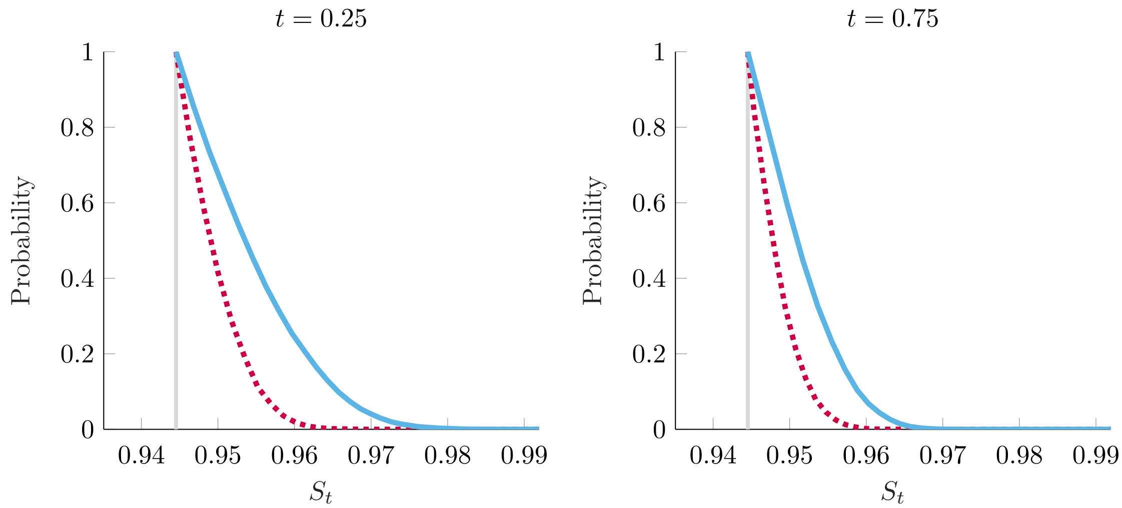

Figure 6: **The "magnet effect".** Conditional probabilities for the stock price to reach the circuit breaker limit within the next 10 minutes. Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

risen above 20%, while the hitting probability without circuit breaker is still close to 0. The gap eventually narrows as both hitting probabilities will converge to 1 as S_t reaches $(1 - \alpha)S_0$.

This is our version of the "magnet effect": the very presence of a circuit breaker raises the probability of the stock price reaching the threshold. Moreover, the pace at which the hitting probability increases as the stock price moves closer to the threshold will be much faster with a circuit breaker than what a "normal price process" would imply. The "magnet effect" is caused by the significant increase in conditional return volatility in the presence of a circuit breaker. Combined with the "time-of-the-day" effect, it is not surprising to see that the "magnet effect" is stronger earlier during the trading day.

Welfare implications In the absence of other frictions, trading halts would reduce investors' abilities to share risk. When the reason to trade is heterogeneous beliefs and the social planner respects the beliefs of individual investors, then any such trading halts will inevitably reduce welfare. The amount of welfare loss depends on the initial wealth distribution (it will be higher when wealth is more evenly distributed). Based on our calibration, the certainty equivalent loss in wealth peaks at close to 3%.

Alternatively, if the planner takes a paternalistic view by evaluating welfare under the correct probability measure, then circuit breakers could protect those investors with irrational beliefs from hurting themselves by trading too much. Such calculations yield certainty equivalent gain in wealth that peaks at about 1%. While this result might appear to provide some justification for implementing circuit breaker rules, its implication would be to prevent trading by irrational investors altogether.

While our framework provides a neoclassical benchmark that highlights some of the negative effects of circuit breakers, it may well be incomplete in providing a full analysis on welfare or the optimal design of circuit breaker rules. To do so requires one to properly take into account the actual frictions such as coordination and information problems, which we leave for future research.

5 Robustness

Our analysis in Section 4 has focused on the case where riskless bonds are in zero net supply ($\Delta \to 0$). In this section, we examine the robustness of these results when riskless bonds are in positive supply. In addition, we also discuss the differences between the continuous-time and discrete-time settings.

5.1 Positive Bond Supply

In the model with positive riskless bond supply, we first consider the problem at the instant before market closure, which will provide us with much of the intuition about the effect of positive bond supply. Suppose the stock market will close at some arbitrary time τ with fundamental value D_τ. There is still trading at time τ, but the agents have to hold onto their portfolios thereafter until time T. The equilibrium conditions are already given in Section 3.2.

For illustration, in Figure 7 we plot the equilibrium stock price as a function of the wealth share of agent A for $\tau = 0.25$ and $D_\tau = 0.97$ (blue solid line), and compare it to the stock price under complete markets (red dotted line). The bond supply is assumed to be

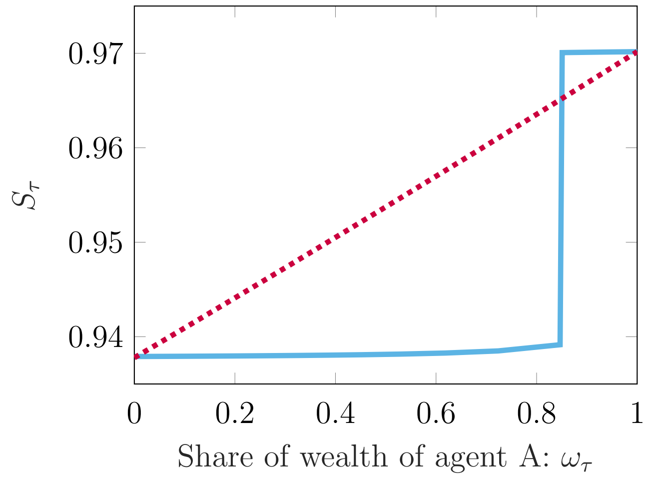

Figure 7: **Stock price upon market closure: positive bond supply.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker.

$\Delta = 0.17$. Notice that because of the one-to-one mapping between D_t and δ_t, we know that agent A is relatively more optimistic for this value of D_τ.

As before, the stock price without circuit breaker is a weighted average of the optimist and pessimist valuations in the case with positive bond supply, with the weight depending on their respective wealth shares. Since agent A is more optimistic, the stock price without circuit breaker is linearly increasing in her wealth share.

We have seen that, when riskless bonds are in zero net supply, the stock price with a circuit breaker will always be equal to the pessimistic valuation at the time of market closure. However, this is no longer the case with $\Delta > 0$. As Figure 7 shows, when agent A's wealth share is not too high, the stock price with circuit breaker is lower than its complete markets counterpart, but the opposite occurs when agent A's wealth share is sufficiently high.

The intuition is as follows. When agent A's wealth share ω_τ^A is not too high, he invests all of his wealth into the stock, but that is still not enough to clear the stock market. In this case, agent A's leverage constraint will be binding, agent B will hold all the riskless bonds and the remaining stock not held by agent A, and the market clearing price has to agree with agent B's (the pessimist) valuation.

This scenario is similar to the case where riskless bonds are in zero net supply, where the pessimist is also the marginal investor. One difference is that the pessimist's valuation here increases with ω_τ^A instead of remaining constant. This is because as agent A gets wealthier, agent B's portfolio will become less risky (he is required to hold less stock relative to the riskless bonds), which makes him value the stock more. However, this effect is quantitatively small.

When agent A's wealth share becomes sufficiently high, he will be able to hold the entire stock market without borrowing. This is possible because not all of the wealth in the economy is in the stock. In such cases, agent A will hold all of the stock and potentially some bonds, while agent B will invest all of his wealth in the bonds. As long as the stock price is above agent B's private valuation, he would want to short the stock, but the short-sales constraint would be binding (an arbitrarily small short position can lead to negative wealth). Consequently, agent A (the optimist) becomes the marginal investor, and the market clearing price has to agree with his valuation.[20] This equilibrium is qualitatively different. The switch of the marginal investor from pessimist to optimist means that the price-dividend ratio will be higher and conditional return volatility lower with circuit breaker.

The above analysis highlights the key differences between the economies with positive and zero bond supply. For a given amount of bond supply Δ, the circuit breaker equilibrium will be similar to what we have seen in the zero bond supply case as long as agent A's wealth share is not too high. However, if agent A's wealth share becomes sufficiently high, the property of the equilibrium changes drastically. Price level will become higher and volatility lower with circuit breaker. Moreover, the region for which this alternative scenario occurs will become wider as bond supply Δ increases (relative to the net supply of the stock).

Figure 8 shows a heat map for the ratio of average daily return volatility in the economies with and without circuit breaker. It is done for a wide range of net bond supply (Δ) and initial wealth share for agent A (ω). The red region indicates volatility amplification by the circuit breaker, while the blue region indicates the opposite. Quantitatively, the volatility amplification effect is stronger when net bond supply is small, and when agent A's initial

[20] There is also a knife-edge case where agent A does not hold any bonds, the stock price is below his valuation, but he faces binding leverage constraint. In this case, the stock price and the wealth distribution have to satisfy the condition $\omega_\tau^A = \frac{S_\tau}{S_\tau + \Delta}$.

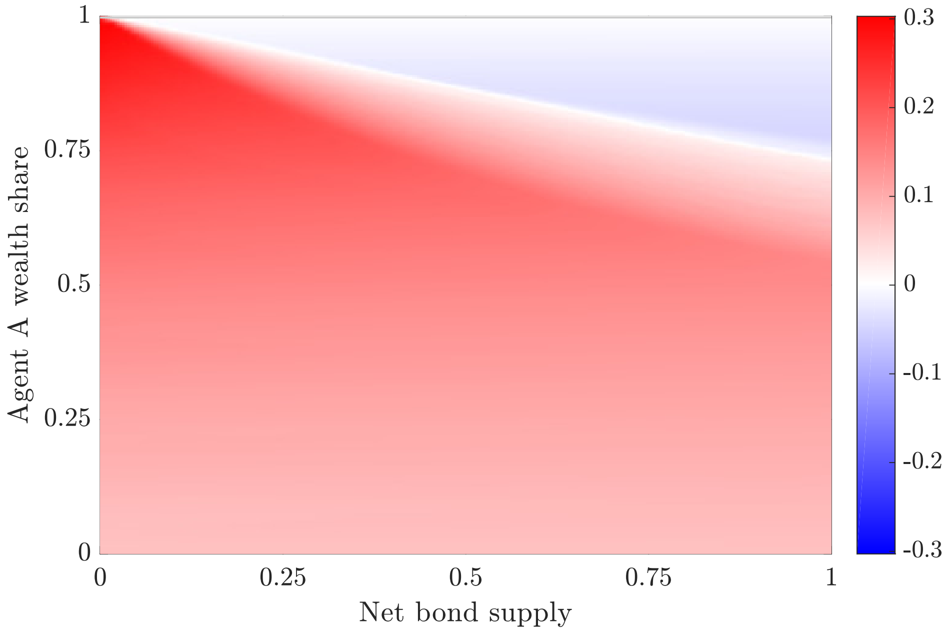

Figure 8: **Heat map: ratio of realized volatilities.** Heat map for the ratio of average daily volatilities in the economies with and without circuit breakers.

wealth share is not too low or too high.

In the data, the net supply of riskless bonds relative to the stock market is likely small. For example, the total size of the U.S. corporate bond market is about \$8 trillion in 2016, while the market for equity is about \$23 trillion. If we assume a recovery rate of 50%, the relative size of the market for riskless bonds would be 0.17. Even if one counts the total size of the U.S. market for Treasuries, federal agency securities, and money market instruments (about \$16.8 trillion in 2016) together with corporate bonds, the relative size of riskless bonds will be less than 1, and we still get volatility amplification for most values of ω.

5.2 Bounded Stock Prices

Our model is set in continuous time with the dividend following a geometric Brownian motion. In a finite time interval $(t, t+s)$, the dividend can in theory take any value on the interval $(0, \infty)$, as does the stock price, no matter how small s is. This feature together with a utility function that satisfies the Inada condition implies that the agents in our model cannot take any levered or short positions in the stock upon market closure. This is why the pessimist has to become the marginal investor upon market closure when bonds are in zero net supply.

In the previous section, we have already seen how the equilibrium could change when riskless bonds are in positive supply. Similar changes could occur if the stock price has a non-zero lower bound during the period of market closure. With bounded prices, the optimistic agent will be able to maintain some leverage when the market closes. Like in the case with positive bond supply, this agent will then be able to hold the entire stock market by himself if his wealth share is sufficiently large, and he might become the marginal investor when the pessimistic agent faces binding short-sales constraint.

Why might the stock price be bounded from below? One reason could be government bailout. It would also effectively capture the fact that bankruptcy is not infinitely costly. In unreported results, we study a discrete-time version of the model where the dividend process is modeled as a binomial tree. We find that when the share of wealth owned by the optimist is not too high, the presence of a circuit breaker lowers prices and increases conditional volatility. The magnitude of the effects is also similar to what we see in the continuous-time model.

6 Extensions

In this section, we extend the model to two-sided circuit breakers. We also discuss how the model can be extended to allow for multi-tier circuit breakers as well as circuit breakers triggered by variables other than price levels.

Two-sided circuit breakers. In some markets, the circuit breaker is triggered when the stock price reaches either the lower bound $(1-\alpha_D)\,S_0$ or the upper bound $(1+\alpha_U)\,S_0$, whichever happens first. This is straightforward to model in our framework. For simplicity, we consider the two-sided circuit breakers in the constant disagreement case, with $\alpha_D = \alpha_U = 5\%$.

Figure 9 shows the results. With the two-sided circuit breaker, the price-dividend ratio is no longer monotonically increasing in D_t; instead, it takes an inverse U-shape. As in the

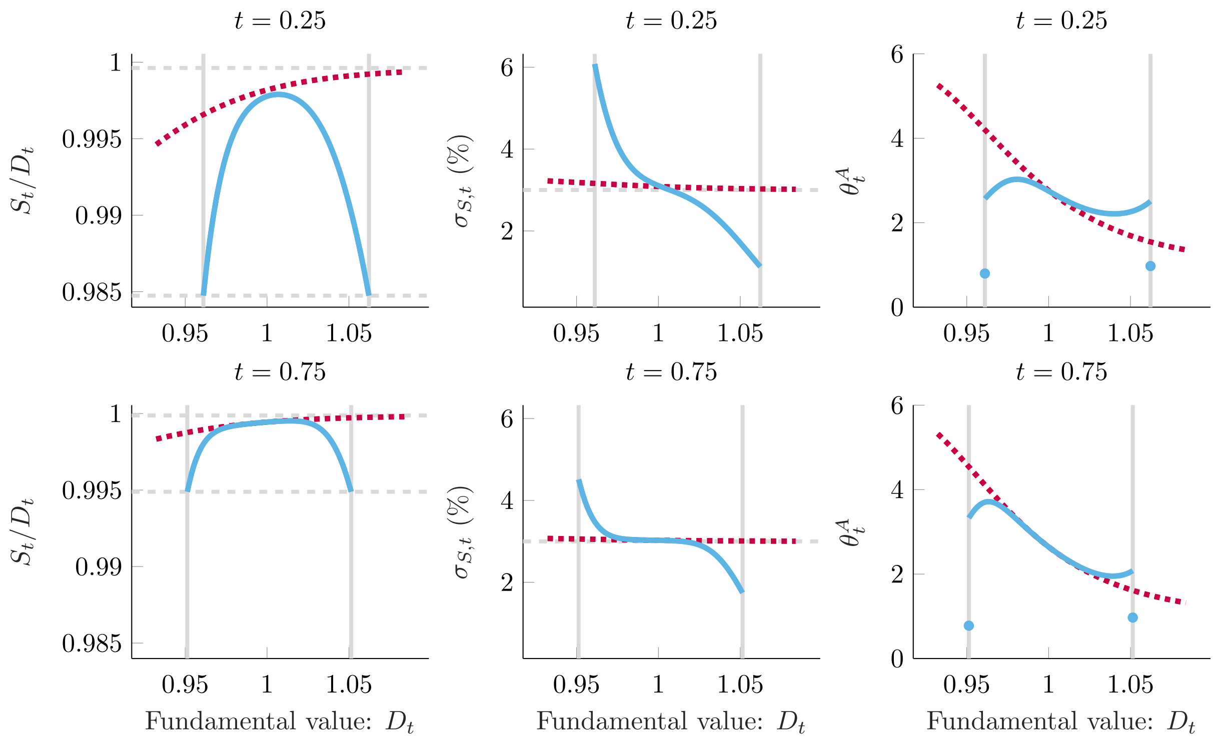

Figure 9: **Price-dividend ratio, conditional return volatility, and agent A's (rational optimist) portfolio holdings in the two-sided circuit breaker case.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the downside threshold $\underline{D}(t)$ and upside threshold $\overline{D}(t)$.

downside circuit breaker case, the price-dividend ratio converges to the pessimist valuation as D_t approaches the downside threshold $\underline{D}(t)$. As D_t rises, the price-dividend ratio also rises and approaches the level in the economy without circuit breaker. The key difference is that as D_t keeps increasing, the price-dividend ratio starts to decline, and eventually converges to the pessimist valuation again as D_t approaches the upside threshold $\overline{D}(t)$.

The reason that the price-dividend ratio converges to the pessimist valuation at the upside circuit breaker threshold $\overline{D}(t)$ is the same as at the downside threshold $\underline{D}(t)$. Since the leverage constraint starts to bind for the optimistic agent upon market closure, the pessimistic agent has to become the marginal investor regardless of whether it is the upside or downside circuit breaker.

While the price-dividend ratio converges to the same value for downside and upside circuit breakers, the implications for conditional return volatility are quite different. As

Figure 9 shows, while the conditional return volatility is amplified (relative to the case without circuit breaker) when D_t is low, the opposite is true when D_t is high, with the volatility dropping to as low as 1% (compared to the fundamental volatility of 3%) at $t = 0.25$. Intuitively, when D_t is high, the direct impact of a positive fundamental shock on the stock price is partially offset by the negative impact due to the increase in the probability of market closure. Thus, the stock price becomes less sensitive to fundamental shocks, which results in lower volatility.

In summary, like the downside circuit breaker, the upside circuit breaker also causes the leverage constraint to bind for the optimistic investor and the pessimistic investor to become the marginal investor. Their impacts on volatility are quite different: conditional return volatility is amplified near the downside circuit breaker threshold but reduced near the upside threshold.

Multi-tier circuit breakers. Circuit breakers implemented on exchanges often have more than one trigger threshold. For example, market-wide circuit breaker in the U.S. stock market can be triggered at three different downside thresholds: 7%, 13%, and 20%. One can solve the model with multi-tier circuit breakers using backward induction, with the starting point being the state when only the highest threshold remains un-triggered (this is identical to the single-tier circuit breaker problem).

To gain some intuition on the effects of multi-tier circuit breakers, consider an example with two downside thresholds, α_1 and α_2, with $\alpha_1 < \alpha_2$, and assume bonds are in zero net supply. Whenever the stock price reaches $(1 - \alpha_1)S_0$ during the trading day, trading is suspended for a period s (e.g., 10 minutes) if there is more than s remaining in the trading day; otherwise the market is closed for the rest of the day. After reopening, if the price reaches $(1 - \alpha_2)S_0$, the market will be closed till the end of the trading day.

When market reopens following the first trading halt, the price dynamics are isomorphic to the case of single-tier circuit breaker. Suppose the first threshold is reached at τ_1 (and we will focus on the interesting case where $\tau_1 + s < T$). Even though the duration of trading halt will be relatively short, both agents will still avoid taking on levered or short positions upon the first trading halt. This again means that the pessimist (agent B) has to become the marginal investor at τ_1. However, a key difference between multi-tier and single-tier circuit breakers is that agent B's valuation of the stock at time τ_1 now depends on his beliefs about the stock price at the time when the market reopens, S_{τ_1+s}.

There are two possible scenarios here. First, if agent B's private valuation at the time of reopening (which depends on D_{τ_1+s}) is higher than the second circuit breaker threshold, $\widehat{S}^B_{\tau_1+s} > (1-\alpha_2)S_0$, then the market will reopen, and the participation of the optimistic agent will raise the market price above agent B's private valuation, $S_{\tau_1+s} > \widehat{S}^B_{\tau_1+s}$. Second, if agent B's private valuation at the time of reopening is lower than the second circuit breaker threshold, the second-tier circuit breaker will be triggered immediately when market reopens. In this case, $S_{\tau_1+s} = \widehat{S}^B_{\tau_1+s}$. Thus, the stock price at the time of reopening following the first trading halt will be on average higher than agent B's private valuation, which means the closing price upon the first trading halt will also be higher than agent B's private valuation, $S_{\tau_1} > \widehat{S}^B_{\tau_1}$.

The above result suggests that the dynamics of the two agents' portfolio holdings will be similar as the market approaches earlier versus later trading halts. However, the impact of trading halt on the price level and return volatility will be weaker near earlier trading halts due to the expectation of market reopening, more so when the duration of trading halt s is shorter.

Circuit breakers based on non-price variables. An appealing property of the solution strategy presented in Section 3.2 is that so long as market closure is characterized by an $\mathcal{F}_t$-stopping time τ, we can determine the stock price upon and prior to market closure in the same way. This means we can use the same solution strategy to study any other types of circuit breakers where the circuit breaker trigger criterion is determined by the history of (D_t, δ_t, η_t) (we will need to search for the stopping rule that is consistent with the circuit breaker criterion, a fixed point problem). For example, we can use this method to solve for models where trading halts are based on conditional return volatility or measures of trading volume (see e.g., Xiong and Yan, 2010).

7 Conclusion

In this paper, we build a dynamic model to examine the mechanism through which market-wide circuit breakers affect trading and price dynamics in the stock market. As we show, a downside circuit breaker tends to lower the price-dividend ratio, reduce daily price ranges, but increase conditional and realized volatility. It also raises the probability of the stock price reaching the circuit breaker limit as the price approaches the threshold (the "magnet effect"). The effects of circuit breakers can be further amplified when some agents' willingness to hold the stock is sensitive to recent shocks to fundamentals, which can be due to behavioral biases, institutional constraints, etc.

Our results demonstrate some of the negative impacts of circuit breakers even without any other market frictions, and they highlight the source of these effects, namely the tightening of leverage constraint when levered investors cannot rebalance their portfolios during trading halts. These results also shed light on the design of circuit breaker rules. Using historical price data from a period when circuit breakers have not been implemented can lead one to significantly underestimate the likelihood of triggering a circuit breaker, especially when the threshold is relatively tight.

Appendix

A Proofs

A.1 Proof of Proposition 1

When there are no circuit breakers, the stock price is

$$\widehat{S}_t = \mathbb{E}_t\left[\frac{\widehat{\pi}_T^A D_T}{\mathbb{E}_t\left[\widehat{\pi}_T^A\right]}\right] = \frac{\mathbb{E}_t\left[\theta + (1-\theta)\,\eta_T\right]}{\mathbb{E}_t\left[D_T^{-1}\left(\theta + (1-\theta)\,\eta_T\right)\right]} = \frac{\theta + (1-\theta)\,\eta_t}{\theta\mathbb{E}_t\left[D_T^{-1}\right] + (1-\theta)\,\mathbb{E}_t\left[D_T^{-1}\eta_T\right]}, \tag{A.1}$$

where

$$\mathbb{E}_t\left[D_T^{-1}\right] = D_t^{-1} e^{-\left(\mu-\sigma^2\right)(T-t)}, \tag{A.2}$$

and

$$\mathbb{E}_t\left[D_T^{-1}\eta_T\right] = \eta_t \mathbb{E}_t\left[D_T^{-1}\frac{\eta_T}{\eta_t}\right] = \eta_t \mathbb{E}_t^B\left[D_T^{-1}\right]. \tag{A.3}$$

Our model fits into the affine disagreement framework of Chen, Joslin, and Tran (2010). Define the log dividend $x_t = \log D_t$. Under measure $\mathbb{P}^B$, the processes for x_t and δ_t are

$$dx_t = \left(\mu - \frac{\sigma^2}{2} + \delta_t\right) dt + \sigma dZ_t^B, \tag{A.4a}$$

$$d\delta_t = \left(\kappa\bar{\delta} + \left(\frac{\nu}{\sigma} - \kappa\right)\delta_t\right) dt + \nu dZ_t^B. \tag{A.4b}$$

Define $X_t = [x_t \ \delta_t]'$, then X_t follows an affine process,

$$dX_t = \left(K_0 + K_1 X_t\right) dt + \sigma_X dZ_t^B, \tag{A.5}$$

with

$$K_0 = \begin{bmatrix} \mu - \frac{\sigma^2}{2} \\ \kappa\bar{\delta} \end{bmatrix},\ K_1 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{\nu}{\sigma} - \kappa \end{bmatrix},\ \sigma_X = \begin{bmatrix} \sigma \\ \nu \end{bmatrix}. \tag{A.6}$$

We are interested in computing

$$g\left(t, X_t\right) = \mathbb{E}_t^B\left[e^{\rho_1' X_T}\right], \quad \text{with } \rho_1 = [-1 \ \ 0]'. \tag{A.7}$$

By applying standard results for the conditional moment-generating functions of affine processes, we get

$$g(t, X_t) = \exp\left(A(t,T) + B(t,T)' X_t\right), \tag{A.8}$$

where

$$\begin{aligned} 0 &= \dot{B} + K_1' B, \quad B(T,T) = \rho_1 & \text{(A.9a)} \\ 0 &= \dot{A} + B' K_0 + \frac{1}{2} tr\left(BB' \sigma_X \sigma_X'\right), \quad A(T,T) = 0. & \text{(A.9b)} \end{aligned}$$

Solving for the ODEs gives:

$$B(t,T) = \left[-1 \quad \frac{1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa} \right]', \tag{A.10}$$

and

$$\begin{aligned} A(t,T) = & \left[\mu - \sigma^2 - \frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} - \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2} \right] (t - T) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3} \left[1 - e^{2\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)} \right] \\ & + \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3} \right] \left[1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)} \right]. \end{aligned} \tag{A.11}$$

After plugging the above results back into (A.1) and reorganizing the terms, we get

$$S_t = \frac{\theta + (1-\theta)\eta_t}{\theta + (1-\theta)\eta_t H(t, \delta_t)} D_t e^{\left(\mu - \sigma^2\right)(T-t)}, \tag{A.12}$$

where

$$\begin{aligned} H(t, \delta_t) &= e^{a(t,T) + b(t,T)\delta_t}, & \text{(A.13a)} \\ a(t,T) &= \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} + \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2} \right] (T - t) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3} \left[1 - e^{2\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)} \right] & \\ &\quad + \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3} \right] \left[1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)} \right], & \text{(A.13b)} \\ b(t,T) &= \frac{1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa}. & \text{(A.13c)} \end{aligned}$$

Finally, to compute the conditional volatility of stock returns, we have

$$\begin{aligned} d\widehat{S}_t &= \widehat{\mu}_{S,t}\widehat{S}_t dt + \widehat{\sigma}_{S,t}\widehat{S} dZ_t \\ &= o(dt) + \widehat{S}_t \frac{dD_t}{D_t} + \eta_t D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{\theta(1-\theta)[1-H(t,\delta_t)]}{\left[\theta+(1-\theta)\eta_t H(t,\delta_t)\right]^2} \frac{d\eta_t}{\eta_t} \\ &\quad - D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{[\theta+(1-\theta)\eta_t](1-\theta)\eta_t H(t,\delta_t) b(t,T)}{\left[\theta+(1-\theta)\eta_t H(t,\delta_t)\right]^2} d\delta_t. \end{aligned}$$

After collecting the diffusion terms, we get

$$\begin{aligned} \widehat{\sigma}_{S,t} = \sigma + \frac{D_t e^{\left(\mu-\sigma^2\right)(T-t)}}{\widehat{S}_t} &\left\{ \frac{\theta(1-\theta)\left[1-H(t,\delta_t)\right]}{\left[\theta+(1-\theta)\eta_t H(t,\delta_t)\right]^2} \frac{\delta_t \eta_t}{\sigma} \right. \\ &\left. - \frac{\left[\theta+(1-\theta)\eta_t\right](1-\theta)\eta_t b(t,T) H(t,\delta_t)}{\left[\theta+(1-\theta)\eta_t H(t,\delta_t)\right]^2} \nu \right\}. \end{aligned} \tag{A.14}$$

A.2 Proof of Proposition 3

Suppose the market closes at time $\tau < T$. The two agents' problems at time τ are specified in (21), together with the portfolio constraints $\theta^i_\tau \geq 0$ and $\phi^i_\tau \geq 0$ as implied by the Inada condition. The Lagrangian for agent i is

$$L = \mathbb{E}^i_\tau \left[\ln\left(\theta^i_\tau D_T + \phi^i_\tau\right)\right] + \zeta^i \left(W^i_\tau - \theta^i_\tau S_\tau - \phi^i_\tau\right) + \xi^i_a \theta^i_\tau + \xi^i_b \phi^i_\tau,$$

and the first order conditions with respect to θ^i_τ and ϕ^i_τ are

$$0 = \mathbb{E}^i_\tau \left[\frac{D_T}{\theta^i_\tau D_T + \phi^i_\tau}\right] - \zeta^i S_\tau + \xi^i_a, \tag{A.15}$$

$$0 = \mathbb{E}^i_\tau \left[\frac{1}{\theta^i_\tau D_T + \phi^i_\tau}\right] - \zeta^i + \xi^i_b. \tag{A.16}$$

Furthermore, the market clearing conditions at time τ are given in (22).

First consider the case when agent A is less optimistic than agent B at time τ. This requires that δ_τ is sufficiently large such that agent A's valuation of the stock in a single-agent economy is higher than that of agent B, $\widehat{S}^A_\tau < \widehat{S}^B_\tau$. This implies the condition

$$\delta_\tau > \underline{\delta} \equiv -\frac{a(t,T)}{b(t,T)}. \tag{A.17}$$

Under this assumption, we can examine the following three scenarios:

1. It can be an equilibrium if the stock price is such that agent A (pessimist) is unconstrained and finds it optimal to hold all of his wealth in the stock (plus the infinitesimal amount of bonds); agent B (optimist) would like to put more than 100% of his wealth in the stock but faces a binding leverage constraint. In this case,

$$\theta_\tau^{i\,*} = \frac{W_{i,\tau}}{S_\tau}, \quad \phi_\tau^{A*} = \Delta \to 0, \quad \phi_\tau^{B*} = 0,$$
$$\xi_a^A = \xi_a^B = 0, \quad \xi_b^A = 0, \quad \xi_b^B > 0.$$

Then from the FOCs (A.15-A.16) of the unconstrained agent A, we get

$$\lim_{\Delta\to 0} S_\tau = \frac{\mathbb{E}_\tau^A\left[\frac{D_T}{\theta_\tau^{A*} D_T + \phi_\tau^{A*}}\right]}{\mathbb{E}_\tau^A\left[\frac{1}{\theta_\tau^{A*} D_T + \phi_\tau^{A*}}\right]} = \frac{1}{\mathbb{E}_\tau^A\left[\frac{1}{D_T}\right]} = D_\tau e^{\left(\mu-\sigma^2\right)(T-\tau)} = \widehat{S}_\tau^A.$$

2. For any $S_\tau < \widehat{S}_\tau^A$, the price is so low that both agents would prefer to take levered positions in the stock. The circuit breaker will constrain both agents from borrowing, but neither will be willing to hold any positive amount of the bond. Thus, the bond market will not clear and this cannot be an equilibrium.

3. For any $S_\tau > \widehat{S}_\tau^A$, agent A will prefer to hold less than 100% of the wealth in the stock. Agent B will need to take levered position in order to clear the stock market but cannot because of the circuit breaker. Thus, this cannot be an equilibrium, either.

Similar arguments apply for the case when agent A is more optimistic than agent B at time τ. In that case, the only equilibrium price will be $S_\tau = \widehat{S}_\tau^B$ in the limit with $\Delta \to 0$.

A.3 Special Case: Constant Disagreement

The stock price can be computed in closed form in the case of constant disagreement, $\delta_t = \delta$. Without loss of generality, we focus on the case where agent B is relatively more optimistic, $\delta \geq 0$. The results are summarized below.

Proposition 4. *Take S_0 as given. With $\delta \geq 0$, the stock price time $t \leq \tau \wedge T$ is*

$$S_t = \left(\omega_t^A \mathbb{E}_t[S_{\tau\wedge T}^{-1}] + \omega_t^B \mathbb{E}_t^B[S_{\tau\wedge T}^{-1}]\right)^{-1}, \tag{A.18}$$

where

$$\mathbb{E}_t[S^{-1}_{\tau\wedge T}] = \frac{1}{\alpha S_0}\left\{N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] + e^{\sigma d_t} N\left[\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\}$$
$$+ D_t^{-1} e^{-(\mu-\sigma^2)(T-t)}\left\{N\left[-\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] - e^{-\sigma d_t} N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\}, \tag{A.19}$$

$$\mathbb{E}^B_t[S^{-1}_{\tau\wedge T}] = \frac{1}{\alpha S_0}\left\{N\left[\frac{d_t - \left(\frac{\delta}{\sigma}+\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right] + e^{\left(\sigma+\frac{2\delta}{\sigma}\right)d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma}+\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\}$$
$$+ D_t^{-1} e^{-(\mu-\sigma^2+\delta)(T-t)}\left\{N\left[-\frac{d_t - \left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right.$$
$$\left.- e^{\left(\frac{2\delta}{\sigma}-\sigma\right)d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\}, \tag{A.20}$$

and

$$d_t = \frac{1}{\sigma}\left[\log\left(\frac{\alpha S_0}{D_t}\right) - \left(\mu-\sigma^2\right)(T-t)\right]. \tag{A.21}$$

Proof. As show in Section 3.2, the stock price at time $t \le \tau\wedge T$ is

$$S_t = \frac{\mathbb{E}_t\left[\pi^A_{\tau\wedge T} S_{\tau\wedge T}\right]}{\pi^A_t} = \frac{\theta+(1-\theta)\eta_t}{\mathbb{E}_t\left[\frac{\theta+(1-\theta)\eta_{\tau\wedge T}}{S_{\tau\wedge T}}\right]}$$
$$= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[\frac{\eta_{\tau\wedge T}}{\eta_t}S^{-1}_{\tau\wedge T}\right]}$$
$$= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right]}. \tag{A.22}$$

The second equality follows from Doob's Optional Sampling Theorem, while the last equality follows from Girsanov's Theorem.

Now consider the case when $\delta_t = \delta$. Taking S_0 as given and imposing the condition for stock price at the circuit breaker trigger, we have

$$\mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] = \frac{1}{\alpha S_0}P_t(\tau\le T) + \mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right], \tag{A.23}$$
$$\mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right] = \frac{1}{\alpha S_0\eta_t}\mathbb{E}_t\left[\eta_\tau \mathbb{1}_{\{\tau\le T\}}\right] + \frac{1}{\eta_t}\mathbb{E}_t\left[\eta_T D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right]. \tag{A.24}$$

The following standard results about hitting times of Brownian motions are helpful for deriving the expressions for the expectations in (A.23)-(A.24) (see e.g., Jeanblanc, Yor, and Chesney, 2009, chap 3). Let Z^μ denote a drifted Brownian motion, $Z^\mu_t = \mu t + Z_t$, with $Z^\mu_0 = 0$. Let

$\mathcal{T}_y^\mu = \inf\{t \geq 0 : Z_t^\mu = y\}$ for $y < 0$. Then:

$$\Pr\left(T_y^\mu \leq t\right) = N\left(\frac{y-\mu t}{\sqrt{t}}\right) + e^{2\mu y} N\left(\frac{y+\mu t}{\sqrt{t}}\right), \tag{A.25}$$

$$E\left[e^{-\lambda T_y^\mu} \mathbb{1}_{\{T_y^\mu \leq t\}}\right] = e^{(\mu-\gamma)y} N\left(\frac{y-\gamma t}{\sqrt{t}}\right) + e^{(\mu+\gamma)y} N\left(\frac{y+\gamma t}{\sqrt{t}}\right), \tag{A.26}$$

where $\gamma = \sqrt{2\lambda + \mu^2}$.

Recall the definition of the stopping time τ in Equation (33), which simplifies in the case with constant disagreement,

$$\tau = \inf\left\{t \geq 0 : D_t = \alpha S_0 e^{-\left(\mu-\sigma^2\right)(T-t)}\right\}. \tag{A.27}$$

Through a change of variables, we can redefine τ as the first hitting time of a drifted Brownian motion for a constant threshold. Specifically, define:

$$y_t = \frac{1}{\sigma} \log\left(e^{-\left(\mu-\sigma^2\right)t} D_t\right), \tag{A.28}$$

then $y_0 = 0$, and

$$y_t = Z_t^{\frac{\sigma}{2}} = \frac{\sigma}{2} t + Z_t. \tag{A.29}$$

Moreover,

$$\mathcal{T}_d^{\frac{\sigma}{2}} = \inf\{t \geq 0 : y_t = d\} \overset{a.s.}{=} \tau, \tag{A.30}$$

where the threshold is constant over time,

$$d = \frac{1}{\sigma} \log\left(\alpha S_0 e^{-\left(\mu-\sigma^2\right)T}\right). \tag{A.31}$$

Conditional on y_t and the fact that the circuit breaker has not been triggered up to time t, the result from (A.25) implies

$$P_t(\tau \leq T) = P_t\left(\mathcal{T}_{d_t}^{\frac{\sigma}{2}} \leq T-t\right) = N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] + e^{\sigma d_t} N\left[\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right], \tag{A.32}$$

where

$$d_t = d - y_t = \frac{1}{\sigma}\left[\log\left(\frac{\alpha S_0}{D_t}\right) - \left(\mu - \sigma^2\right)(T-t)\right]. \tag{A.33}$$

The threshold d_t is normalized with respect to y_t so as to start the drifted Brownian motion $Z^{\frac{\sigma^2}{2}}$

from 0 at time t. Next,

$$\begin{aligned}
\mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right] &= D_t^{-1}e^{-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}\mathbb{E}_t\left[e^{-\sigma(Z_T-Z_t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1}e^{-\left(\mu-\sigma^2\right)(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1}e^{-\left(\mu-\sigma^2\right)(T-t)}\left\{N\left[-\frac{d_t+\frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]-e^{-\sigma d_t}N\left[\frac{d_t-\frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\}. \qquad \text{(A.34)}
\end{aligned}$$

The second equality follows from Girsanov's Theorem, and the third equality again follows from (A.25). Under $\mathbb{Q}$, $Z_t^{\sigma}=Z_t+\sigma t$ is a standard Brownian motion, and

$$y_t=-\frac{\sigma}{2}t+Z_t^{\sigma}. \qquad \text{(A.35)}$$

Next, it follows from (A.28) and the definition of τ that

$$y_\tau=y_t+\frac{\sigma}{2}(\tau-t)+(Z_\tau-Z_t)=d. \qquad \text{(A.36)}$$

Thus,

$$Z_\tau-Z_t=d_t-\frac{\sigma}{2}(\tau-t). \qquad \text{(A.37)}$$

With these results, we can evaluate the following expectation:

$$\begin{aligned}
\mathbb{E}_t\left[\eta_\tau\mathbb{1}_{\{\tau\le T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_\tau-Z_t)-\frac{\delta^2}{2\sigma^2}(\tau-t)}\mathbb{1}_{\{\tau\le T\}}\right] \\
&= \eta_t e^{\frac{\delta d_t}{\sigma}}\mathbb{E}_t\left[\exp\left(-\left(\frac{\delta}{2}+\frac{\delta^2}{2\sigma^2}\right)(\tau-t)\right)\mathbb{1}_{\{\tau\le T\}}\right] \\
&= \eta_t\left\{N\left[\frac{d_t-\left(\frac{\delta}{\sigma}+\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]+e^{\left(\sigma+\frac{2\delta}{\sigma}\right)d_t}N\left[\frac{d_t+\left(\frac{\delta}{\sigma}+\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\},
\end{aligned}$$

where the last equality follows from an application of (A.26). Finally,

$$\begin{aligned}
&\mathbb{E}_t\left[\eta_T D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right] = \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_T-Z_t)-\frac{\delta^2}{2\sigma^2}(T-t)}D_t^{-1}e^{-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)-\sigma(Z_T-Z_t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= \eta_t D_t^{-1}e^{-\left(\mu-\sigma^2+\delta\right)(T-t)}\mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[\mathbb{1}_{\{\tau>T\}}\right] \\
&= \eta_t D_t^{-1}e^{-\left(\mu-\sigma^2+\delta\right)(T-t)}\left\{N\left[-\frac{d_t-\left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]-e^{\left(\frac{2\delta}{\sigma}-\sigma\right)d_t}N\left[\frac{d_t+\left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\}.
\end{aligned}$$

The second equality follows from Girsanov's Theorem, and the third equality follows from (A.25).

Under $\widetilde{\mathbb{Q}}$, $Z_t^{\sigma-\frac{\delta}{\sigma}} = Z_t + \left(\sigma - \frac{\delta}{\sigma}\right) t$ is a standard Brownian motion, and

$$y_t = \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right) t + Z_t^{\sigma-\frac{\delta}{\sigma}}. \tag{A.38}$$

$\square$

B Numerical Solution

Now we outline the numerical algorithm used to solve the model for the $\Delta > 0$ case. Time interval $[0, T]$ is discretized using a grid $\{t_0, t_1, ..., t_n\}$, where $t_0 = 0$ and $t_n = T$. For every point t_i on the time grid we construct grids for a set of state variables which uniquely determine fundamental value D_{t_i}, Radon-Nikodym derivative η_{t_i} and disagreement δ_{t_i} [21]. We will denote by $\Theta_{i,k} = (t_i, \Xi_k)$ a tuple (which we will call a "node") that summarizes the state of the economy at time t_i and k here indexes a particular point of the discretized state-space. We assign probabilities of the transition $\Theta_{i,k} \to \Theta_{i+1,j}$ for all i, j, k to match expected value and dispersion of one-step changes in state variables with their continuous time counterparts, namely, drifts and diffusions.

Using this structure we can solve for the equilibrium in complete markets in the following way.

1. For time point $t_n = T$: use equations (8a)—(9) to calculate consumption allocations and state-price density in all nodes $\Theta_{n,k}$; set stock price equal to D_T.

2. For time point t_{n-1}: use transition probabilities and the fact that $\pi_t S_t$, $\pi_t \widehat{W}_t^A$, π_t are martingales to calculate the state-price density, stock price and wealth of agent A in nodes $\Theta_{n-1,k}$ for all k.

3. Proceding backwards repeat the above step $n-1$ times to obtain the state-price density, stock price and wealth of agent A in every point $\Theta_{i,k}$.

4. Using transition probabilities calculate drift and diffusion of the stock price process and portfolio holdings of each agent.

In the case with circuit breakers we want to use the algorithm similar to the one above

[21] One obvious potential choice of state variables is D_t, η_t and δ_t. However, depending on specific assumptions of the model the state-space dimensionality can be reduced, e.g. in the case of constant disagreement one need to keep track of t and D_t only since they uniquely determine disagreement δ_t and the Radon-Nikodym derivative.

initialized it at time $[\tau \wedge T]$ instead of time T. The problem is that τ is endogenous itself. The following steps show how we solve the problem:

1. In every node $\Theta_{i,k}$: pick a grid for $\omega_{t_i} = W^A_{t_i}/(W^A_{t_i} + W^B_{t_i})$ spanning the interval $[0, 1]$ and solve the problem (21)—(22) for every point of the grid[22]. Solution to this problem yields "stop" prices $\underline{S}_{i,k,j}$ and marginal utilities of wealth of agents which we will denote by $V^{A\prime}_{i,k,j}$ and $V^{B\prime}_{i,k,j}$, where j indexes grid points for ω_{t_i}. Using the planner's problem (24) first order condtion we can define,
$$\lambda_{i,k,j} = \frac{\eta V^{B\prime}_{i,k,j}}{V^{A\prime}_{i,k,j} + \eta V^{B\prime}_{i,k,j}}.$$

2. Now pick an initial guess for the planner's weight λ_g and price threshold $\underline{S}$. In every node $\Theta_{i,k}$ using the values $\lambda_{i,k,j}$ and $S_{i,k,j}$ from the previous step find the "stop" price $\underline{S}_{i,k}$ in point λ_g by interpolation. If $\underline{S}_{i,k} \leq \underline{S}$ then the node $\Theta_{i,k}$ will be either a "stop" node or a node that will never be reached in the equilibrium with circuit breakers. For "stop" nodes we define stock price to be equal to $\underline{S}_{i,k}$ and state-price density to be proportional to $V^{A\prime}_{i,k}$ (which is also obtained by interpolation of $V^{A\prime}_{i,k,j}$ in point λ).

The above procedure effectively defines the stopping time rule τ corresponding to the economy with threshold $\underline{S}$ and planner's weight λ_g. Now we can use the backward procedure described for the case of complete markets to obtain the state-price density, stock price and wealth of every agent in every node $\Theta_{i,k}$. Note that initial wealth share of agent A in this economy will be different from both λ_g and ω. The final step is to find λ_g and $\underline{S}$ so that initial wealth share in the resulting economy is equal to ω and $\underline{S} = (1-\alpha)S_0$. This can be done using the standard bisection method.

[22] In equilibrium given initial wealth distribution the value $W^A_{t_i}/(W^A_{t_i} + W^B_{t_i})$ is uniquely pinned down by the Radon-Nikodym derivative η_{t_i} and fundamental value D_{t_i}. Since this relationship is endogenous and is not known before the model is solved our algorithm requires solving the problem for a wide range of wealth distributions.

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