

ASSET PRICING AND THE CREDIT MARKET

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Abstract

We study asset pricing and trading behavior in an exchange economy populated by two agents with different risk aversion. We show that the credit market plays a central role in the risk sharing between the two agents. It allows the less-risk-averse agent to borrow in order to take on levered positions in the stock and thus bear more risk. Optimal risk sharing results in the more-risk-averse agent effectively selling covered “call” options to the less-risk-averse agent. As the state of the economy changes, the equilibrium amount of credit in the market also fluctuates, which in turn influences expected stock returns, stock return volatility, the term structure of interest rates, and trading activity in the stock market. We further explore the immediate empirical implication that variation in the size of the credit market is related to variation in expected stock returns. Using various measures of changes in the size of the credit market, we find that they have significant power in forecasting one-year excess returns of the stock market. Our results suggest that the credit sector is of fundamental importance to the behavior of asset prices.

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1. INTRODUCTION

The simple representative-agent setting has become one of the most-popular models for studying equilibrium asset pricing (e.g., Lucas (1978) and Cox, Ingersoll and Ross (1985)). In such a setting, the credit market plays only a virtual role. For the representative agent, equilibrium borrowing and lending, which equals the aggregate supply of credit, is always zero. Hence, models with only one representative agent are largely silent about quantity variables in the market and their connection with prices. In particular, they make no predictions about the actual behavior of the credit sector and how it influences asset pricing.

In reality, however, the credit sector plays a central role in the financial markets. Recent events such as the subprime-mortgage crisis and the subsequent liquidity shocks in the financial markets indicate that the availability of credit (or the lack thereof) can have first-order effects on asset prices. Thus, introducing a meaningful credit sector into the standard asset-pricing framework is a natural extension that might provide new insights about the workings of financial markets.

Using a model similar to Wang (1996), we extend the canonical asset-pricing framework to allow for two classes of investors, each with a different level of risk aversion. In this setting, the two classes of investors borrow from and lend to each other. This results in a significant credit sector that expands and contracts in response to shocks to the economy. We solve the model in closed form and provide explicit solutions for equilibrium prices and investors' optimal consumption and trading strategies. We then examine the role of the credit market and its connection with asset prices. Especially, we explore how the amount of credit, determined endogenously in the market, facilitates risk sharing among investors and influences the behavior of stock and bond prices.

A number of interesting results emerge from this analysis. First, we show that the equilibrium consumption allocation between the two agents, each being the representative investor of his class, is such that the more-risk-averse agent's consumption is less risky than the aggregate endowment (or consumption) while the less-risk-averse agent's consumption is more risky. As a result, the more-risk-averse agent ends up with the lion's share of total consumption in bad states of the economy (when aggregate consumption is low) while giving up much of his share in the good states. Such an allocation is equivalent to the more-risk-averse agent selling a covered "call" option on the aggregate endowment or the stock market to the less-risk-averse agent, shifting more risk from the former to the latter. Trading activity in the model is driven by the dynamic replication strategies associated with these convex option-like consumption allocations.

Second, the credit market is essential in facilitating this optimal risk sharing. In particular, the more-risk-averse agent provides liquidity in the form of credit to the less-risk-averse agent, allowing him to take on levered positions in the stock and thus bear more risk. In return, the more-risk-averse agent switches part of his portfolio into debt, receiving a stream of safe cash flows in the form of interest payments. As a result, the size of the credit market varies drastically with market "demographics," i.e., the wealth distribution between the two agents. When the wealth is too skewed toward one agent, which is the case when the economy is in extremely good or bad states, the credit market becomes minuscule. This is because the agent with little wealth can no longer accommodate the borrowing or lending needs of the other agent. For intermediate states of the economy, however, the size of the credit market becomes substantial, allowing sufficient leverage for the less-risk-averse agent to take on more risk.

Moreover, we show that the relative size of the credit market is closely related to the behavior of asset prices. Under calibrated parameter values, we find that stock return volatility comoves

with the market’s leverage ratio, defined as the total amount of credit in the market normalized by the total size of the market. This is in part because as leverage reaches its maximum, agents’ wealth and consumption shares become most sensitive to changes of the economy, which leads to more volatile stock prices. In fact, we show that when both agents are present, despite the expanded risk-sharing opportunities provided by the credit and stock market, the equilibrium stock price volatility can be higher than its level when only one of the agents is present. In addition, we find that when the leverage ratio is maximized, the interest rate becomes more stable, which also leads to an overall upward-sloping term structure for interest rates.

Furthermore, we show that under i.i.d. shocks to the economy, the resulting market demographics, shaped by agents’ risk-sharing strategies, evolve in non-trivial ways. This leads to rich patterns in stock and bond returns. For example, the dividend yield, risk premium, and Sharpe ratio of the stock typically behave countercyclically. In extremely bad states of the economy, however, the stock’s risk premium can turn procyclical, becoming negatively correlated with dividend yield. Under certain parameter values, the dividend yield and the expected return of the stock can both be nonmonotonic with respect to the level of the market and behave procyclically. The stock returns also displays interesting forms of heteroscedasticity. The volatility is highly persistent over time, procyclical in low states of the economy but countercyclical in high states.

A key contribution of the paper is to establish a fundamental link between asset prices and quantities in the market, especially the amount of credit. The primary empirical implication of the model is that changes in the size of the credit sector are highly informative about shifts in the demographics of the market, which, in turn, drive the behavior of asset prices. More specifically, changes in the size of the credit sector should be linked to time variation in the equity premium. Thus, information about the size of the credit market may prove useful in forecasting excess stock returns.

We test this empirical implication using the standard predictive regression framework familiar from the asset-pricing literature. Specifically, we regress one-year (nonoverlapping) excess returns on the CRSP value-weighted index for the 1953 to 2006 period on a number of variables that previous research has suggested may have predictive ability for excess stock returns: lagged stock returns, the dividend yield, and Lettau and Ludvigson’s (2001) *cay* measure. We then introduce several measures of the size of the credit sector into the regression and examine how the predictive ability of the regression changes.

The results are striking. By themselves, the lagged stock return, dividend yield, and *cay* variables result in a predictive regression for the one-year horizon with an adjusted R^2 of about 17 percent. When the credit sector variables are introduced, however, the adjusted R^2 for the regression increases to over 40 percent. These levels of predictability for stock returns are far higher than any previously documented in the literature. These results demonstrate both the theoretical and empirical importance of the credit sector in asset-pricing models.

Several papers have considered the impact of heterogeneity in investors’ risk aversion on asset pricing. Dumas (1989) and Wang (1996) use a two-agent setting to examine how heterogeneity gives rise to time-varying risk aversion in the aggregate and the resulting behavior of interest rates. Further allowing the feature of “catching-up with the Joneses” in investor preferences in a similar setting, Chan and Kogan (2002) consider the dynamics of stock prices.

Our paper extends this line of research in three important directions. First, while these papers focus on the behavior of prices, we focus on both prices and quantities, especially the interaction between the two. To link prices with quantities is of essential importance to models with hetero-

geneous investors—it is where these models can produce new, distinctive, and testable predictions beyond those from a representative model. After all, in a complete market, there always exists a representative-agent model for an economy with heterogeneous agents that yields identical pricing relations from the fundamentals. Thus, empirical tests of these models have to turn to the additional predictions, which must concern disaggregated variables such as quantities. By solving and analyzing in closed-form investors’ portfolio behavior together with prices, we are able to produce various predictions directly connecting the two. Next, we explore some of the empirical predictions of the model. In particular, we show that the amount of credit endogenously generated in the market contains useful information about the risk premium of the stock. Another important feature of our paper is its analysis of the credit market, in particular, how the amount of credit is related to the degree of risk sharing and the resulting stock and bond prices. In addition, even with limited degrees of freedom, we are able to produce a rich set of results concerning both the dynamics of stock and bond prices that are compatible with the empirical findings, at least for some sets of calibrated parameters.

This paper is organized as follows. Section 2 describes the model. Section 3 presents a single-agent version of the model as a benchmark for comparison. Section 4 solves the equilibrium for the two-agent model. Section 5 discusses the equilibrium consumption allocation. Section 6 analyzes how the credit market helps the two agents achieve the optimal risk sharing. Section 7 examines the behavior of asset prices in the model and the interactions with borrowing and lending in the credit market. Section 8 considers the trading activity in the stock market. Section 9 reports our exploratory empirical work on the link between the size of the credit market and stock market returns. Section 10 concludes the paper. All proofs are provided in the Appendix.

2. THE MODEL

The model used throughout this paper parallels Wang (1996). Specifically, we consider a pure exchange economy in which there is a single perishable consumption good that also serves as the numeraire. The economy is endowed with a flow of the consumption good. We denote the rate of endowment flow as X_t and assume that it follows a geometric Brownian motion,

$$dX_t = \mu X_t dt + \sigma X_t dZ_t, \tag{1}$$

where $X_0 > 0$, $\mu \geq 0$ and $\sigma > 0$ are constants, and Z_t is a standard Wiener process.¹ The process X_t is positive with probability one and, conditional on X_t , $X_{t+\tau}$ with $\tau \geq 0$ is lognormally distributed.

There exists a market where shares of the aggregate endowment (the “stock”) are traded. A share of the stock yields a dividend flow at rate X_t . The total number of shares of the stock in the economy then equals one. In addition, there exists a “money market” where a locally riskless security is traded (i.e., investors can borrow from or lend to each other without default). As is standard, we assume that this riskless security is in zero net supply in the economy. Let P_t denote the price of the stock and r_t the instantaneous riskless interest rate.

Investors in this economy can trade competitively in the securities market and consume the proceeds. Let C_t be an investor’s consumption rate, N_t his holdings of the stock, and M_t his

¹Throughout the paper, equalities or inequalities involving random variables are always in the sense of almost surely with respect to the underlying probability measure.

holdings of the riskless security. The consumption and trading strategies $\{C_t, (N_t, M_t)\}$ are adapted processes satisfying the standard integrability conditions, that is, $\forall T \in [0, \infty)$,

$$\int_0^T C_t dt < \infty, \quad (2a)$$

$$\int_0^T |M_t r_t dt + N_t (X_t dt + dP_t)| < \infty, \quad (2b)$$

$$\int_0^T N_t^2 d[P_t] < \infty, \quad (2c)$$

where $[P_t]$ denotes the quadratic variation process of P_t .² The investor's wealth process, defined by $W_t = M_t + N_t P_t$, must be positive with probability one, and conform to the stochastic differential equation,

$$dW_t = r_t M_t dt + (X_t dt + dP_t) N_t - C_t dt. \quad (3)$$

The requirement that wealth be positive is to rule out arbitrage opportunities (following Dybvig and Huang (1988)). Let Θ denote the set of consumption/trading strategies that satisfy the above conditions.

There are two classes of identical investors in the economy, denoted as 1 and 2. Both classes are initially endowed with only shares of the stock. The initial endowment of shares for the classes of investors are $1 - n$ and n , respectively. The initial number of shares optimally chosen by each class at time zero, of course, need not equal their initial endowments. Investors in each class choose their consumption and investment strategies to maximize their lifetime expected utility. The preferences of the two classes of investors are

$$E_t \left[\int_0^\infty e^{-\rho\tau} \frac{C_{1,t+\tau}^{1-\gamma}}{1-\gamma} d\tau \right], \quad (4a)$$

$$E_t \left[\int_0^\infty e^{-\rho\tau} \frac{C_{2,t+\tau}^{1-2\gamma}}{1-2\gamma} d\tau \right], \quad (4b)$$

respectively, where γ is a positive constant. $C_{1,t}$ and $C_{2,t}$ denote the total consumption of the first and second classes of investors, respectively. Thus, the first and second classes of investors have constant relative risk aversion (CRRA) of γ and 2γ , respectively.

We further impose several conditions on the model's parameter values. The first condition is the growth condition,

$$\rho > \max \{0, (1-\gamma)(\mu - \frac{1}{2}\gamma\sigma^2), (1-2\gamma)(\mu - \gamma\sigma^2)\}. \quad (5)$$

It ensures that investors' expected utilities are uniformly bounded given the aggregate consumption process in Equation (1). In addition, we need the following set of conditions

$$\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2} + (\mu - \frac{1}{2}\sigma^2) - 2\gamma\sigma^2 > 0, \quad (6a)$$

$$\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2} - (\mu - \frac{1}{2}\sigma^2) + (\gamma - 1)\sigma^2 > 0. \quad (6b)$$

²See Karatzas and Shreve (1988) for a discussion of the quadratic variation process of a given stochastic process.

These conditions guarantee that the stock and bond prices behave properly.³

In specifying the securities markets, we have only introduced the stock and the locally riskless security as traded securities. As will be shown later, the stock and the riskless security are sufficient to dynamically complete the securities market in the sense of Harrison and Kreps (1979). Arbitrary consumption plans (satisfying certain integrability conditions) can be financed by continuous trading in the stock and the riskless security. Allowing additional securities will not affect the nature of the equilibrium. Thus, in deriving the market equilibrium, we will consider the securities market as consisting of only the stock and the riskless security. Simple arbitrage arguments can then be used to price other securities if they exist.

We have assumed that there are only two classes of investors in the economy and that they behave competitively in the market. Since investors within each class have the same isoelastic preferences, we can represent each class with a single representative investor who has the same preferences as the individual investors and the total endowment of each class (see, for example, Rubinstein (1974)). In deriving the equilibrium, we can then treat the economy as populated with the two representative investors who behave competitively. In the remainder of the paper, we treat the two representative investors generically and simply refer to them as the more-risk-averse and less-risk-averse agents.

Market equilibrium in this economy consists of a pair of price processes $\{P_t, r_t\}$ and the consumption-trading strategies $\{C_{i,t}, (N_{i,t}, M_{i,t}), i = 1, 2\}$ such that the agents' expected lifetime utilities are maximized subject to their respective wealth dynamics in Equation (3), and the securities markets clear:

$$N_{1,t} + N_{2,t} = 1, \tag{7a}$$

$$M_{1,t} + M_{2,t} = 0. \tag{7b}$$

3. THE SINGLE-AGENT EQUILIBRIUM

Before presenting the results for the two-agent model, we first review the trading and asset-pricing implications of the familiar single-representative-agent model in our setting. In fact, as shown by Stapleton and Subrahmanyam (1990), this is the well-known situation considered by Black and Scholes (1973). The results from the single-agent model then provide a benchmark for comparison to those from the two-agent model.

The single-agent model is nested within the two-agent model by assuming that only one of the two agents, say the less-risk-averse agent, is present in the market. Thus, the less-risk-averse agent is initially endowed with all of the shares of stock in the economy, $1 - n = 1$ or $n = 0$.⁴ The agent maximizes his expected lifetime utility through consumption and investment choices $\{C_t, (N_t, M_t)\}$.

³Given CRRA preferences and the process for the aggregate endowment, both agents' marginal utility and stock payoffs are unbounded from above. Thus, parameter restrictions are needed to ensure the prices of certain securities such as the stock and bonds are well defined.

⁴The parallel case where the more-risk-averse agent is endowed with all the shares of stock is given by simply replacing γ with 2γ throughout all of the formulas in this section.

Here, for brevity, we have omitted the subscript i in denoting agent i . In equilibrium, however, the agent's consumption C_t must equal the aggregate amount of dividends X_t . Similarly, market clearing implies that the agent holds all of the shares of the stock and does not borrow or lend; $N_t = 1$ and $M_t = 0$. This latter feature makes the trading implications of the single-agent model simple as there is no trading in equilibrium and the agent never changes the number of shares of the stock or the riskless asset in his portfolio.

The equilibrium price of the stock can be obtained directly from the Euler equation associated with N_t ,

$$P_t = E_t \left[\int_0^\infty e^{-\rho\tau} \left(\frac{C_{t+\tau}}{C_t} \right)^{-\gamma} X_{t+\tau} d\tau \right] = X_t E_t \left[\int_0^\infty e^{-\rho\tau} \left(\frac{X_{t+\tau}}{X_t} \right)^{1-\gamma} d\tau \right], \quad (8)$$

where the second equality follows from $C_t = X_t$. As shown in the Appendix, this expression reduces to

$$P_t = \frac{1}{\rho - \kappa} X_t, \quad (9)$$

where

$$\kappa = (1 - \gamma)(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}(1 - \gamma)^2\sigma^2. \quad (10)$$

Thus, the price-dividend ratio, defined as

$$Y_t = P_t/X_t, \quad (11)$$

is constant in the single-agent economy, i.e., $Y_t = 1/(\rho - \kappa)$. Its inverse $y_t = X_t/P_t$ is simply the dividend yield of the stock, which is $\rho - \kappa$.

An application of Itô's Lemma to Equation (9) gives the dynamics for the stock price,

$$dP_t = P_t (\mu dt + \sigma dZ_t). \quad (12)$$

Thus, the stock price inherits the geometric Brownian motion dynamics of the underlying dividend process. In this single-agent economy, the agent's wealth W_t equals the value of his stock holdings, $W_t = M_t + N_t P_t = P_t$.

In a similar manner, the riskless rate can be determined from the Euler equation associated with M_t . The Appendix shows that the riskless rate is given by

$$r_t = \rho + \mu\gamma - \frac{1}{2}\gamma(1 + \gamma)\sigma^2, \quad (13)$$

which is a constant.

The single-agent market exhibits some simple properties. For example, the interest rate is constant and the stock returns are i.i.d. In particular, the expected return on the stock is $\mu + y_t = \mu + \rho - \kappa$ and its return volatility is σ , both constant over time. Moreover, stock returns are serially uncorrelated. As we see below, these are no longer true when both agents are present in the market.

4. THE TWO-AGENT EQUILIBRIUM

In this section, we present the closed-form solutions for equilibrium consumption, asset prices, and portfolio choices for the two-agent model. We explore the economic intuition and implications of

these results more fully in the subsequent sections, and provide the proofs and derivations in the Appendix.

The equilibrium is derived in three steps. First, relying on the complete securities market in our model, we solve for the equilibrium allocation of consumption between the two agents from its Pareto optimality. Second, using the Euler equation for the agents we compute the equilibrium stock price and interest rate that support the equilibrium allocation. Finally, by analyzing the agents' portfolio policies financing their consumption, we obtain their equilibrium holdings of the stock and the riskless security.

4.1 Consumption

In the two-agent economy, the sum of the agents' consumption streams must equal the aggregate dividends, i.e. $C_{1,t} + C_{2,t} = X_t$. An allocation $C_{1,t}, C_{2,t}$ is Pareto optimal if, and only if, there exists $\alpha \in [0, 1]$ such that $C_{1,t}, C_{2,t}$ solves the problem

$$\max_{C_{1,t} + C_{2,t} \leq X_t} E_0 \left[\int_0^\infty e^{-\rho t} \left[\alpha \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + (1-\alpha) \frac{C_{2,t}^{1-2\gamma}}{1-2\gamma} \right] dt \right]. \quad (14)$$

The solution is

$$C_{1,t} = X_t - \frac{2}{b} \left(\sqrt{1 + bX_t} - 1 \right), \quad C_{2,t} = X_t - C_{1,t} = \frac{2}{b} \left(\sqrt{1 + bX_t} - 1 \right), \quad (15)$$

where $b = 4\left(\frac{\alpha}{1-\alpha}\right)^{1/\gamma}$. To simplify notation, we denote the more-risk-averse agent's consumption simply as C_t . Thus, in equilibrium, the less-risk-averse agent's consumption is $X_t - C_t$.

The demographics of the market are characterized by the relative consumption levels of the two agents. Let s_t denote the less-risk-averse agent's share of total consumption. We have

$$s_t = \frac{X_t - C_t}{X_t} = \frac{\sqrt{1 + bX_t} - 1}{\sqrt{1 + bX_t} + 1}. \quad (16)$$

As we will see below, s_t is an important variable in characterizing the behavior of the economy. We return to analyze its properties in Section 5.

4.2 Asset Prices

Given the equilibrium consumption allocation, we now compute the stock price and the interest rate that support the equilibrium. The Euler equation for the more-risk-averse agent with respect to his stock holding reads

$$P_t = E_t \left[\int_0^\infty e^{-\rho\tau} \left(\frac{C_{t+\tau}}{C_t} \right)^{-2\gamma} X_{t+\tau} d\tau \right]. \quad (17)$$

Substituting the expression for C_t into Equation (17) leads to the following closed-form solution for the equilibrium price-dividend ratio of the stock

$$\frac{P_t}{X_t} = Y_t = a_1 - a_2 s_t F(1, -\theta, 3 + \theta - 2\gamma; s_t) - a_3 (1 - s_t) F(1, \lambda, 2\gamma + 2\lambda; 1 - s_t), \quad (18)$$

where $F(a, b, c; z)$ is the standard hypergeometric function (see, for example, Abramowitz and Stegun (1970), Chapter 15),

$$\theta = \frac{\psi + (\mu - \frac{1}{2}\sigma^2)}{\sigma^2}, \quad \lambda = \frac{\psi - (\mu - \frac{1}{2}\sigma^2)}{\sigma^2}, \quad \psi = \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}, \quad (19)$$

and

$$a_1 = \frac{\theta + \lambda - \gamma}{\psi(\gamma + \lambda - 1)(1 + \theta - 2\gamma)}, \quad (20a)$$

$$a_2 = \frac{2\gamma}{\psi(1 + \theta - 2\gamma)(2 + \theta - 2\gamma)}, \quad (20b)$$

$$a_3 = \frac{\gamma}{\psi(\gamma + \lambda - 1)(2\gamma + 2\lambda - 1)}. \quad (20c)$$

The equilibrium riskless rate is obtained from the Euler equations in a manner similar to that for the stock price,

$$r_t = \rho + \frac{2\mu\gamma}{(1 + s_t)} - \frac{\gamma(2\gamma + 3)\sigma^2}{(1 + s_t)^2} + \frac{2\gamma\sigma^2}{(1 + s_t)^3}. \quad (21)$$

4.3 Optimal Portfolios

In equilibrium, the more-risk-averse agent's wealth W_t is simply the present value of his consumption stream. Substituting C_t into the corresponding Euler equations implies that

$$\begin{aligned} \frac{W_t}{X_t} = w_t = & b_1 (1 - s_t) - b_2 s_t(1 - s_t) F(1, 1 - \theta, 3 + \theta - 2\gamma; s_t) \\ & + b_3 s_t(1 - s_t) F(1, 1 + \lambda, 2\gamma + 2\lambda; 1 - s_t), \end{aligned} \quad (22)$$

where

$$b_1 = \frac{\theta + \lambda}{\psi(2\gamma + \lambda - 1)(1 + \theta - 2\gamma)}, \quad (23a)$$

$$b_2 = \frac{2\gamma - 1}{\psi(1 + \theta - 2\gamma)(2 + \theta - 2\gamma)}, \quad (23b)$$

$$b_3 = \frac{2\gamma - 1}{\psi(2\gamma + \lambda - 1)(2\gamma + 2\lambda - 1)}. \quad (23c)$$

From the wealth of the more-risk-averse agent, we can derive his optimal holdings of the stock and the riskless security, denoted by N_t and M_t , respectively. Market clearing then implies that the less-risk-averse agent will hold $1 - N_t$ shares of the stock and $-M_t$ units of the riskless security.

By definition, the more-risk-averse agent's wealth equals the value of his portfolio holdings, $W_t = M_t + N_t P_t$. Following Cox and Huang (1989) and Wang (1996), this implies that $dW_t = N_t dP_t$ after imposing the self-financing constraint $dM_t + P_t dN_t = 0$. Thus, the ratio of the diffusion coefficients in the dynamics of W_t and P_t can be used to solve for N_t . Once N_t is determined, M_t can be obtained directly from the expression for wealth. As shown in the Appendix, the optimal portfolio holdings for the more-risk-averse agent are

$$N_t = \frac{\Psi_t}{\Phi_t}, \quad (24a)$$

$$M_t = W_t - N_t P_t, \quad (24b)$$

where

$$\begin{aligned}\Psi_t &= \frac{(1+s_t)}{s_t(1-s_t)}w_t - b_1 \\ &- b_2(1-2s_t)F(1,1-\theta,3+\theta-2\gamma;s_t) - b_2s_t(1-s_t)F'(1,1-\theta,3+\theta-2\gamma;s_t) \\ &+ b_3(1-2s_t)F(1,1+\lambda,2\gamma+2\lambda;1-s_t) - b_3s_t(1-s_t)F'(1,1+\lambda,2\gamma+2\lambda;1-s_t),\end{aligned}\quad (25a)$$

$$\begin{aligned}\Phi_t &= \frac{(1+s_t)}{s_t(1-s_t)}\frac{1}{y_t} \\ &- a_2F(1,-\theta,3+\theta-2\gamma;s_t) - a_2s_tF'(1,-\theta,3+\theta-2\gamma;s_t) \\ &+ a_3F(1,\lambda,2\gamma+2\lambda;1-s_t) + a_3(1-s_t)F'(1,\lambda,2\gamma+2\lambda;1-s_t),\end{aligned}\quad (25b)$$

y_t is the dividend yield ($y_t = 1/Y_t$), and $F'(a,b,c;z) = (ab/c)F(a+1,b+1,c+1,z)$.

5. CONSUMPTION ALLOCATION

From the market equilibrium given in the previous section, we now examine the allocation of consumption (and risk) between the two agents, how this allocation is achieved through their trading in the stock and the credit market, and how their trading activity determines the behavior of asset prices.

When only a single agent is present, he consumes the aggregate endowment X_t . When two agents are present, they have to share the aggregate endowment. Given the difference in their preferences, each will not simply share a constant portion of the aggregate endowment. From Equation (15), the equilibrium allocation is such that the more-risk-averse agent consumes the lion's share in bad states, i.e., states with low aggregate endowment, and the less-risk-averse agent consumes the major share in the good states.

In illustrating the results of the paper, we will use a baseline calibration throughout to make the results easier to compare. Specifically, we assume that the expected dividend growth rate μ is 0.03 and that the volatility of dividend growth σ is 0.12. These values are consistent with the historical properties of imputed corporate dividends (for example, see Longstaff and Piazzesi (2004)). We also assume that the subjective time discount rate ρ is 0.01 and the less-risk-averse agent has logarithmic preferences, i.e., $\gamma = 1$. As we will see in Section 7, these parameter values lead to asset prices that are broadly compatible with what we see in the data. For example, the interest rate ranges between 2.56 and 2.68 percent, the stock risk premium ranges between 1.4 and 3.0 percent, stock return volatility ranges between 12 and 14 percent, and the term premium of interest rates is close to zero. Finally, we assume that the initial dividend level X_0 and the share allocation between the two agents n are such that $\alpha = \frac{1}{2}$.

Figure 1 plots the two agents' shares of aggregate consumption for different levels of aggregate consumption (endowment). Also plotted is the aggregate consumption X_t as a function of the less-risk-averse agent's share s_t . The left panel of Figure 1 shows that at a given time t , the share of the less-risk-averse agent's consumption s_t monotonically increases with the aggregate level of consumption X_t . It starts at zero as X_t is close to zero, but increases as X_t increases and approaches one as X_t goes to infinity. This consumption allocation across different states of the economy is intuitive. As the aggregate endowment decreases, the marginal utility of the more-risk-averse agent increases faster than that of the less-risk-averse agent. On the other hand, as the

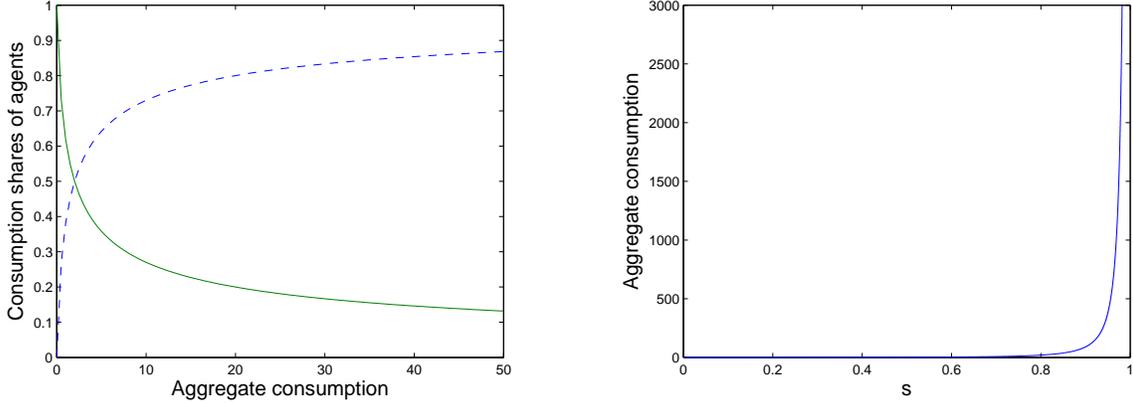


Figure 1. Aggregate consumption and agents' consumption shares. The left panel plots the agents' consumption shares as functions of aggregate consumption, the dashed line for the less-risk-averse agent and the solid line for the more-risk-averse agent, and the right panel plots aggregate consumption as a function of the less-risk-averse agent's consumption share s_t . The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, $\gamma = 1.00$, and $\alpha = 0.50$.

aggregate endowment increases, the marginal utility of the more-risk-averse agent decreases faster than that of the less-risk-averse agent. The optimal consumption is reached when the marginal utilities of the two agents are equal. This is achieved when the more-risk-averse agent consumes a relatively larger share in the bad states by claiming a relative smaller share in the good states. The right panel of Figure 1 shows that aggregate consumption X_t is very small in absolute value in the states when the more-risk-averse agent dominates the economy (when s_t is small), while the opposite is true when the less-risk-averse agent dominates the economy.

Over time, as X_t changes, so does s_t . In particular, s_t evolves as follows

$$ds_t = \mu_{s,t} dt + \sigma_{s,t} dZ_t, \quad (26)$$

where

$$\mu_{s,t} = \frac{s_t(1-s_t)}{1+s_t} \left[(\mu - \sigma^2) + \frac{1}{(1+s_t)^2} \sigma^2 \right], \quad (27a)$$

$$\sigma_{s,t} = \frac{s_t(1-s_t)}{1+s_t} \sigma. \quad (27b)$$

One immediate observation is that in addition to itself, the dynamics of s_t depend only on the parameters governing the aggregate consumption process, i.e., μ and σ ; the dynamics do not depend on the initial condition of the economy (i.e., X_0 and n) which only fixes the initial value of s_t . More importantly, the dynamics of s_t do not depend on the parameters concerning the agents' preferences, i.e., ρ and γ . They do, however, depend on the fact that the ratio between the two agents' relative risk aversion is two. This property comes from the fact that given the dynamics of total consumption, the dynamics of s_t are determined by the sharing rule between the two agents.

Given that the two agents have constant relative risk aversion and the same time discount rate ρ , the sharing rule depends only on the ratio of their relative risk aversion coefficients. In our model, this ratio is two.

The drift and volatility of s_t imply that it follows a process similar to the class of Wright-Fisher diffusions used in genetics and many other contexts (see, for example, Karlin and Taylor (1981)). The drift of this process is a ratio of simple polynomials. Depending on parameter values, the drift can be uniformly positive (when $\mu > \frac{3}{4}\sigma^2$), uniformly negative (when $\mu < 0$), or can be positive for values of s_t below some threshold and negative for values greater than that threshold (when $0 < \mu < \frac{3}{4}\sigma^2$). This latter situation implies a certain type of mean-reverting behavior for the process. However, the process does not have a stationary distribution in this situation. The volatility of the process takes its maximum value at $s_t = \sqrt{2} - 1$.

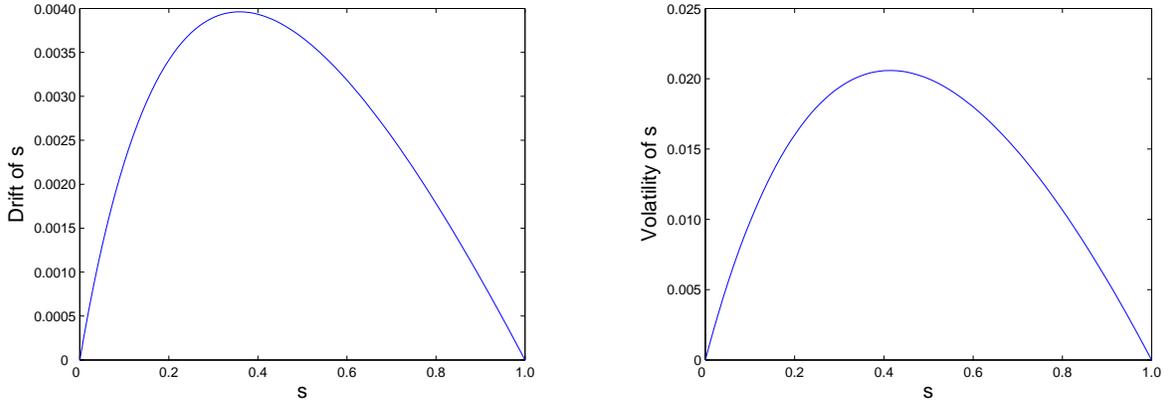


Figure 2. Dynamics of the consumption share of the less-risk-averse agent.

The left panel plots the drift of the consumption share of the less-risk-averse agent $\mu_{s,t}$ as a function of s_t and the right panel plots the volatility of the consumption share $\sigma_{s,t}$. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

Figure 2 plots both the drift (the left panel) and the volatility of s_t (the right panel) for the baseline parameter values. Clearly, in this case where $\mu - \frac{3}{4}\sigma^2 = 0.03 - \frac{3}{4}(0.12)^2 = 0.0192 > 0$, the drift of s_t is always positive, indicating that the less-risk-averse agent is steadily gaining share of the economy. The drift increases steeply with s_t for small values of s_t . It peaks as s_t approaches 0.4 and then declines quickly. The volatility of s_t has a simple humped shape. It is zero at the two extreme ends, i.e., when s_t is equal to zero or one, when one of the agents owns the whole economy. It peaks when s_t is around 0.4. As we will see below, the dynamics of s_t are very much related to the risk sharing between the two agents and the resulting market behavior.

6. RISK SHARING AND THE CREDIT MARKET

The consumption share of the two agents, as shown in Figure 1, reveals a striking pattern. The consumption of the less-risk-averse agent is a convex function of aggregate consumption, while

that of the more-risk-averse agent is a concave function. This represents the optimal risk sharing between the two agents given their preferences. In fact, the more-risk-averse agent shifts a large part of the aggregate risk, given by the uncertainty in X_t , to the less-risk-averse agent. As a result, the risk profile of the less-risk-averse agent actually exceeds that of the overall economy.

6.1 Leverage and Risk Sharing

Risk sharing is achieved through the two agents' trading in the securities market. In particular, it is facilitated by the lending of the more-risk-averse agent in the credit market to the less-risk-averse agent. As a result, the more-risk-averse agent is able to switch his stock holdings into riskless debt, thus to maintain a less-risky wealth profile. This is accommodated by the less-risk-averse agent, who issues debt to the more-risk-averse agent to finance his own levered purchase of additional stock shares.

Thus, the credit market plays a critical role in allowing optimal risk sharing among agents with different risk preferences. In the absence of a credit market, each of the agents would have to hold the same portfolio consisting of a 100 percent weight in the stock. The presence of the credit market allows agents to modify the risk profile of their portfolios by borrowing and lending and thus allocate risk optimally.

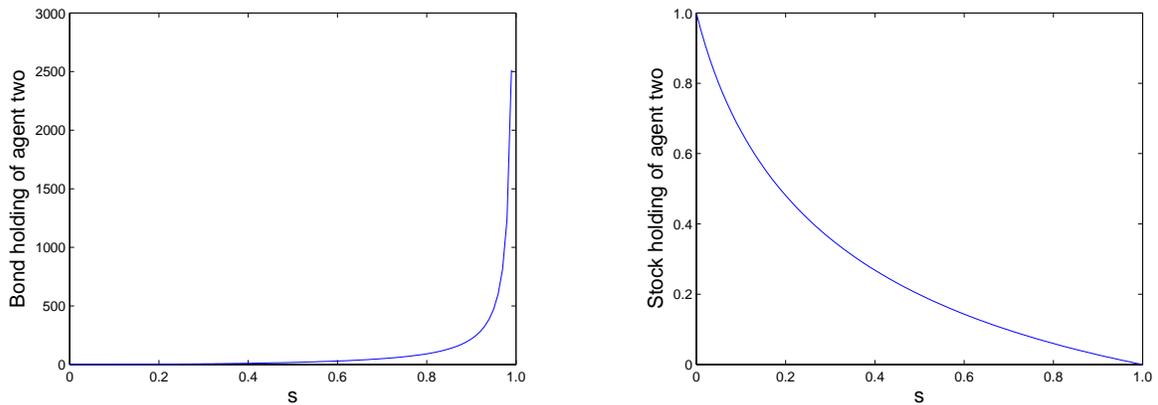


Figure 3. Bond and stock holdings of the more-risk-averse agent. The left panel plots the amount of riskless debt the more-risk-averse agent (agent two) holds as a function of s_t and the right plots the number of stock shares he holds. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

Figure 3 plots the debt and stock shares held by the more-risk-averse agent as a function of s_t . The left panel shows M_t , the total amount of “short-term” debt, in the form of instantaneous credit, held by the more-risk-averse agent. As we can see, for all possible states of the economy (i.e., the whole range of s_t), M_t is positive. That is, the more-risk-averse agent is always the lender in the market, lending money to the less-risk-averse agent in exchange for safe future payoffs. Of course, the bond position of the less-risk-averse agent is simply $-M_t$, which is always negative.

At low levels of X_t , the consumption share of the less-risk-averse agent s_t is close to zero.

In these states, the more-risk-averse agent owns most of the economy and consumes most of the aggregate endowment. As the left panel of Figure 3 shows, the level of debt is small in these states as the less-risk-averse agent has little wealth to use as collateral in borrowing. As X_t increases, the overall wealth of the economy increases. Moreover, the less-risk-averse agent also has more wealth. Consequently, he can take on more debt by issuing more bonds to the more-risk-averse agent. Indeed, we see that M_t rises quickly with X_t or equivalently s_t .

While the increase in the lending of the more-risk-averse agent represents a shift in his wealth from the stock to bond, the increase in the borrowing of the less-risk-averse agent is used to increase his stock positions. The right panel of Figure 3 plots the stock shares held by the more-risk-averse agent N_t . Since the total number of stock shares is normalized to one, the number of shares held by the less-risk-averse agent is simply $1 - N_t$. Clearly, at low levels of X_t , the more-risk-averse agent holds most of the stock. In fact, as mentioned above, he owns most of the economy and consumes the lion's share of the aggregate consumption. As X_t increases, however, his stock holding monotonically decreases. When X_t approaches infinity, s_t approaches one (the less-risk-averse agent consumes most of the economy), and N_t approaches zero. In those states, the more-risk-averse agent holds most of his wealth bonds.

As shown in Cox and Huang (1989), an agent's portfolio rebalancing can be interpreted as the dynamic trading strategy that generates "derivative" contracts that deliver the optimal consumption for each date and state. From this perspective, the portfolio strategy of the more-risk-averse agent, which sells stock shares for bonds as the stock price rises and buys when the stock price falls, is exactly to achieve the negative convexity in his desired consumption profile. Thus, through the dynamic rebalancing of his stock and bond positions, he is synthetically "selling" call options to the less-risk-averse agent. In the vocabulary of the options market, the risk-averse agent is short "gamma," while the less-risk-averse agent is long "gamma."

Finding that the risk-averse agent sells options to the less-risk-averse agent may seem counterintuitive at first. After all, selling options is generally viewed as a highly risky enterprise. In this equilibrium, however, the more-risk-averse agent is not simply selling options outright with a potentially unbounded downside. Rather, the risk-averse agent follows a much more conservative "covered call" strategy by selling options against an underlying stock position. Obviously, the credit market is crucial to allow him to achieve this through his portfolio strategy.

6.2 Optimal Portfolio Weights

In addition to describing the agents' stock and bond holdings in absolute terms, we also examine them in relative terms. In particular, we consider the relative weight of stock in both agents' portfolios $w_{i,t}$, where

$$w_{1,t} = \frac{(1 - N_t)P_t}{(1 - N_t)P_t - M_t}, \quad w_{2,t} = \frac{N_t P_t}{N_t P_t + M_t}. \quad (28)$$

The relative weight of bond in agent i 's portfolio is simply $1 - w_{i,t}$. Figure 4 plots $w_{1,t}$ and $w_{2,t}$ as a function of s_t .

Facilitated by the credit market and the possibility of leverage, the difference between the two agents' portfolios is striking. For the less-risk-averse agent, the weight of stock in his portfolio is always above one, reflecting the fact that he is always levered. For small values of s_t , which corresponds to low levels of X_t or bad states of the economy, the stock weight in his portfolio is close to two. In other words, he pledges all his wealth as collateral to borrow. In these states, it is

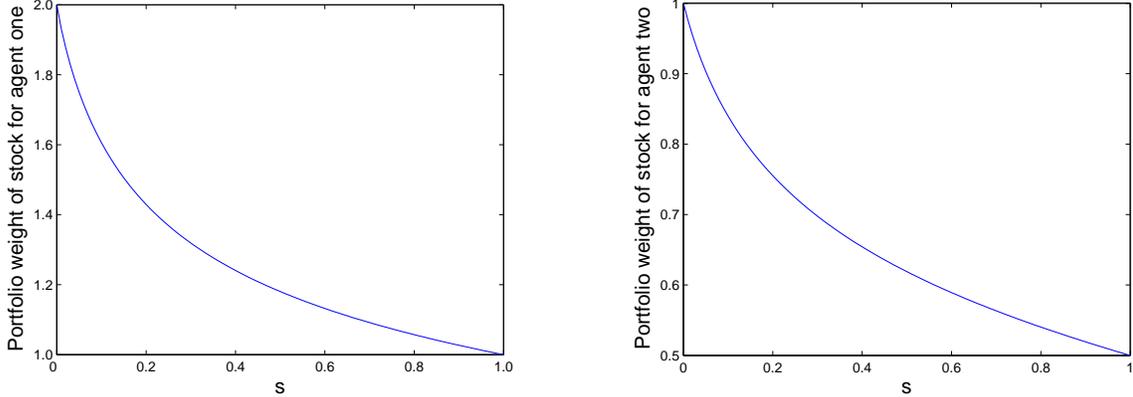


Figure 4. Weight of stock in agents' portfolios. The left panel plots the weight of stock in the portfolio of the less-risk-averse agent (agent one) against s_t , and the right panel plots that of the more-risk-averse agent (agent two). The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

the more-risk-averse agent who is more wealthy, and he can fully accommodate the leverage needs of the less-risk-averse agent. From Figure 3, we see that the absolute size of debt is small for small s_t . But it is a large percentage of the less-risk-averse agent's portfolio. As s_t increases, the economy moves into good states, in which the less-risk-averse agent gains a larger share of the total wealth (and consumption). Despite his preference for leverage, less debt would be available as the wealth share of the more-risk-averse agent dwindles. Consequently, he is forced to reduce leverage and the weight of the stock in his portfolio decreases. When s_t approaches one, the less-risk-averse agent owns most of the economy, which is the stock. The weight of the stock in his portfolio approaches one.

For the more-risk-averse agent, the weight of stock in his portfolio is always between zero and one since he holds part of his portfolio in the riskless bond. For small values of s_t (i.e., X_t), he owns most of the economy and thus most of the stock. The debt he holds is only a trivial part of his total portfolio. In these bad states of the economy, the opportunity for risk sharing is very limited and $w_{2,t}$ is close to one. As s_t increases, his share of the total economy decreases. He shifts to safer asset allocations, investing a smaller fraction of his wealth in the stock while investing more in the bond. When s_t approaches one, the less-risk-averse agent dominates the economy. In these states, $w_{2,t}$ approaches zero and the more-risk-averse agent ends up with all his portfolio in the bond. In other words, he completely avoids the risk of the economy.

The above analysis reveals the central role the credit market plays. The risk sharing between the two agents is achieved by allowing the less-risk-averse agent to bear a larger share of the aggregate risk, which is fully reflected in the risk of the stock market. Such a shift is facilitated by the credit market, which allows the less-risk-averse agent to borrow capital and take on levered positions in the stock. The amount he can borrow, however, depends on the amount of collateral, i.e., wealth, he has. In the bad states (i.e., s_t is close to zero), he has less wealth as collateral and risk sharing is limited. In the good states (i.e., s_t is close to one), the less-risk-averse agent controls most of the wealth and thus has abundant collateral. In these states, the risk sharing is

more complete as he bears all the risk of the economy and the more-risk-averse agent's wealth is all in bonds.

6.3 The Market-Leverage Ratio

Given the importance of the credit market, we now consider the ratio of aggregate credit in the market to the total value of assets held by the agents. This ratio, which we denote the market-leverage ratio, is simply M_t/P_t . Intuition suggests that there are some common-sense bounds on the values that this ratio can take in equilibrium. In particular, the market-leverage ratio should be bounded below by zero given the agents' risk-sharing incentives. On the other hand, if the interest-rate payments on the debt exceed the dividend payments received by the less-risk-averse agent, he would only be able to avoid default by borrowing further. Even this expedient would appear to have a limit since the total debt payments made by the less-risk-averse agent could not exceed the aggregate dividend payments generated by the stock, the positive-net-supply asset in the economy.

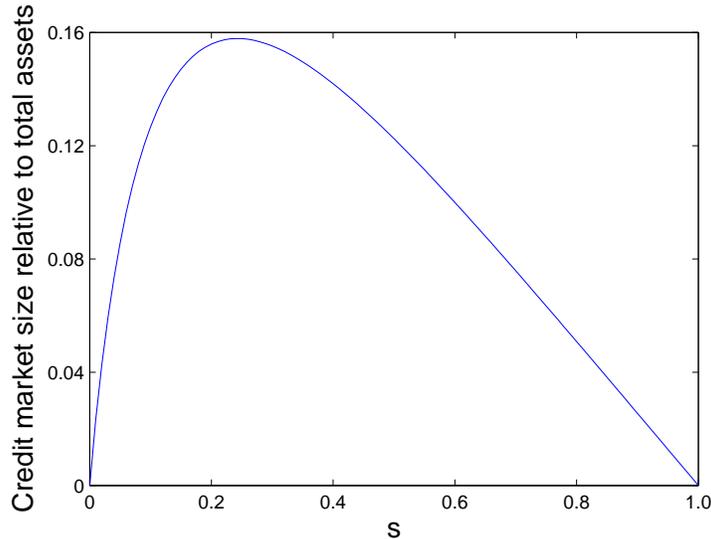


Figure 5. Market-leverage ratio. The figure plots the ratio of the amount of debt outstanding to the total value of the assets in the economy for different values of s_t . The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

Figure 5 plots the market-leverage ratio as a function of s_t . As expected, the market-leverage ratio approaches zero as s_t approaches either zero or one. This follows simply because the aggregate amount of debt in the market depends on the relative size of each agent in the market. If there is effectively only one agent in the market, little or no debt can occur. For intermediate values of s_t , however, the amount of debt in the economy can be substantial. The maximum market-leverage ratio of close to 16 percent occurs around $s_t = 0.25$.

The maximum market-leverage ratio implied by the model is very consistent with the U.S. historical experience. Using the Federal Reserve's Z.1 statistical data for the flow of funds accounts

of the United States from 1953 to 2006, we find that the market-leverage ratio ranges from a low of about 8 percent in 1953, to a high of slightly more than 19 percent in 2005. This historical high of 19 percent agrees closely with maximum value implied by our model.

Comparing the market-leverage ratio plotted in Figure 5 for different states of the economy, which are fully characterized by s_t , with the volatility of the less-risk-averse agent's consumption share plotted in Figure 2 (the right panel), we find a striking similarity between the two. In particular, the market-leverage ratio peaks at around 0.25 and so does the volatility of s_t (the exact locations of their maxima are slightly different). This is not surprising. Given that optimal risk sharing induces the less-risk-averse agent to load up on risk by leveraging, the amount of risk he bears reaches a maximum with the amount of leverage. At this point, his wealth as well as consumption are also most volatile.

7. ASSET PRICES

We now examine how risk sharing between the two agents influences the behavior of asset prices. Especially, we focus on the role of the credit market and its impact on asset prices.

To put the asset-pricing implications of leverage and risk sharing into perspective, recall that the interest rate is constant and stock returns are i.i.d. through time in the single-agent economy. In contrast, the moments of returns are generally time varying in the two-agent economy. From Equations (18) and (21), we see that both r_t and y_t (the dividend yield) vary with s_t . We will first examine the behavior of the stock price and then the properties of interest rates.

7.1 The Stock Price and Its Dynamics

Figure 6 plots the price-dividend ratio Y_t of the stock (the left panel) and its dividend yield $y_t = 1/Y_t$ (the right panel) as functions of s_t for the baseline parameter values. The left panel shows that the valuation ratio for dividends increases in a slightly nonlinear way as the economy expands. Conversely, the dividend yield decreases with s_t in this case. In other words, the dividend yield behaves countercyclically—it increases when the economy expands and decreases when the economy shrinks.

Since $P_t = X_t Y_t$ where Y_t is the price-dividend ratio given in Equation (18), Itô's Lemma implies that stock price dynamics dP_t/P_t can be expressed in terms of dX_t/X_t and dY_t/Y_t . From Equation (1), however, the moments of the dividend process dX_t/X_t are constant since the dividend follows an i.i.d. geometric Brownian motion. As a result, any variation in the return moments is due entirely to variation in the valuation ratio Y_t , which is fully determined by s_t .

Given Equation (18) and the dynamics of s_t in Equations (26-27), we obtain the following expression for the stock price dynamics (see Appendix for details)

$$\frac{dP_t}{P_t} = \left(\phi_t \mu_{s,t} + \frac{1}{2} \xi_t \sigma_{s,t}^2 \right) dt + \phi_t \sigma_{s,t} dZ_t, \quad (29)$$

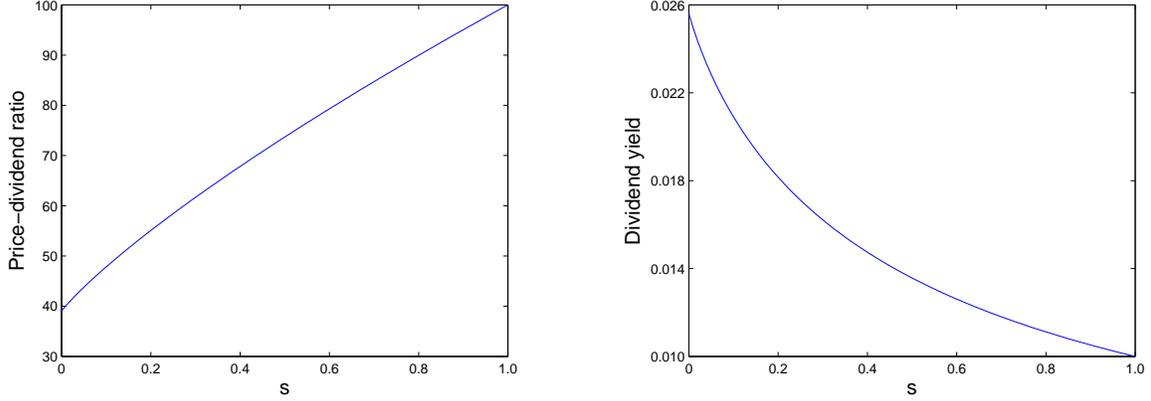


Figure 6. Stock price-dividend ratio and dividend yield. The left panel plots the price-dividend ratio as a function of s_t and the right panel plots the stock's dividend yield. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

where

$$\begin{aligned} \phi_t &= (1 + s_t)/[s_t(1 - s_t)] \\ &- a_2 y_t F(1, -\theta, 3 + \theta - 2\gamma; s_t) - a_2 y_t s_t F'(1, -\theta, 3 + \theta - 2\gamma; s_t) \\ &+ a_3 y_t F(1, \lambda, 2\gamma + 2\lambda; 1 - s_t) + a_3 y_t (1 - s_t) F'(1, \lambda, 2\gamma + 2\lambda; 1 - s_t), \end{aligned} \quad (30a)$$

$$\begin{aligned} \xi_t &= 2 \phi_t (1 + s_t)/[s_t(1 - s_t)] - 2/[s_t^2(1 - s_t)^2] \\ &- 2a_2 y_t F'(1, -\theta, 3 + \theta - 2\gamma; s_t) - a_2 y_t s_t F''(1, -\theta, 3 + \theta - 2\gamma; s_t) \\ &- 2a_3 y_t F'(1, \lambda, 2\gamma + 2\lambda; 1 - s_t) - a_3 y_t (1 - s_t) F''(1, \lambda, 2\gamma + 2\lambda; 1 - s_t), \end{aligned} \quad (30b)$$

and the derivatives F' and F'' are given by the simple differentiation formula for hypergeometric functions, $F'(a, b, c; z) = (ab/c)F(a + 1, b + 1, c + 1; z)$. Although there are numerous terms in this expression, it does show that the stock price dynamics are explicitly a function of s_t , the market demographics.

7.2 The Expected Return and Return Volatility of the Stock

Given the stock price dynamics, we now analyze the behavior of stock returns, especially their conditional distributions. Since the stock price follows a diffusion process and the dividend yield follows a smooth process, the instantaneous stock returns are conditionally normal. Thus, we need only to focus on their first two moments, the expected return and return volatility.

The expected return for the stock, which we denote by q_t , is just the sum of the dividend yield y_t and the expected price appreciation $E[dP_t/P_t]$. From Equation (29), this can be expressed as

$$q_t = y_t + \phi_t \mu_{s,t} + \frac{1}{2} \xi_t \sigma_{s,t}^2. \quad (31)$$

Given the smooth nature of the dividend, stock return volatility comes solely from price volatility. From Equations (29-30), it is given by $\sigma_t = |\phi_t \sigma_{s,t}|$. For the baseline parameters, Figure 7 plots

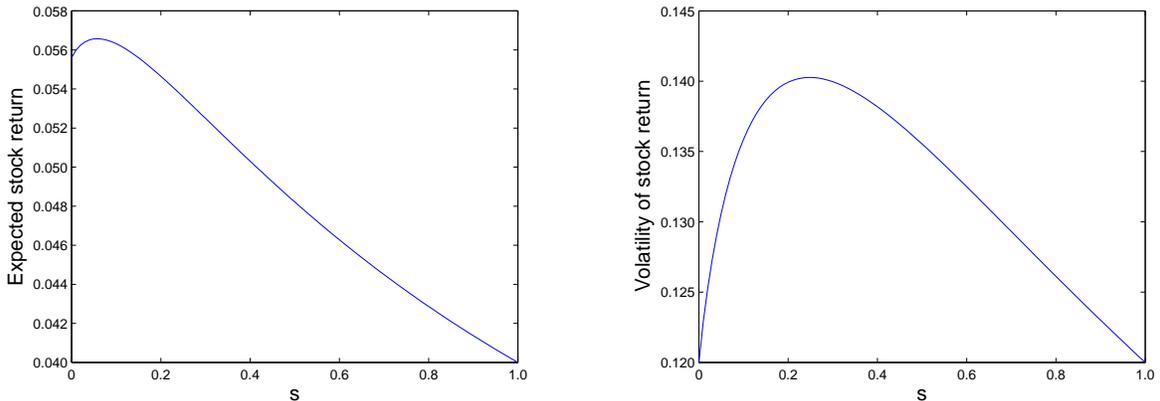


Figure 7. Expected return and volatility of the stock. The left panel plots the expected return of the stock as a function of s_t and the right panel plots the stock return volatility. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

both the expected return of the stock (the left panel) and its return volatility (the right panel) as a function of s_t .

For most values of s_t , in particular for s_t greater than 0.05, the stock’s expected return decreases with s_t . That is, the expected return behaves in a countercyclical manner for most of the states of the economy. This behavior also implies a negative correlation between the expected return and the return of the stock itself. However, this pattern is not uniform. When the economy falls into very low consumption states (when s_t falls below 0.05), the expected stock return can behave procyclically—it comoves with the overall level of the market. The possibility of this rich relation between the stock’s expected return and price levels should be taken into account when analyzing the empirical behavior of these two variables.

Next, we examine stock return volatility and how it varies with the state of the economy. Recall that in the single-agent economy, the volatility of stock returns is just the volatility of the dividend process σ . This is true independent of the level of risk aversion of the representative single agent. In contrast, the stock return volatility can differ significantly from the volatility of dividends when both agents are present in the market.

The right panel of Figure 7 plots the volatility of stock returns as a function of s_t . As expected, stock return volatility approaches the volatility of the dividend process, which is 0.12, as s_t approaches either of the limiting values of zero or one. For all other values of s_t , however, the volatility of stock returns diverges from the volatility of dividends. In fact, in this case, stock return volatility is always higher than the dividend volatility of 0.12. This implies that when both agents are present in the market and have more risk-sharing, the stock price actually becomes more volatile than if only one of them is present. The volatility of stock returns reaches its maximum of over 0.14 when s_t is between 0.20 and 0.30.

The nonmonotonic behavior of stock return volatility with s_t (and thus X_t) implies a rich pattern of heteroscedasticity for stock returns. In particular, in the region of s_t exceeding 0.25,

stock return volatility is negatively correlated with changes in the stock price. That is, the volatility increases as the market drops. This is compatible with the empirical relation between stock market returns and return volatility (see, e.g., Black (1976) and Nelson (1991)). Figure 5 also shows, however, that for small values of s_t (less than 0.25), i.e., when the economy is in low consumption states, the correlation between stock volatility and return can be positive.

Comparing the behavior of stock return volatility with that of the market-leverage ratio shown in Figure 5, we see that the two are closely related. Both exhibit a unimodal, humped-shaped pattern, reaching their maxima when s_t is roughly 0.25. In this case, stock return volatility moves together with the market-leverage ratio. Such a strong positive correlation between these two variables suggests that even though leverage helps to achieve optimal risk sharing overall, it can substantially increase the local volatility of the stock market.

7.3 The Risk Premium and Sharpe Ratio of the Stock

We now examine the expected excess return or (instantaneous) risk premium on the stock, which is defined by

$$\pi_t = q_t - r_t, \tag{32}$$

where r_t is the instantaneous interest rate, and its Sharpe ratio π_t/σ_t . Figure 8 plots these two quantities as functions of s_t . As we see from the left panel, for a wide range of s_t , in particular, when $s_t > 0.05$, the expected excess return decreases with the level of the market. This suggests that the time variation of the risk premium is also countercyclical. Comparing the behavior of the expected excess return of the stock and that of the dividend yield, we see a positive relation between the two. This is consistent with the empirical evidence on the positive correlation between dividend yield and future stock returns (see Fama and French (1988) and Campbell and Shiller (1988a, b), among others).

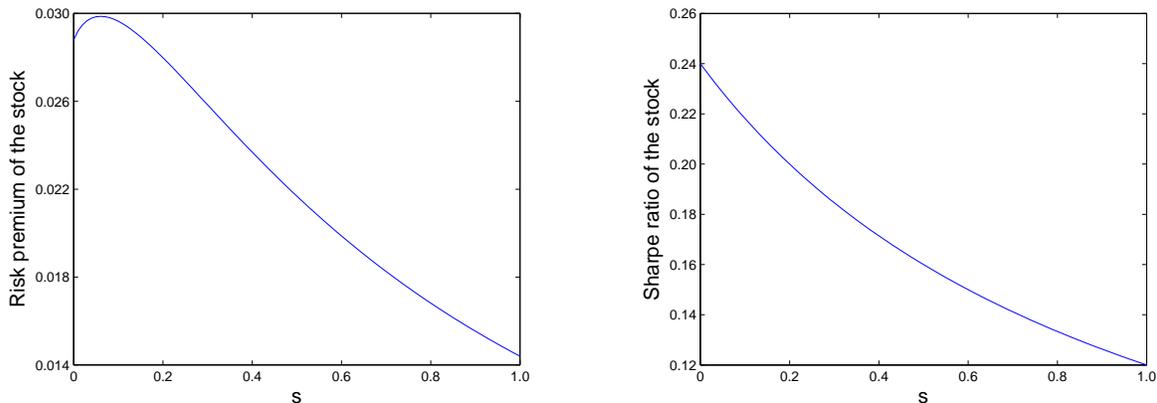


Figure 8. Expected excess return and Sharpe ratio of the stock. The left panel plots the expected excess return of the stock as a function of s_t and the right panel plots the Sharpe ratio. The parameters are at the baseline values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

From Figure 8 we also see that the expected excess return on the stock is not uniformly

countercyclical. For states with very low consumption levels, it can be positively correlated with changes in the stock price. In other words, it can turn procyclical in these states. Our results clearly suggest that the relation between the risk premium and the price level of the stock market can be quite complex.

The right panel of Figure 8 further shows that in contrast to the more-complex behavior of the expected excess return, the Sharpe ratio of the stock exhibits a simple countercyclical pattern. Among the parameter values we have explored, the countercyclical behavior of the Sharpe ratio seems to be quite robust. This is consistent with the empirical evidence presented by Ferson and Harvey (1991), among others.

7.4 Further Discussions on Stock Returns

Under the baseline parameter values our model produces relatively simple patterns for stock price behavior that are largely compatible with the empirical observations. However, these patterns are by no means unique. In fact, under different parameter values our model can lead to a variety of behaviors for stock return dynamics.

Instead of presenting an extensive analysis of various possible return patterns in our model, we consider another set of parameter values, which are also reasonable in matching the data, and show that they can lead to quite different properties of the equilibrium. Our purpose here is merely to illustrate the richness in the model's predictions. In particular, we let the growth rate of the aggregate dividend μ be 0.02 and its volatility σ be 0.15. Moreover, we let the time discount rate ρ be 0.04 and the relative risk aversion of the less-risk-averse agent remain at $\gamma = 1$. For these parameter values, our model leads to an interest rate between 1.25 and 3.75 percent, a risk premium for the stock between 2.25 and 4.50 percent, and a stock return volatility around 15 percent.

Figure 9 illustrates the various properties of the stock price under this set of parameter values. The top left panel plots the stock's price-dividend ratio Y_t for different states of the economy. In contrast to the baseline case, the price-dividend ratio is no longer monotonic in s_t in this case. In fact, it is contracyclical for all s_t less than 0.75. That is, the price-dividend ratio can actually decrease with the level of the market. But for extremely good states of the economy, i.e., when s_t exceeds 0.75, the correlation between the two turns positive. The price-dividend ratio has a minimum value of roughly 25 at around $s_t = 0.75$.

The top right panel shows the behavior of the expected return on the stock q_t . It shows a similar dependence on s_t as the dividend yield but with some differences in the low states of the economy. In particular, for s_t between 0.05 and 0.75, the expected return of the stock is procyclical—it increases with the level of the market. For s_t less than 0.05 or greater than 0.75, however, the expected return becomes countercyclical.

This nonmonotonic behavior makes it clear that the expected return in the two-agent economy is not just a weighted average of the expected returns of the two extreme cases when only the more-risk-averse agent or the less-risk-averse agent populates the market. In these cases, the expected stock return would be 0.06 and 0.0575, respectively. Figure 9 shows that the expected return of the stock in the two-agent model can lie outside the bounds given by the limiting one-agent economies implied by allowing s_t to approach zero or one. This result parallels those described in Wang (1996) for the riskless interest rate.

The bottom left panel of Figure 9 plots the expected excess return of the stock against s_t . It is interesting that it decreases monotonically with s_t . The difference in the behavior of the

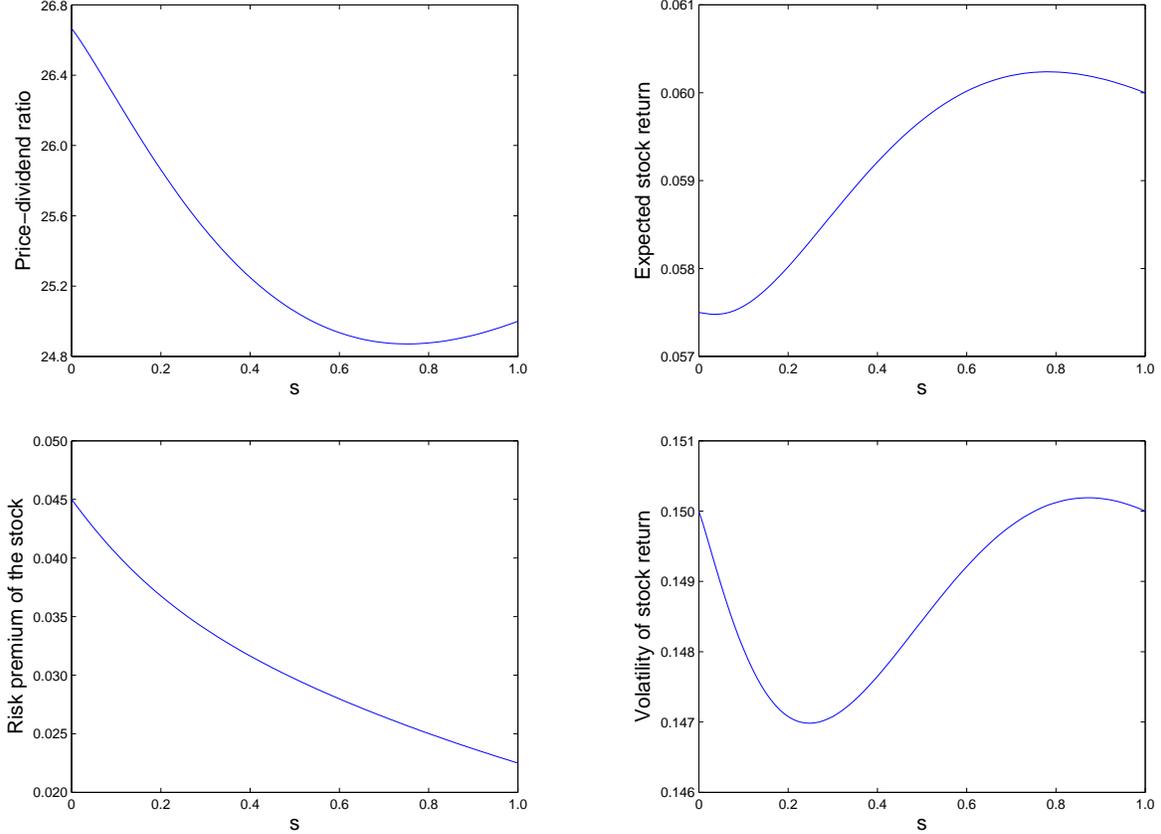


Figure 9. Stock price behavior under alternative parameter values. The top two panels plot the price-dividend ratio (the left panel) and the expected return (the right panel) of the stock, respectively, as functions of s_t . The bottom two panels plot the expected excess return (the left panel) and return volatility (the right panel) of the stock, respectively. The parameters are at the alternative values: $\mu = 0.02$, $\sigma = 0.15$, $\rho = 0.04$, and $\gamma = 1.00$.

expected excess return and that of the expected return is caused by the riskless interest rate, which is monotonically increasing with s_t under the current parameter values.

What is the most striking is the behavior of stock return volatility, which is shown in the bottom right panel of Figure 9. Its dependence on the state of the economy is highly nonlinear. When s_t is small, the volatility decreases with the level of the stock market. It reaches a minimum when s_t is around 0.25. For s_t between 0.25 and 0.85, the volatility of the stock is positively related to the level of its price. After reaching its maximum at s_t around 0.85, the volatility becomes negatively related to s_t again. The fact that stock return volatility can be lower than the fundamental volatility σ , its value in the single-agent economy, shows that risk sharing between the two agents can help to reduce price volatility under certain circumstances.

7.5 Interest Rates and Bond Prices

We now turn our attention to the behavior of interest rates and bond prices. The instantaneous interest rate is given in Equation (21). The behavior of interest rates in a market with heterogeneous agents is analyzed in detail by Wang (1996) in a model similar to ours. Although the interest rate stays constant when only one of the agents is present in the market, it becomes stochastic when both are present. The solution we obtain for the agents' equilibrium portfolio policies allows us to further link quantities in the market such as the total amount of credit with the behavior of interest rates and bond prices.

In addition to the instantaneous interest rate, we can also compute the prices of long-term bonds and their yields as in Wang (1996). Especially, we want to consider the price of a consol bond which pays a continuous interest flow at a rate of one. Its price can be computed in closed form from the Euler equation,

$$\begin{aligned}
 B_t &= E_t \left[\int_0^\infty e^{-\rho\tau} \left(\frac{C_{t+\tau}}{C_t} \right)^{-2\gamma} d\tau \right], \\
 &= a'_1 - a'_2 s_t F(1, 1-\theta, 2+\theta-2\gamma; s_t) - a'_3 (1-s_t) F(1, \lambda+1, 2\gamma+2\lambda+2; 1-s_t), \quad (33)
 \end{aligned}$$

where

$$a'_1 = \frac{\theta + \lambda - \gamma}{\psi(\gamma + \lambda)(\theta - 2\gamma)}, \quad (34a)$$

$$a'_2 = \frac{2\gamma}{\psi(1 + \theta - 2\gamma)(\theta - 2\gamma)}, \quad (34b)$$

$$a'_3 = \frac{\gamma}{\psi(\gamma + \lambda)(2\gamma + 2\lambda + 1)}. \quad (34c)$$

The yield to maturity on the consol bond, which represents an average yield on long-term bonds, can then be defined by

$$l_t = \frac{1}{B_t}. \quad (35)$$

The difference between the long-term bond yield and the instantaneous interest rate r_t , $l_t - r_t$, gives a measure of the term spread for bond yields.

Figure 10 illustrates the behavior of short- and long-term interest rates under the baseline parameter values. The top left panel plots the instantaneous interest rate as a function of s_t . Note that at $s_t = 0$, r_t reaches the limiting interest rate $r^{(2)}$ when only the more-risk-averse is present in the market, which is 0.0268. Similarly, at $s_t = 1$, r_t approaches the limiting interest rate $r^{(1)}$ when only the less-risk-averse agent is present, which is 0.0256. Overall, in this case r_t decreases with the level of aggregate consumption (or stock price).⁵

However, it is worth noticing that for a significant range of s_t , namely, from 0.15 to 0.30, the interest rate remains relatively constant. To better understand this phenomenon, let us compare the behavior of r_t with that of the market-leverage ratio shown in Figure 5. We observe that the

⁵This overall relation between r_t and s_t is sensitive to the parameter values, which determine $r^{(1)}$ and $r^{(2)}$. If $r^{(2)} > r^{(1)}$, as in the case here, r_t decreases with X_t overall (but not necessarily monotonically). If $r^{(2)} < r^{(1)}$, the opposite is true.

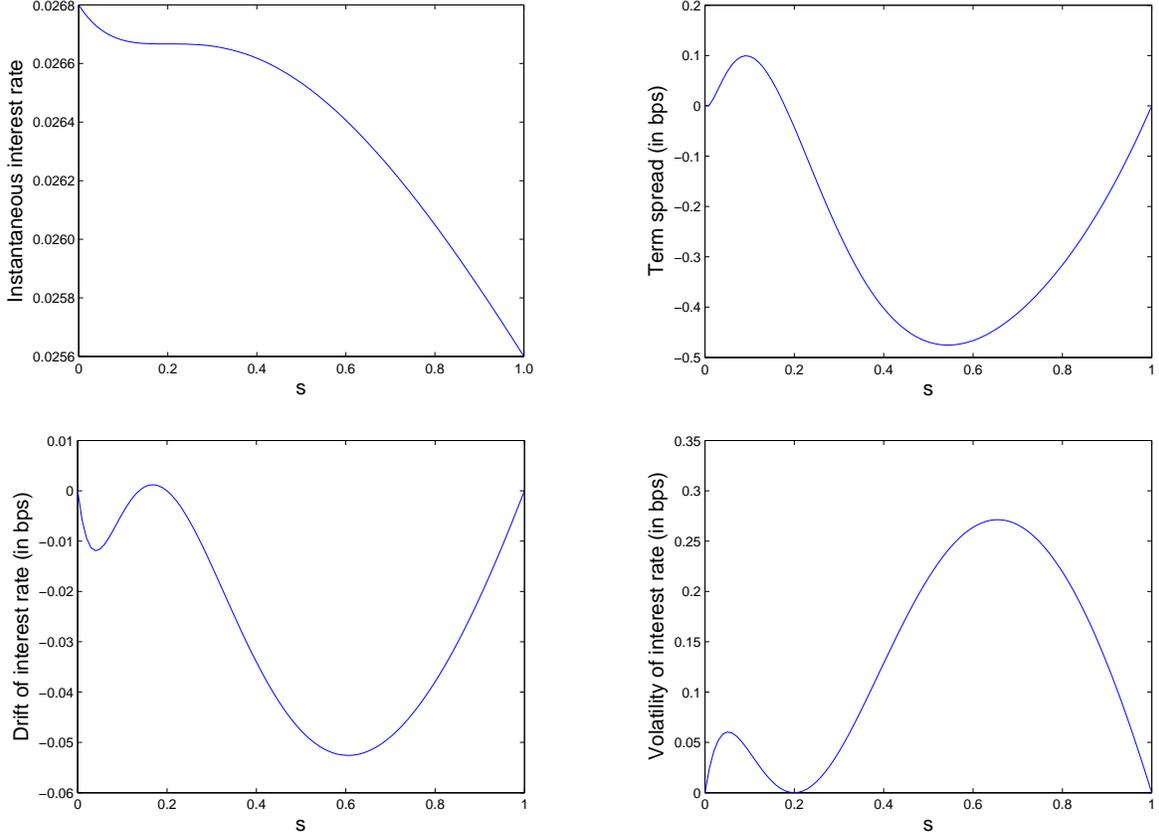


Figure 10. Bond yields and interest rate dynamics. The top two panels plot the instantaneous interest rate (the left panel) and the term spread of interest rates (the right panel) against s_t , respectively. The bottom two panels plot the drift (the left panel) and volatility (the right panel) of the instantaneous interest rate, respectively. The parameters are at the baseline values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

market-leverage ratio reaches its maximum when s_t is around 0.25. This suggests that when the market-leverage ratio reaches its peak, the demand and the supply of credit are both maximized. At this point, they also become less variable with respect to changes in the state of the economy. As a result, the interest rate is less volatile. We further confirm this intuition below when we examine the interest-rate dynamics. When s_t is outside this range, both the market-leverage ratio and the interest rate change significantly when the aggregate endowment varies. This implies a positive correlation between absolute changes in the market-leverage ratio and interest rate.⁶

The top right panel of Figure 10 shows the term spread of interest rates for different states of the economy. First recall that $s_t = 0$ and 1 correspond to the single-agent economy when the term spread is zero because in these cases the interest rate is constant and thus the term structure of

⁶The direction of changes in the leverage ratio and interest rate is ambiguous, depending upon which side of the maximum leverage ratio the market is on and the overall relation between the interest rate and aggregate consumption. See also the previous footnote.

interest rates is flat. When both agents are present, this is no longer the case. For small values of s_t , i.e., when it is less than 0.18, the term spread is positive. Given that we have not computed the interest rates for all maturities, we cannot make detailed statements about the shape of the term structure. However, the positive difference between the yield on the consol and the instantaneous interest rate indicates that overall the term structure is upward sloping. The term spread reaches a maximum of 0.10 basis points when s_t is around 0.08. As s_t increases further, the term spread starts to decrease. It turns negative when s_t exceeds 0.18. This implies an overall downward-sloping term structure of interest rates. The term spread has a minimum of -0.48 basis points at $s_t = 0.55$ and approaches zero as s_t approaches one.

Obviously, the term spread we observe in this case is very small in magnitude. This is in part because for the parameter values we use, the level and the variability of the interest rate are both low, roughly consistent with the data. The low volatility of the interest rate will limit the slope of term structure in general. Given the limited degrees of freedom we have in the model, we focus less on the magnitudes of various effects and more on their qualitative features.

Our results indicate that the relation between the term spread and the aggregate state of the economy can be quite rich. Within certain ranges of the economy (i.e., $s_t > 0.55$), it behaves procyclically, while for other ranges of the economy, it can be countercyclical.

Two factors drive the shape of the term structure, the expectation about future interest rates and the risk premium associated with their uncertainty. In order to see how expectations about future interest rates behave, we plot in the bottom two panels of Figure 10 the drift of the instantaneous interest rate (the left panel) and its volatility (the right panel), respectively. It is not surprising that the drift of the interest rate is overall negative. It is highly nonlinear, however. In particular, at around $s_t = 0.17$, it turns to slightly positive (it is positive for $0.14 \leq s_t \leq 0.20$).

Examining the various panels in Figure 10, we notice that the interest rate is relatively insensitive to s_t and the term spread is positive at around the same range of s_t where the drift of r_t is near zero. From the bottom right panel of Figure 10, we further confirm that the volatility of the interest rate becomes zero around $s_t = 0.20$. This suggests that in these states of the economy, the risk premium for long-term bonds tends to be positive.

It is again useful to consider the interest-rate dynamics jointly with the amount of credit generated in the market. As can be seen from Figures 5 and 10, the region over which both the drift and volatility of the interest rate approach zero also overlaps with the region where the leverage ratio of the market is maximized. This reinforces the intuition that quantities and prices are closely related.

When s_t exceeds 0.20, the economy is in high consumption states and the drift of the interest rate is always negative. The expectation about future decreases in interest rates seems to dominate the term spread and makes it negative for a wide range of s_t , i.e., when $0.20 < s_t < 1$.

As in the discussion of stock returns, the behavior of the instantaneous interest rate and the term spread in our model also depends on the parameter values. For example, for the alternative set of parameter values ($\mu = 0.02$, $\sigma = 0.15$ and $\rho = 0.04$), $r^{(2)} < r^{(1)}$, and the interest rate increases with s_t . The term spread is always negative in this case, however, suggesting a negative risk premium for long-term bonds. For brevity, we omit a more-detailed discussion of various possible interest-rate dynamics that can emerge from our model.

8. TRADING ACTIVITY

As we discussed in Sections 5 and 6, risk-sharing between the two agents with different risk preferences is achieved through their trading in the securities market. In particular, in our model it is accomplished by allowing the less-risk-averse agent to borrow in the credit market and then take on a levered position in the stock market. Moreover, such a levered position is not static, but rather dynamic. As the economy evolves, the desire for borrowing and lending also changes for the less- and more-risk-averse agents. Consequently, both agents follow dynamic trading strategies to replicate their desired consumption profiles. Their equilibrium trading strategies not only drive asset prices as we elaborated in the previous section, but also the trading activities both in the credit and the stock market. In Section 6, we examined the amount of credit generated endogenously in the market and its behavior. In this section, we turn our attention to the stock market and analyze our model's implications for stock trading activity and how it behaves.

In a continuous-time setting like ours with the diffusive nature of the information flow, trading volume in the conventional sense is not properly defined. In fact, it would be infinite. This is because the local variation of the underlying shocks is unbounded, and so is the agents' security holdings. Trading costs have to be part of the analysis in order to study volume in a rigorous manner (see, for example, Lo, Mamaysky and Wang (2004)). Such a treatment is beyond the scope of this paper. Instead, we use an alternative measure for the amount of trading activity in the market. In particular, given the stock holding of an agent N_t (e.g., the more-risk-averse agent), we use its absolute volatility $\sigma_{N,t}$ to gauge his trading activity. Given that the less-risk-averse agent's stock holding is $1 - N_t$, our measure of trading activity does not depend on which agent we follow.

From the stock holdings of the more-risk-averse agent given in Equations (24-25), some algebra yields the following expression for $\sigma_{N,t}$,

$$V_t = \sigma_{N,t} = \left| \left[\frac{1}{\Phi_t} \frac{d\Psi_t}{ds_t} - \frac{\Psi_t}{\Phi_t^2} \frac{d\Phi_t}{ds_t} \right] \sigma_{s,t} \right|, \quad (36)$$

where $\sigma_{s,t}$ is the volatility of s_t given in Equation (27). Figure 11 plots our measure of stock trading activity V_t for different values of s_t .

Not surprisingly, trading activity exhibits the same unimodal pattern as the market-leverage ratio. In the two extremes, i.e., when $s_t = 0$ or 1 , the market is dominated by one of the agents and there is no trading. Somewhere in the middle range of s_t , trading is most intense as both agents have large needs to share risk and are also compatible in size to accommodate each other.

The behavior of stock trading activity shown in Figure 11 has several interesting implications. First, the level of trading activity evolves smoothly in the state space, but can differ substantially in different parts of the state space. This implies that it can be highly persistent over time. When the economy moves into those states with high trading activity, say, when s_t falls between 0.05 and 0.30, it will stay there for a while and so will trading activity. Second, in some states of the economy, in particular when s_t is relatively small, trading is procyclical, while in other states, i.e., when s_t is relatively large, it can be countercyclical. This rich relation between trading activity and changes in the price level of the stock market may help explain the complex empirical patterns between them (see, for example, Karpoff (1987) and Gallant, Rossi and Tauchen (1992)). Third, comparing the behavior of trading activity and stock return volatility, we see a strong positive relation between the two. Trading is particularly active when return volatility is high. This is one of the most robust patterns about trading activity observed in the data (see Karpoff). Fourth,

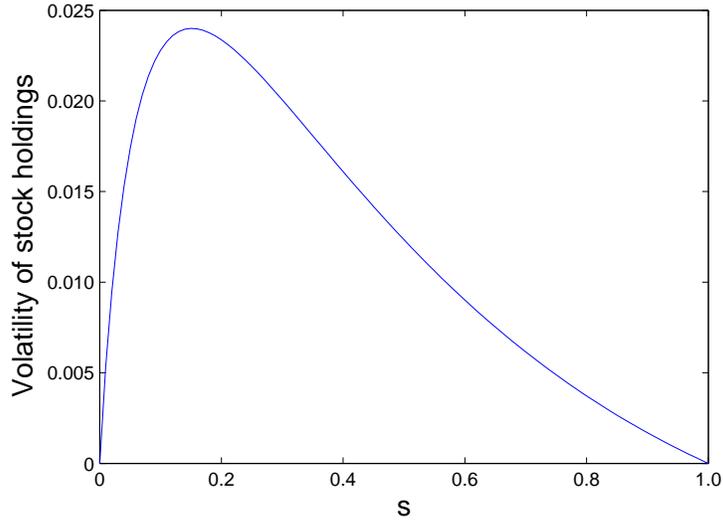


Figure 11. Stock trading activity. The figure plots the volatility of agents’ stock holdings as a measure of trading activity for different values of s_t . The parameters are at the baseline values: $\mu = 0.03$, $\sigma = 0.12$, $\rho = 0.01$, and $\gamma = 1.00$.

comparing the behavior of stock trading activity and leverage in the market (Figure 5), we also see a close relation between these two variables. In particular, trading in the stock market peaks as the market-leverage ratio approaches its maximum. This is intuitive given that in our model leverage is used by the less-risk-averse agent to finance his stock purchases.

9. EMPIRICAL RESULTS

The key difference between the standard single-agent framework and the two-agent model developed in this paper is that the distribution of wealth among agents becomes an important state variable that drives the equilibrium. While the notion that heterogeneity affects asset pricing is certainly not new, taking heterogeneous-agent models to the data has traditionally proven difficult precisely because agent heterogeneity is not directly observable, at least at the aggregate level.

In this paper, we have shown that the credit market allows for risk sharing among the agents in the model. In general, the more equal the distribution of wealth in the economy, the greater is the amount of leverage. An immediate corollary of our results is that changes in the size of the credit sector (which are observable) provide direct information about changes in the relative wealth of the two classes of agents (which are not directly observable). Thus, the model delivers the testable empirical implication that changes in the size of the credit sector should be associated with changes in key asset-pricing measures such as expected returns.

To explore this empirical implication of the model, we focus on the relation between the equity premium and the size of the credit sector. Since the equity premium is itself not directly observable, we will use the standard approach of estimating predictive vector autoregressions (VARs) in which

ex post excess stock market returns are regressed on ex ante credit sector measures. Intuitively, if time variation in the equity premium is correlated with changes in the size of credit sector, then these measures should have predictive power for subsequent excess stock returns

As the measure of excess stock market returns, we use the excess return on the CRSP value-weighted index. The data consist of the annual excess (non-overlapping) returns for the 55-year period from 1952 to 2006 (data provided by courtesy of Ken French).

There is an extensive and rapidly-growing literature on stock return predictability that is far too lengthy for us to review in depth. We note, however, that there are a number of economic measures identified in the literature that appear to have some predictive power for excess stock returns. Our approach will be to include three of the variables that appear prominently in the forecasting literature, and then evaluate whether credit sector information has incremental forecasting power for excess returns in the VARs.

These three variables are: the lagged excess return for the CRSP value-weighted index, the Lettau and Ludvigson (2001) *cay* measure, and the annual dividend yield for the CRSP value-weighted index. The inclusion of the lagged excess return is motivated by the extensive empirical literature on the returns from momentum strategies. Examples of this literature include DeBondt and Thaler (1985), Lo and MacKinlay (1988), Poterba and Summers (1988), Jegadeesh and Titman (1993), and many others.⁷ The inclusion of the *cay* measure is motivated by the evidence in Lettau and Ludvigson that the consumption-wealth ratio is a strong predictor of stock returns. Finally, the inclusion of the dividend yield in the VARs is motivated by the results of Fama and French (1988), Goyal and Welch (2003), Cochrane (2007), Lettau and Van Nieuwerburgh (2007), and many others.

To capture changes in the size of the credit sector, we use two variables in the VARs that reflect the size of the credit market in the economy. In particular, the first measure captures the relative size of cash flows in the credit and stock markets. It is defined as the ratio of aggregate personal interest income to aggregate personal dividend income as reported in lines 14 and 15 of Table 2.1 “Personal Income and its Disposition” reported in the National Income and Product Accounts (NIPA) by the Bureau of Economic Analysis.⁸ The second measure can be viewed as a measure of debt service and is defined as the ratio of aggregate personal income from interest income to the total value of household assets. The total value of household assets appears as line 1 of Table B.100 “Balance Sheet of Households and Nonprofit Organizations” reported in statistical release Z.1 “Flow of Funds Accounts of the United States” by the Federal Reserve Board.

Table 1 reports the results from the VARs for the excess return predictive regressions. The first regression specification includes only the lagged excess return, *cay* measure, and dividend yield. Consistent with the results in the literature, we find that the combination of these three ex ante variables has significant in-sample predictive power for the CRSP value-weighted excess returns at the one-year horizon. The R^2 for the regression is 0.2185; the adjusted R^2 is 0.1716.

The predictive power for excess stock returns increases dramatically when the two credit sector measures are included in the VAR specification. When the first lagged values are included, the R^2 becomes 0.3451 and the adjusted R^2 becomes 0.2769. When the first two lagged values of these variables are included, the R^2 increases to 0.4837, while the adjusted R^2 increases to 0.4033. To our knowledge, these in-sample R^2 s far exceed any that have been previously reported in the literature

⁷Also see Grinblatt, Titman, and Wermers (1995) and Chan, Jegadeesh, and Lakonishok (1996).

⁸Data reported as of September 5, 2007.

Table 1. Predictive Regressions for Excess Stock Returns with Leverage Variables. This table reports the results from the regression of annual excess returns for the CRSP value-weighted index on the ex ante predictive variables. $ExRet_t$ denotes the excess return on the CRSP value-weighted index, Cay_t is the Lettau-Ludvigson Cay measure, $DivYld_t$ is the dividend yield for the CRSP value-weighted index, Int/Div_t is the ratio of total interest received to total dividends received, where these values are from the BEA National Income and Product Accounts, $Int/Assets_t$ is the ratio of total interest received to total household assets, where the total household value is from the Federal Reserve Board's Flow of Funds Z.1 Report. The p -value is for the F -statistic for significance of the regression. The data consist of annual observations for the 1952-2006 period ($N = 55$).

Predictive Variable	Regression Coefficients [Newey-West t -Statistics]			
Intercept	-0.0113 [-0.15]	-0.1607 [-1.56]	-0.2900 [-2.72]	-0.2821 [-2.08]
$ExRet_{t-1}$	-0.0425 [-0.39]	-0.0466 [-0.45]	-0.2774 [-2.41]	-0.1681 [-2.38]
Cay_{t-1}	5.3096 [3.16]	4.0419 [2.30]	4.3209 [2.20]	5.0453 [3.44]
$DivYld_{t-1}$	2.7013 [1.29]	7.5216 [3.62]	12.1719 [4.57]	10.7310 [3.12]
Int/Div_{t-1}		-0.2166 [-2.57]	0.0249 [0.17]	
Int/Div_{t-2}			-0.3615 [-2.63]	-0.2380 [-2.39]
Int/Div_{t-3}				-0.1160 [-1.16]
$Int/Assets_{t-1}$		32.1657 [2.26]	-24.1033 [-0.87]	
$Int/Assets_{t-2}$			74.6362 [3.10]	35.9002 [1.55]
$Int/Assets_{t-3}$				18.6555 [0.77]
R^2	0.2185	0.3451	0.4837	0.4354
Adjusted R^2	0.1716	0.2769	0.4033	0.3456
p -value	0.0060	0.0008	0.0001	0.0004

for one-year horizons.⁹

Since the NIPA and Federal Reserve Flow of Funds data are not necessarily available in the market at the end of year $t - 1$ to use in forecasting excess returns for year t , it is important to address the issue of look-ahead bias in this analysis. To this end, we reestimate the VARs using only the second and third lags of the credit sector variables. This diagnostic check is a very conservative one since we are limiting the regression to credit sector variables that are roughly one year out of date. Despite this large handicap, the results indicate that the credit sector variables still provide significant incremental predictive power. In particular, the R^2 for the regression is 0.4354 while the adjusted R^2 is 0.3456.

Finally, Table 1 also reports Newey-West t -statistics for the explanatory variables. These t -statistics need to be interpreted carefully, however, given the well-known biases associated with lagged persistent variables in predictive regressions (see Stambaugh (1999)).

10. CONCLUSION

The credit market does not play a significant role in the standard single-representative-agent model in asset pricing. In this paper, we allow for two classes of investors with different levels of risk aversion and solve in closed form for equilibrium consumption levels, portfolio choices, and asset prices. In this setting, agents borrow and lend to each other to achieve optimal risk sharing and a meaningful credit sector arises.

In this model, the “demographics” of the market, as measured by the relative wealth of the agents, emerges as a key state variable driving the market. We show that borrowing and lending between the less- and more-risk-averse investors facilitates risk sharing between them and shapes the evolution of their relative wealth, which induces significant time variation in expected asset returns and return volatility. An immediate implication of this market interaction is that changes in the amount of credit in the market reveal information about changes in the relative wealth of the agents, and therefore, about changes in return moments. We take this empirical implication to the data and show that variables measuring changes in the size of the credit sector have significant predictive power for excess stock returns, even after controlling for previously documented predictive variables. Our results provide strong support for the empirical implications of the model.

⁹These R^2 s are also on the same order of magnitude as those reported by Cochrane and Piazzesi (2005) in forecasting one-year excess returns on Treasury bonds using a vector of forward rates.

REFERENCES

- Abramowitz, Milton, and Irene A. Stegun, 1970, *Handbook of Mathematical Functions*, New York, Dover Publications.
- Black, Fischer, 1976, Studies of Stock Price Volatility Changes, in *Proceedings of the 1976 Meetings of the Business and Economic Statistics Section, American Statistical Association*, 171-181.
- Black, Fischer, and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637-654.
- Campbell, John, and Robert Shiller, 1988a, The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors, *Review of Financial Studies* 1, 195-227.
- Campbell, John, and Robert Shiller, 1988b, Stock Prices, Earnings, and Expected Dividends, *Journal of Finance* 43, 661-676.
- Chan, Louis, Narasimhan Jegadeesh, and Josef Lakonishok, 1996, Momentum Strategies, *Journal of Finance* 51, 1681-1713.
- Chan, Y., and Leonid Kogan, 2002, Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices, *Journal of Political Economy* 110, 1255-1285.
- Cochrane, John H., 2007, The Dog that did not Bark: A Defense of Return Predictability, *Review of Financial Studies*, forthcoming
- Cochrane, John H., and Monika Piazzesi, 2005, Bond Risk Premia, *American Economic Review* 95, 138-160.
- Cox, John C., and Chi-fu Huang, 1989, Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process, *Journal of Economic Theory* 49, 33-83.
- Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985, An Intertemporal General Equilibrium Model of Asset Prices, *Econometrica* 53, 363-384.
- DeBondt, Werner, and Richard Thaler, 1985, Does the Stock Market Overreact?, *Journal of Finance* 40, 793-805.
- Dumas, Bernard, 1989, Two-Person Dynamic Equilibrium in the Capital Market, *Review of Financial Studies* 2, 157-188.
- Dybvig, Phillip and Chi-fu Huang, 1988, Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Streams, *Review of Financial Studies* 1, 377-401.
- Fama, Eugene, and Kenneth French, 1988, Dividend Yields and Expected Stock Returns, *Journal of Financial Economics* 22, 3-27.
- Ferson, Wayne, and Campbell Harvey, 1991, The Variation of Economic Risk Premium, *Journal of Political Economy* 99, 385-415.
- Gallant, Ron, Peter Rossi, and George Tauchen, 1992, Stock Prices and Volume, *Review of Financial Studies* 5, 199-242.
- Goyal, Amit, and Ivo Welch, 2003, Predicting the Equity Premium with Dividend Ratios, *Management Science* 49, 639-654.

- Gradshteyn, Izrail S., and Iosif M. Ryzhik, 2000, *Tables of Integrals, Series, and Products* (6th ed.), New York, Academic Press.
- Grinblatt, Mark, Sheridan Titman, and Russ Wermers, 1995, Momentum Investment Strategies, Portfolio Performance, and Herding: A Study of Mutual Fund Behavior, *American Economic Review* 85, 1088-1105.
- Harrison, J. Michael, and David M. Kreps, 1979, Martingales and Arbitrage in Multi-Period Securities Markets, *Journal of Economic Theory* 20, 381-408.
- Jegadeesh, Narasimhan, and Sheridan Titman, 1993, Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency, *Journal of Finance* 48, 65-91.
- Karlin, Samuel, and Howard M. Taylor, 1981, *A Second Course in Stochastic Processes*, Academic Press, Inc., London, UK.
- Karatzas, Ioannis, and Steven E. Shreve, 1988, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.
- Karpoff, Jonathan, M., 1987, The Relation Between Price Changes and Trading Volume: A Survey, *Journal of Financial and Quantitative Analysis* 22, 109-126.
- Lettau, Martin, and Sydney Ludvigson, 2001, Consumption, Wealth, and Expected Returns, *Journal of Finance* 55, 815-849.
- Lettau, Martin, and Stijn Van Nieuwerburgh, 2007, Reconciling the Return Predictability Evidence, *Review of Financial Studies*, forthcoming.
- Lo, Andrew, Harry Mamaysky, and Jiang Wang, 2004, Asset Prices and Trading Volume under Fixed Transaction Costs, *Journal of Political Economy* 112, 1054-1090.
- Lo, Andrew, and Craig MacKinlay, 1988, Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test, *Review of Financial Studies* 1, 41-66.
- Longstaff, Francis A. and Monika Piazzesi, 2004, Corporate Earnings and the Equity Premium, *Journal of Financial Economics* 74, 401-421.
- Lucas, Robert E., 1978, Asset Prices in an Exchange Economy, *Econometrica* 46, 1429-1445.
- Nelson, Daniel, 1991, Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59, 347-370.
- Poterba, James, and Lawrence Summers, 1988, Mean Reversion in Stock Prices: Evidence and Implications, *Journal of Financial Economics* 22, 27-60.
- Rubinstein, Mark, 1974, An Aggregation Theorem for Securities Markets, *Journal of Financial Economics* 1, 225-244.
- Stambaugh, Robert F., 1999, Predictive Regressions, *Journal of Financial Economics* 54, 375-421.
- Stapleton, Richard, and Marti Subrahmanyam, 1990, Risk Aversion and the Intertemporal Behavior of Asset Prices, *Review of Financial Studies* 3, 677-693.
- Wang, Jiang, 1996, The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogeneous Investors, *Journal of Financial Economics* 41, 75-110.

APPENDIX

A1. Solution to the Single-Agent Model.

In equilibrium, $C_t = X_t$ for the representative agent. Substituting this into the Euler equation gives,

$$\frac{P_t}{X_t} = \int_0^\infty e^{-\rho\tau} \frac{E_t[X_{t+\tau}^{1-\gamma}]}{X_t^{1-\gamma}} d\tau. \quad (A1)$$

From Equation (1),

$$X_{t+\tau} = X_t \exp \left[\left(\mu - \frac{1}{2}\sigma^2 \right) \tau + \sigma (Z_{t+\tau} - Z_t) \right], \quad (A2)$$

which implies

$$E_t[X_{t+\tau}^{1-\gamma}] = X_t^{1-\gamma} \exp \left[(1-\gamma) \left(\mu - \frac{1}{2}\sigma^2 \right) \tau + \frac{1}{2}(1-\gamma)^2 \sigma^2 \tau \right]. \quad (A3)$$

Substituting into Equation (A1) gives

$$\frac{P_t}{X_t} = \int_0^\infty e^{-(\rho-\kappa)\tau} d\tau, \quad (A4)$$

where $\rho > \kappa$, which then implies Equation (9).

Denote the value of a riskless zero-coupon bond with maturity $\Delta\tau$ as $e^{-r\Delta\tau}$. The agent's first-order conditions imply

$$e^{-r\Delta\tau} = e^{-\rho\Delta\tau} \frac{E[X_{t+\Delta\tau}^{-\gamma}]}{X_t^{-\gamma}}. \quad (A5)$$

Using Equation (A2) to represent $X_{t+\Delta\tau}^{-\gamma}$ and taking expectations gives an expression similar to Equation (A3) which is then substituted into Equation (A5),

$$e^{-r\Delta\tau} = e^{-\rho\Delta\tau} \exp \left[-\gamma \left(\mu - \frac{1}{2}\sigma^2 \right) \Delta\tau + \frac{1}{2}\gamma^2 \sigma^2 \Delta\tau \right]. \quad (A6)$$

Taking the logarithm and letting $\Delta\tau \rightarrow 0$ gives Equation (13).

A2. Solution to the Two-Agent Model.

A. Equilibrium Consumption Allocation

To be a solution for the problem in Equation (14), an allocation $C_{1,t}, C_{2,t}$ must satisfy the optimality condition,

$$E_0 \left[\int_0^\infty e^{-\rho t} \left[\alpha C_{1,t}^{-\gamma} - (1-\alpha) C_{2,t}^{-2\gamma} \right] dt \right] = 0, \quad (A7)$$

for all $t > 0$. Substituting in the solutions for $C_{1,t}$ and $C_{2,t}$ given in Equation (15) shows that they satisfy this optimality condition. The relative weight of the two agents, α , is determined by the initial conditions of the economy, in particular X_0 and agents' endowment of shares, given by n . The condition to determine α is given later.

B. The Stock Price

To solve for the stock price P_t , we substitute the more-risk-averse agent's optimal consumption into Equation (17),

$$P_t = (\sqrt{1 + bX_t} - 1)^{2\gamma} E_t \left[\int_0^\infty e^{-\rho\tau} \frac{X_{t+\tau}}{(\sqrt{1 + bX_{t+\tau}} - 1)^{2\gamma}} d\tau \right], \quad (A8)$$

$$= b^{-2\gamma} (\sqrt{1 + bX_t} - 1)^{2\gamma} E_t \left[\int_0^\infty e^{-\rho\tau} X_{t+\tau}^{1-2\gamma} (\sqrt{1 + bX_{t+\tau}} + 1)^{2\gamma} d\tau \right]. \quad (A9)$$

We rewrite $X_{t+\tau}$ as $X_t e^u$ where u is normally distributed with mean $\hat{\mu}\tau = (\mu - \frac{1}{2}\sigma^2)\tau$ and variance $\sigma^2\tau$. Substituting in the normal density gives

$$P_t = b^{-2\gamma} (\sqrt{1 + bX_t} - 1)^{2\gamma} X_t^{1-2\gamma} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{(1-2\gamma)u} (\sqrt{1 + bX_t e^u} + 1)^{2\gamma} \int_0^\infty e^{-\rho\tau} \tau^{-1/2} \exp \left[\frac{-u^2 + 2\hat{\mu}u\tau - \hat{\mu}^2\tau^2}{2\sigma^2\tau} \right] d\tau du, \quad (A10)$$

$$= b^{-2\gamma} (\sqrt{1 + bX_t} - 1)^{2\gamma} X_t^{1-2\gamma} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{(1-2\gamma)u} (\sqrt{1 + bX_t e^u} + 1)^{2\gamma} e^{\hat{\mu}u/\sigma^2} \int_0^\infty \tau^{-1/2} \exp \left(\frac{-u^2}{2\sigma^2} \frac{1}{\tau} - \frac{(\hat{\mu}^2 + 2\sigma^2\rho)\tau}{2\sigma^2} \right) d\tau du, \quad (A11)$$

$$= b^{-2\gamma} (\sqrt{1 + bX_t} - 1)^{2\gamma} X_t^{1-2\gamma} \int_{-\infty}^\infty \frac{2}{\sqrt{2\pi\sigma^2}} e^{(1-2\gamma + \hat{\mu}/\sigma^2)u} (\sqrt{1 + bX_t e^u} + 1)^{2\gamma} \sqrt{\frac{|u|}{\psi}} K_{1/2} \left(\frac{|u| \psi}{\sigma^2} \right) du, \quad (A12)$$

where $\psi = \sqrt{\hat{\mu}^2 + 2\rho\sigma^2}$, $K_{1/2}(\cdot)$ is the modified Bessel function (see Abramowitz and Stegun (1970), Chapter 10), and the last expression follows from Gradshteyn and Ryzhik (2000) 3.471.9. In turn, Gradshteyn and Ryzhik 8.469.3 implies

$$K_{1/2} \left(\frac{|u| \psi}{\sigma^2} \right) = \sqrt{\frac{\pi\sigma^2}{2|u| \psi}} \exp \left(-\frac{|u| \psi}{\sigma^2} \right). \quad (A13)$$

Substituting this into Equation (A12) gives,

$$P_t = \frac{b^{-2\gamma}}{\psi} (\sqrt{1 + bX_t} - 1)^{2\gamma} X_t^{1-2\gamma} (I_1 + I_2), \quad (A14)$$

where

$$I_1 = \int_0^\infty \exp[(1 - 2\gamma - \lambda)u] (\sqrt{1 + bX_t e^u} + 1)^{2\gamma} du, \quad (A15)$$

$$I_2 = \int_{-\infty}^0 \exp[(1 - 2\gamma + \theta)u] (\sqrt{1 + bX_t e^u} + 1)^{2\gamma} du, \quad (A16)$$

and

$$\lambda = \frac{\psi - \hat{\mu}}{\sigma^2} \geq 0, \quad \theta = \frac{\psi + \hat{\mu}}{\sigma^2} \geq 0. \quad (A17)$$

Define a new variable $w = \sqrt{1 + bX_t e^u} - 1$, and let $\eta = \sqrt{1 + bX_t}$. Changing variables gives

$$I_1 = 2^{1-\lambda} (bX_t)^{2\gamma+\lambda-1} \int_{\eta-1}^{\infty} (w+1) (1 + \frac{1}{2}w)^{-\lambda} w^{-\lambda-2\gamma} dw, \quad (A18)$$

$$I_2 = 2^{\theta+1} (bX_t)^{2\gamma-\theta-1} \int_0^{\eta-1} (w+1) (1 + \frac{1}{2}w)^\theta w^{\theta-2\gamma} dw. \quad (A19)$$

In turn,

$$\begin{aligned} I_1 + I_2 &= 2^{1-\lambda} (bX_t)^{2\gamma+\lambda-1} \int_{\eta-1}^{\infty} w^{-\lambda-2\gamma} (1 + \frac{1}{2}w)^{-\lambda} dw \\ &\quad + 2^{1-\lambda} (bX_t)^{2\gamma+\lambda-1} \int_{\eta-1}^{\infty} w^{1-\lambda-2\gamma} (1 + \frac{1}{2}w)^{-\lambda} dw \\ &\quad + 2^{\theta+1} (bX_t)^{2\gamma-\theta-1} \int_0^{\eta-1} w^{\theta-2\gamma} (1 + \frac{1}{2}w)^\theta dw \\ &\quad + 2^{\theta+1} (bX_t)^{2\gamma-\theta-1} \int_0^{\eta-1} w^{1+\theta-2\gamma} (1 + \frac{1}{2}w)^\theta dw. \end{aligned} \quad (A20)$$

Applying Gradshyeyn and Ryzhik 3.194.1 and 3.194.2, which requires $1 + \theta - 2\gamma > 0$ and $2\gamma + 2\lambda - 2 > 0$, and then using Abramowitz and Stegun (1970) 15.3.4, we have

$$\begin{aligned} I_1 + I_2 &= \frac{2(\eta+1)^{2\gamma-1}}{2\gamma+2\lambda-1} F(1, \lambda, 2\gamma+2\lambda; 2/(\eta+1)) \\ &\quad + \frac{2(\eta+1)^{2\gamma-1}(\eta-1)}{2\gamma+2\lambda-2} F(1, \lambda, 2\gamma+2\lambda-1; 2/(\eta+1)) \\ &\quad + \frac{2(\eta+1)^{2\gamma-1}}{1+\theta-2\gamma} F(1, -\theta, 2+\theta-2\gamma; (\eta-1)/(\eta+1)) \\ &\quad + \frac{2(\eta+1)^{2\gamma-1}(\eta-1)}{2+\theta-2\gamma} F(1, -\theta, 3+\theta-2\gamma; (\eta-1)/(\eta+1)). \end{aligned} \quad (A21)$$

To simplify the expression, we apply Abramowitz and Stegun 15.2.20,

$$\begin{aligned} F(1, \lambda, 2\gamma+2\lambda-1; 2/(\eta+1)) &= \\ &= \frac{\eta+1}{\eta-1} - \frac{2(2\gamma+\lambda-1)}{(\eta-1)(2\gamma+2\lambda-1)} F(1, \lambda, 2\gamma+2\lambda; 2/(\eta+1)), \end{aligned} \quad (A22)$$

$$\begin{aligned} F(1, -\theta, 2+\theta-2\gamma; (\eta-1)/(\eta+1)) &= \\ &= \frac{\eta+1}{2} - \frac{(1+\theta-1\gamma)(\eta-1)}{2+\theta-2\gamma} F(1, -\theta, 3+\theta-2\gamma; (\eta-1)/(\eta+1)). \end{aligned} \quad (A23)$$

Substituting these expressions into the solution for $I_1 + I_2$, substituting $I_1 + I_2$ into (A14), and

then collecting terms gives

$$\begin{aligned}
P_t &= \frac{X_t}{\psi(\gamma + \lambda - 1)} + \frac{X_t}{\psi(1 + \theta - 2\gamma)} \\
&\quad - \frac{2\gamma(X_t - C_t)}{\psi(1 + \theta - 2\gamma)(2 + \theta - 2\gamma)} F(1, -\theta, 3 + \theta - 2\gamma; 1 - C_t/X_t) \\
&\quad - \frac{\gamma C_t}{\psi(\gamma + \lambda - 1)(2\gamma + 2\lambda - 1)} F(1, \lambda, 2\gamma + 2\lambda; C_t/X_t), \tag{A24}
\end{aligned}$$

after substituting out η . Dividing this expression through by X_t and using the definition of s_t gives the price-dividend ratio in Equation (18) and defines the constants a_1 , a_2 , and a_3 .

C. The Instantaneous Interest Rate and Consol Price

The riskless rate r is again given by the more-risk-averse agent's first-order condition for a short-term riskless bond,

$$e^{-r\Delta\tau} = e^{-\rho\Delta\tau} E_t \left[\left(\frac{C_{t+\Delta\tau}}{C_t} \right)^{-2\gamma} \right], \tag{A25}$$

which becomes

$$e^{-r\Delta\tau} = e^{-\rho\Delta\tau} \left[\frac{E_t[(\sqrt{1 + bX_{t+\Delta\tau}} - 1)^{-2\gamma}]}{(\sqrt{1 + bX_t} - 1)^{2\gamma}} \right]. \tag{A26}$$

Applying Itô's Lemma to $(\sqrt{1 + bX_{t+\Delta\tau}} - 1)^{-2\gamma}$, taking expectations in the numerator above, and then allowing $\Delta\tau \rightarrow 0$ gives

$$r_t = \rho + \frac{4\mu\gamma(1 + bX_t) - \gamma b\sigma^2 X_t}{2(1 + bX_t)(2 - C_t/X_t)} - \frac{\gamma(2\gamma + 1)\sigma^2}{(2 - C_t/X_t)^2}. \tag{A27}$$

Expressing X_t in terms of s_t and substituting in gives Equation (21).

To solve for the price of a consol bond, we substitute the solution for C_t into Equation (33) which gives

$$B_t = (\sqrt{1 + bX_t} - 1)^{2\gamma} E_t \left[\int_0^\infty e^{-\rho\tau} (\sqrt{1 + bX_t e^u} - 1)^{-2\gamma} d\tau \right]. \tag{A28}$$

This expression is very similar to Equation (A8) and can be evaluated by following the same steps used in deriving P_t above. In doing this, the additional parameter restrictions $\theta > 2\gamma$ and $\gamma + \lambda > 0$ are required to insure the existence of a finite solution for the consol price. Combining these parameter restrictions with those following Equation (A20), we have the sufficient parameter conditions in Equation (6) to guarantee finite stock and consol prices.

D. Optimal Portfolios

The more-risk-averse agent's wealth W_t is the present value of his consumption stream

$$W_t = E_t \left[\int_0^\infty e^{-\rho s} \left(\frac{C_{t+s}}{C_t} \right)^{-2\gamma} C_{t+s} ds \right]. \tag{A29}$$

After substituting in for C_t , this becomes

$$W_t = \frac{2}{b} (\sqrt{1 + bX_t} - 1)^{2\gamma} E_t \left[\int_0^\infty e^{-\rho\tau} \left(\sqrt{1 + bX_t e^u} - 1 \right)^{1-2\gamma} d\tau \right]. \quad (\text{A30})$$

This expression is similar to Equation (A8) and the closed-form solution in Equation (22) can be obtained by following the same steps used in deriving P_t above.

To solve for N_t we note that the ratio of the diffusion coefficients for W_t and P_t is simply W_X/P_X . These derivatives are easily obtained from Equations (18) and (22) using the differentiation formula for the hypergeometric function, $F'(a, b, c; z) = (ab/c)F(a + 1, b + 1, c + 1; z)$. The value of M_t follows from the identity $M_t = W_t - N_t P_t$.

E. Stock-Price Dynamics

To obtain stock price dynamics, we note that $P_t = X_t Y_t$. Furthermore,

$$X_t = \frac{4}{b} \frac{s_t}{(1 - s_t)^2}. \quad (\text{A31})$$

Thus, P_t can be expressed exclusively as a function of s_t . A straightforward application of Itô's Lemma gives the expressions in Equations (29-30).

F. The Determination of α

The initial wealth of the more-risk-averse agent is nP_0 . From Equation (A29),

$$nP_0 = W_0 = \frac{2}{b} (\sqrt{1 + bX_0} - 1)^{2\gamma} E_0 \left[\int_0^\infty e^{-\rho t} \left(\sqrt{1 + bX_t} - 1 \right)^{1-2\gamma} dt \right]. \quad (\text{A32})$$

Substituting in Equation (A8) for P_0 , we have

$$n \left[\int_0^\infty e^{-\rho t} (\sqrt{1 + bX_t} - 1)^{-2\gamma} X_t dt \right] = \frac{2}{b} E_0 \left[\int_0^\infty e^{-\rho t} \left(\sqrt{1 + bX_t} - 1 \right)^{1-2\gamma} dt \right]. \quad (\text{A33})$$

Since the conditional expectations on the two sides of this equation depend only on X_0 and b , this equation determines b in terms of n and X_0 . Since

$$b = 4 \left(\frac{\alpha}{1 - \alpha} \right)^{1/\gamma}, \quad (\text{A34})$$

we obtain α .

G. The Measure of Trading Activity $\sigma_{N,t}$

From the stock holding of the more-risk-averse agent given in Equations (24-25) and the dynamics of s_t given in Equations (26-27), by Itô's lemma we have $\sigma_{N,t} = N_s \sigma_{s,t}$.