Introduction to Game Theory

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(based on work with Dan Loeb and Jerrod Ankenman)

What is a game?

A "Game" is a situation in which "players" have choices. The outcome of the game can be more or less favorable to each player depending on the interaction of those choices (and possibly an element of luck).

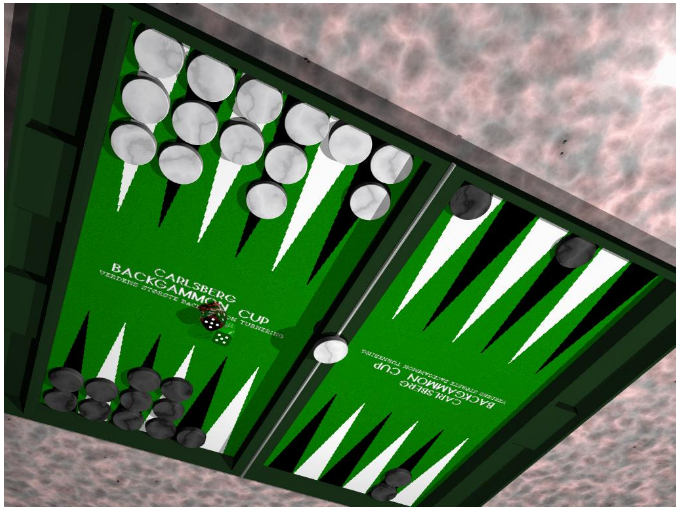
- Number of Players (1, <u>2</u>, many)
- Chance in Rules (random, <u>deterministic</u>)
- Information (hidden, <u>common knowledge</u>)
- Order of play (sequential, <u>simultaneous</u>)
- **Zero-sum or Cooperative Play**

Combinatorial Games

Chess, Checkers, Go, Tic-tac-toe, Nim, Dots-and-Boxes, ...

- 2 players
- Deterministic
- Common knowledge
- **Sequential play**
- Zero-sum. Win-lose, Draw-draw, Lose-win.

Backgammon



- 2 players
- **Random (dice)**
- Common knowledge
- Sequential play
- Zero-sum. Loser pays winner amount on the doubling cube (possibly times 2 or 3)

Poker

- Could be 2 players (heads-up) or more
- **Random (cards)**
- **Hidden information**
- Sequential play
- Usually zero-sum (unless house is taking a percentage of the pot or in some tournaments)

Bridge

- 2 "players" (actually each is a 2-person team)
- Random (cards)
- **Hidden information. (Even within a team!)**
- Sequential play
- Zero-sum (even when duplicate)

North dealer.
Both sides vulnerable.

NORTH
♠ A 7 4
♥ A 6
♦ A K Q 10 7 2
♣ 10 3

WEST
♠ Q 8 3
♥ Q J 10 8
♦ 8
♣ J 9 7 6 4

EAST
♠ 10 9 5 2
♥ 7 5 4 3
♦ J 9 6 5 3
♣ —

SOUTH
♠ K J 6
♥ K 9 2
♦ 4
♣ A K Q 8 5 2

The bidding:

North	East	South	West
1 ♦	Pass	3 ♣	Pass
3 ♦	Pass	3 NT	Pass
6 NT			

Opening lead — queen of hearts.

Zero-Sum Matrix Games

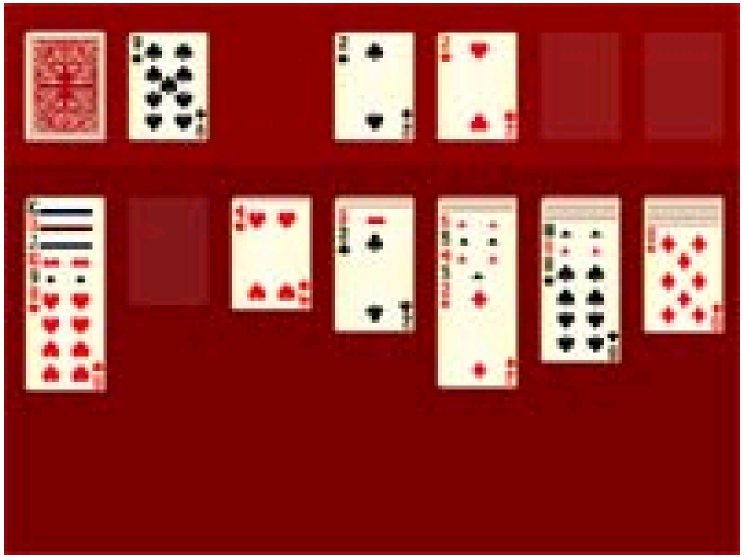
Odds-Evens, Roshambo

- 2 players
- Deterministic. (Strategies can be random!)
- Common knowledge.
- Simultaneous play.
- Zero-sum

We will begin by studying Zero-sum matrix games.

Solitaire

- 1 player, or 1 player against "nature"
- Random (cards)



Voting Systems

Congress & President form coalitions.
Winning coalition (if any) divides US Budget amongst themselves...

- Many players (100+435+1+1)
- Deterministic?
- Common knowledge. (Wheeling-dealing behind closed doors, but voting in public.)
- Sequential play.
- "Cooperative play"

Stock Market

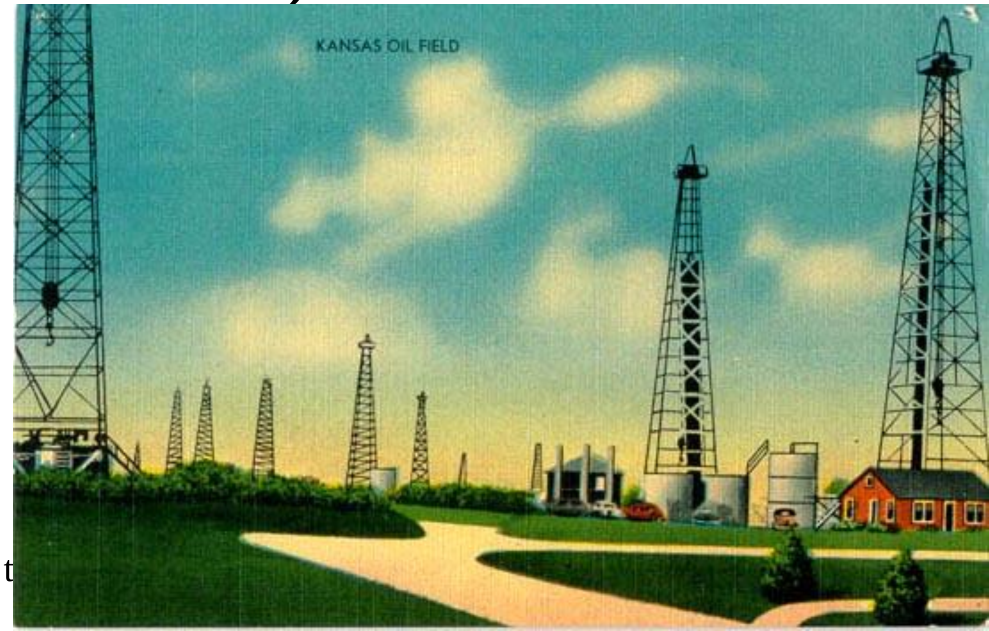
- Many players (~10^8, firms like SIG, Mom&Pop investors, etc.)
- Random (news)
- Hidden information!
- Simultaneous (asynchronous) play.
- Zero-sum (modulo taxes, commissions, SEC …).

(We will give a couple of examples: 0-sum and cooperative.)

Auctions

Each player can spend money for research to get estimate of value of oil field. Each player places a sealed bid. High bidder wins. (Beware of the winner's curse…)

- 2 or more players
- Random (estimation error, depends on $ spent research)
- Hidden information (research)
- Simultaneous play.
- Not zero-sum.



Introduction to Game Theory

0-sum 2-player Games.
- Odd-even game
- Cops & Robbers
- Market-making

Cooperative 2-player Games.
- Prisoner's Dilemma
- Outbidding
- Chicken / MAD
- Position Dumping

Multiplayer Games
- Highway construction
- Cost of Sharing a Secret

Zero-Sum Two Player Games

Zero-sum Two-player Games. Odd-even game

- Players: Rose Todd vs. Colin Stevens
- Each player simultaneously displays 1 or 2 fingers. Stevens wins $1 if total is **even** (both players display same number of fingers). Todd wins $1 if total is **odd**.
- Payoffs to Colin shown in table.

  Rose chooses **Row** and
  Colin chooses **Column**.

Odd-Even Game			
		Colin Stevens	
G(x,y)		C1	C2
Rose Todd	R1	+1	-1
	R2	-1	+1

Odd-Even game

Theory of Moves (Steven J. Brams)

- Suppose Colin and Rose both are thinking of choosing 1.
- Then Rose should switch to 2.
- If she does, then Colin should switch to 2.
- So Rose should switch to 1.
- So Colin should switch to 1.
- We are back we started.

  There seems to be no stable solution!

Odd-Even Game

		Colin Stevens	
G(x,y)		C1	C2
Rose Todd	R1	+$1	-$1
	R2	-$1	+$1

Nash Equilibrium

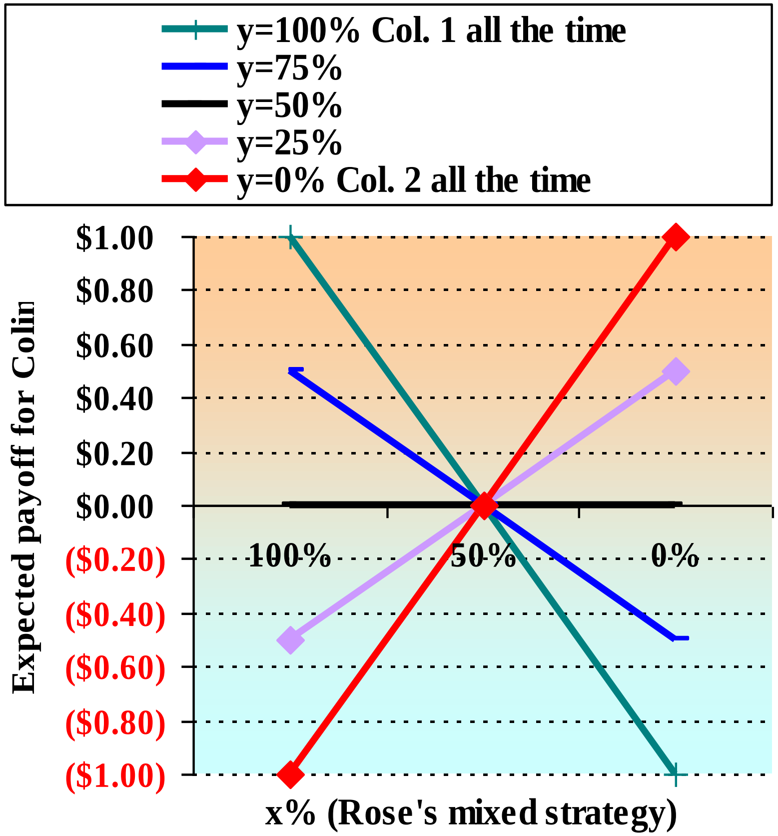
- John F. Nash, Nobel Prize 1994 for "pioneering analysis of equilibria in the theory of non-cooperative games."
- Nash's work extended earlier idea of John Von Neumann and Oskar Morgenstern.
- A (Nash) Equilibrium is a set of strategies:

  One strategy for each player such that no player has an incentive to unilaterally change his strategy.
- In Odd-Even, Rose & Colin each have 2 possible "pure" strategies. None of the four possible combinations is a Nash equilibrium.

Mixed Strategies

Odd-Even Game			
		Colin Stevens	
G(x,y)		C1	C2
Rose Todd	R1	+$1	-$1
	R2	-$1	+$1

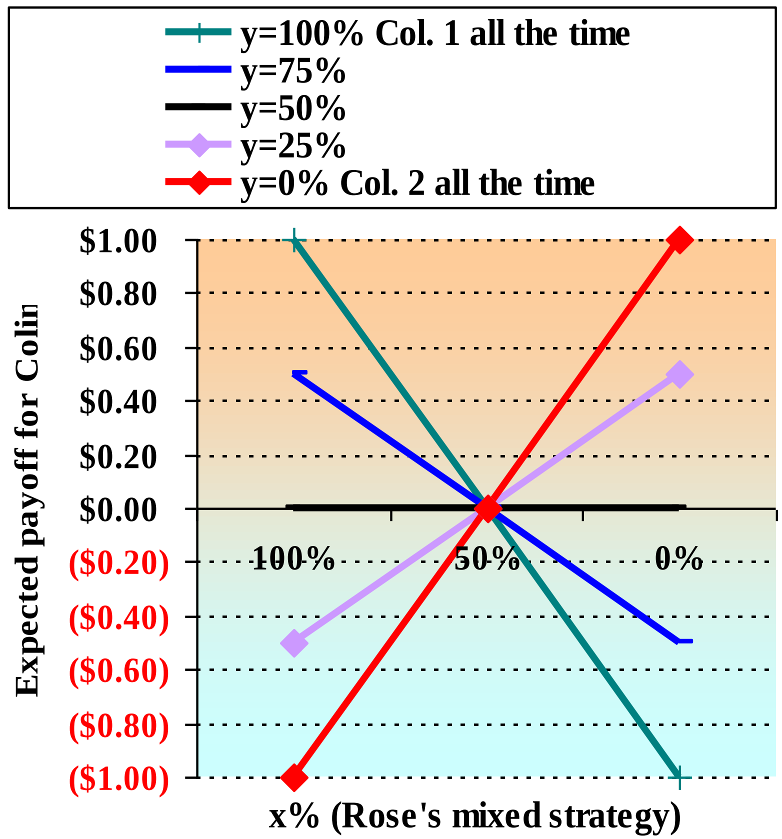
- In order to avoid being outguessed, choose a random combination of strategies.
- Rose selects R1, x% of the time and R2, (100-x)% of the time.
- Colin chooses C1, y% of the time.



Mixed Strategies

Odd-Even Game			
		Colin Stevens	
G(x,y)		C1	C2
Rose Todd	R1	+$1	-$1
	R2	-$1	+$1

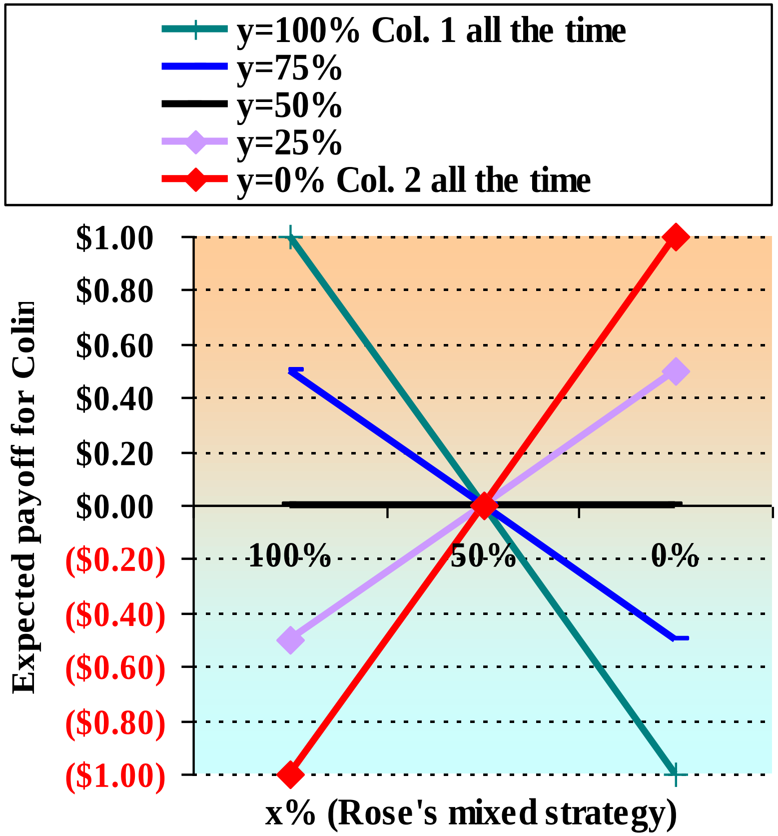
- y<50%, If Colin chooses C1 less than half of the time, Rose may pick R1 (x=100%) and Colin loses on average.
- y>50%, similarly, Rose picks R2 (x=100%) and Colin loses on average.
- Setting y=50%, guarantees that Colin breaks even.
- Similarly x=50% guarantees that Rose breaks even.



Mixed Strategies

Odd-Even Game			
		Colin Stevens	
G(x,y)		C1	C2
Rose Todd	R1	+$1	-$1
	R2	-$1	+$1

- x=50%, y=50% is a Nash equilibrium. Both players break even.
- If either player unilaterally changes his strategy, he is not better off. (In fact, as is usually the case, he is indifferent.)
- The symmetry of the solution reflects the symmetry of the game.
- Let's change the rules so solution is not so obvious.



Cops & Robbers Variant

G(x,y)		Cop Colin	
		C1	C2
Robber Rose	R1	+$1	-$1
	R2	-$1	**$0**

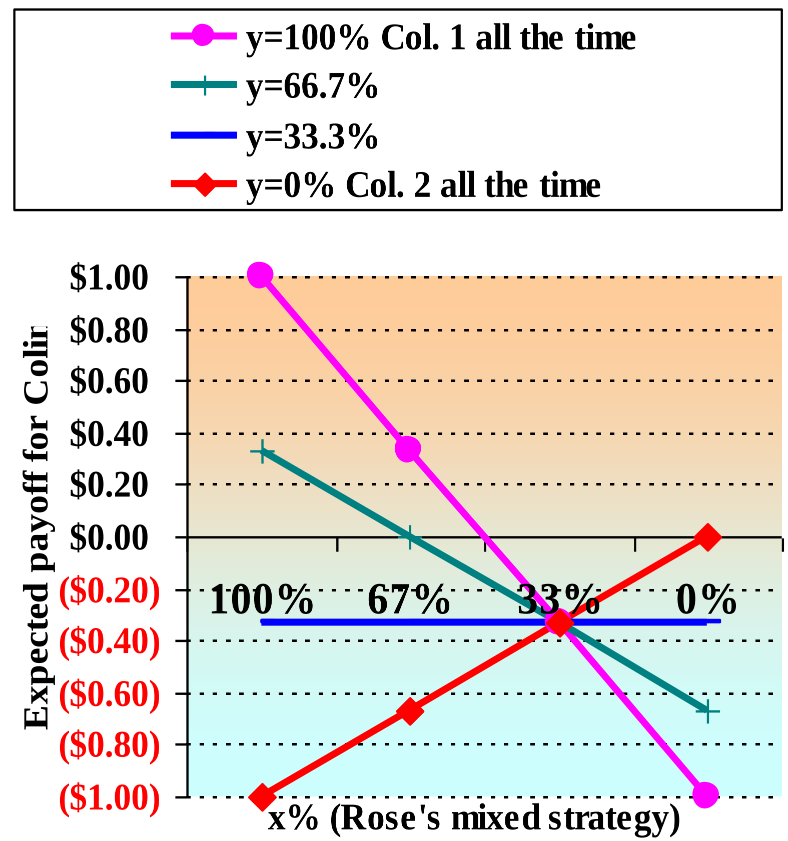
- Rose is a sociopathic Robber who can try to steal or not (1 or 2). Colin is a Cop who can try to catch Rose or not (1 or 2).
- If neither does anything, the payoff is zero. If Rose steals successfully, she gets $1, if Colin catches Rose, he gets $1.
- Rose has it in for Colin, so if Colin tries to catch Rose when she is not stealing, Rose gets $1
- The game is the same as the Odd/Even except the R2/C2 outcome of +$1 is changed to 0 in Rose's favor.

- The game favors Rose since she can guarantee at least $0 by not stealing. (Row 2).
- If Colin catches on to this he will just not try to catch Rose (Col 2) and break even.
- So Rose can try to steal occasionally, but 1/2 the time is too much, since then Colin will just try catching her all of the time (Col 1) and break even.

Cops & Robbers Variant

		Cop Colin	
G(x,y)		C1	C2
Robber Rose	R1	+$1	-$1
	R2	-$1	**$0**

- If Colin tries to catch Rose (C1) with y>1/3, then Rose's best play is to not steal (R2) and Colin's expectation will be -y < -1/3.
- Similarly if Colin tries to catch Rose with y<1/3, Rose can just steal and Colin's expectation will be y-(1-y) = 2y-1 < -1/3.
- Conversely, Colin can try to catch Rose exactly 1/3 of the time and guarantee an expectation of -1/3.
- The optimal solution is Rose Steals 1/3 of the time and Colin tries to catch her 1/3 of the time.



Mike Caro's AKQ Game

- Game invented by Mike Caro [*Card Player*, 1995]
- I'm corrupting the legend a bit but… Two cave men wanted to play poker but cards hadn't been invented yet. So then one of them invents the Ace. They really couldn't figure out much of a game with it, so then the other one created the King. They could now shuffle and deal but the game was uninteresting since there was no point in betting. They didn't know what to do until one day one of them got married and his wife came up with the idea of the Queen. She has regretted it ever since.

Mike Caro's AKQ Game

- Each player, Raisin' Rose and Callin' Colin, antes $1.
- Both players are dealt one hole card from a 3 card-deck which includes an Ace, King, and Queen. Each player may look at his card but not his opponent's.
- Raisin' Rose may bet $1. If Rose checks, there is a showdown and the ordering of cards is Ace > King > Queen, and the winner gets the pot and nets $1.
- If Rose bets, Colin has a chance to Call or Fold. If Colin folds, Rose wins $1, if Colin calls there is a showdown with $2 going to the winner.

Actions	Rose	Colin
Ace	Always Bets, will always win if called.	Always Calls, will always win
King	Never Bets, will never win if called.	May Call (C1) or Fold (C2).
Queen	May Bet (R1) or Check (R2).	Never Calls, will never win.

Mike Caro's AKQ Game

- Rose and Colin both have at most two viable strategies.
- Let R1 be Rose's strategy of Betting (stealing) with a Queen.
- Let R2 be Rose's strategy of Checking with a Queen.
- For both R1 and R2, Rose will Bet with an Ace and Check with a King.
- Let C1 be Colin's strategy of Calling a bet with a King (being the "Table Cop").
- Let C2 be Colin's the strategy of Folding a bet with a King.
- For both C1 and C2, Colin will always Call with an Ace and Fold a Queen.

Actions	Rose	Colin
Ace	Always Bets, will always win if called.	Always Calls, will always win
King	Never Bets, will never win if called.	May Call (C1) or Fold (C2).
Queen	May Bet (R1) or Check (R2).	Never Calls, will never win.

Callin' Colin's Expected Value

R's Strategy	A	A	K	K	Q	Q	**C's Cards**	**Total**	**C's EV**
R1 Bet Q	1	2	-2	2	-1	-1	C1 Call K	1	1/6
R2 Check Q	1	1	-2	1	-1	-1	C1 Call K	-1	- 1/6
R1 Bet Q	1	2	-1	-1	-1	-1	C2 Fold K	-1	- 1/6
R2 Check Q	1	1	-1	1	-1	-1	C2 Fold K	0	0
R's Cards	K	Q	A	Q	A	K	**C's Strategy**		

Mike Caro's AKQ Game

AKQ Matrix Form		
G(x,y)	C1	C2
R1	1/6	- 1/6
R2	- 1/6	0

		Cop Colin	
G(x,y)		C1	C2
Robber	R1	+$1	-$1
Rose	R2	-$1	**$0**

- Here, we have reduced the game to matrix form, which is rather interesting since we took a sequential play poker game and mapped it into 2x2 matrix game.
- Notice that we have already solved the game as the matrix is the same as for Cops and Robbers scaled by 1/6. Hence Rose should bluff 1/3 of the time with a Q and Colin should call 1/3 of the time with a K.

Callin' Colin's Expected Value

R's Strategy	A	A	K	K	Q	Q	**C's Cards**	**Total**	**C's EV**
R1 Bet Q	1	2	-2	2	-1	-1	C1 Call K	1	1/6
R2 Check Q	1	1	-2	1	-1	-1	C1 Call K	-1	- 1/6
R1 Bet Q	1	2	-1	-1	-1	-1	C2 Fold K	-1	- 1/6
R2 Check Q	1	1	-1	1	-1	-1	C2 Fold K	0	0
R's Cards	K	Q	A	Q	A	K	**C's Strategy**		

Optimal vs. Exploitive Play

So why play optimally in poker? Isn’t poker all about reading your opponents, and then adjusting?

• Optimal play is akin to using standard market metrics. You can evaluate an option price based on the current price, volatility, interest rate, etc, without any prediction on the performance of the underlying equity.

• It helps to know what optimal play is even though you are going to deviate from it.

• Sometimes you are against someone who is a top player because of his reading skills, where you don’t want to get into that type of a contest with him.

These are situations where I would deviate from optimal play:

• If I see some of my opponent’s hole cards. Unfortunately this seems to go on at both the poker table and the financial world.

• If I have a tell on my opponent, an indication by his mannerisms that he has a particular type of hand. Jay Sipelstein pointed out often you can get tells on other traders.

•If I have information on my opponent’s past behavior. For example in the last million hands we have played, he has never tried bluffing a 4th time after having been caught 3 times in a row.