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# SYMPLECTIC CORRECTORS FOR CANONICAL HELIOCENTRIC *n*-BODY MAPS

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## ABSTRACT

Symplectic correctors are developed for *n*-body maps (symplectic integrators) in canonical heliocentric coordinates. Several correctors are presented explicitly.

*Key words:* celestial mechanics — gravitation — methods: analytical — methods: numerical — solar system: general

## 1. INTRODUCTION

Symplectic correctors, introduced by Wisdom et al. (1996), can dramatically reduce the error of *n*-body integrations that use the *n*-body mapping method of Wisdom & Holman (1991). In that method the Hamiltonian for the *n*-body problem is written in terms of Jacobi coordinates and split into two parts: the Keplerian part, which describes the interaction of each planet with the central mass, and the interaction part, which describes the gravitational interaction among the planets. In the mapping method, the evolution of the full Hamiltonian is approximated by interleaving the evolution under the interaction and Keplerian Hamiltonians. Mappings can also be developed for the *n*-body problem in canonical heliocentric coordinates (Wisdom 1992; Touma & Wisdom 1994). Indeed, the planetary orbit part of our numerical integrations that showed that the obliquity of Mars evolves chaotically (Touma & Wisdom 1993) was carried out in canonical heliocentric coordinates. Duncan et al. (1998) and Chambers (1999) argue that canonical heliocentric coordinates have advantages over Jacobi coordinates when dealing with close encounters, and use them in their integration algorithms, which are widely used. In canonical heliocentric coordinates the Hamiltonian for the *n*-body problem is split into three parts: a Keplerian part, an interaction part, and an indirect part. Unfortunately, with a general splitting into three parts, the symplectic correctors derived in Wisdom et al. (1996) are not applicable. But with special splittings, the original symplectic correctors can be used for maps developed in canonical heliocentric coordinates.

I first review the basic idea of the symplectic correctors. I then develop the *n*-body Hamiltonian and the symplectic correctors in canonical heliocentric coordinates. Correctors of several different orders are presented explicitly. The use of the correctors is illustrated in 100 Myr integrations of the outer planets.

## 2. BACKGROUND ON SYMPLECTIC CORRECTORS

The idea of the symplectic correctors is best understood, and was originally understood, in terms of the $\delta$-function formulation of the symplectic mapping method, also known as symplectic integration. Assume that the Hamiltonian for a problem can be split into two solvable (or efficiently computable) Hamiltonians:

$$H = H_A + H_B. \tag{1}$$

For the corrector to work, $H_A \gg H_B$, but to make a map this is not necessary.

To make a mapping for this problem, high-frequency terms are added to the Hamiltonian so that $H_B$ is effectively multiplied by a periodic sequence of Dirac $\delta$-functions:

$$H_{\rm Map} = H_A + 2\pi\delta_{2\pi}(\Omega t - \pi)H_B, \tag{2}$$

where $\delta_{2\pi}$ is a periodic sequence of $\delta$-functions spaced by $2\pi$ in its argument. With argument $\Omega t - \pi$, the period of the map (the integration step) is $2\pi/\Omega$, and the $\delta$-function kick occurs midway through the integration step. This map is second order in the step.

The rationale that leads to this integrator is that the high-frequency terms that are added to the Hamiltonian to turn it into the mapping Hamiltonian are unimportant for the long-term evolution because their effects tend to average out. However, the high-frequency terms generate short-term oscillatory effects in the evolution. In Wisdom et al. (1996), it was shown how these short-term periodic effects could be removed by canonical perturbation theory. A similar trick was used earlier in our papers (Tittemore & Wisdom 1987a, 1987b, 1990) on the tidal evolution of the Uranian satellites. The result is a canonical transformation from "mapping coordinates" to "real coordinates" and vice versa. The resulting symplectic correctors dramatically reduce the error in integrations carried out with these mappings (symplectic integrators).

Wisdom et al. (1996) also showed how to implement the correctors in terms of Lie series. So the correctors can be computed by interleaving the same components as are used to carry out the integrations. Explicit formulae for a number of correctors of various orders are presented in the Appendix.

## 3. *n*-PLANET HAMILTONIANS

In the *n*-body problem, the corrector idea applies to problems with a dominant central mass. The Hamiltonian for the *n*-planet problem is

$$H = \sum_{i=0}^{n} \frac{p_i^2}{2m_i} - \sum_{0 \le i < j}^{n} \frac{Gm_i m_j}{r_{ij}}, \tag{3}$$

where $i = 0$ for the central mass, $m_i$ are the masses of the bodies, $p_i = m_i v_i$ for velocity $v_i$, $G$ is the gravitational constant, and $r_{ij}$ is the distance between bodies $i$ and $j$.

An elegant description of the Jacobi coordinates (including the hierarchical Jacobi coordinates) is given in Sussman & Wisdom (2001). I do not repeat that here. In the familiar Jacobi coordinates, each Jacobi coordinate $x_i'$ for $0 < i \le n$ refers to the center of mass of bodies with smaller indices, and one of the new coordinates is the center of mass of the whole system. (In the hierarchical Jacobi coordinates, the coordinate tree can be more complicated.) Let $p_i'$ be the conjugate momenta.

An important property of the Jacobi coordinates is that the kinetic energy remains diagonal in the momenta:

$$\sum_{i=0}^{n} \frac{p_i^2}{2m_i} = \sum_{i=1}^{n} \frac{(p_i')^2}{2m_i'} + \frac{P^2}{2M}, \tag{4}$$

where $m_i'$ are the Jacobi masses, $P$ is the total momentum of the system, and $M$ is the total mass. The potential energy does not depend on the center of mass, so the center-of-mass degree of freedom is ignorable. The Hamiltonian for the $n$-body problem can be written in the form

$$H = H_{\rm K} + H_I, \tag{5}$$

where $H_{\rm K}$ is the sum of $n$ Keplerian Hamiltonians,

$$H_{\rm K}^i = \frac{(p_i')^2}{2m_i'} - \frac{\mu_i'}{r_i'}, \tag{6}$$

the factor $\mu_i'$ depends on the particular splitting chosen, and

$$H_I = \left(\frac{\mu_i'}{r_i'} - \frac{Gm_0 m_i}{r_i}\right) - \sum_{0<i<j}^{n} \frac{Gm_i m_j}{r_{ij}}. \tag{7}$$

The first term is an "indirect" term, which depends on the coordinates, and the sum is the gravitational potential of the planets with one another. Symplectic maps can be made from these components by interleaving the evolution governed by these two Hamiltonians (Wisdom & Holman 1991). We used these symplectic maps to verify that the motion of Pluto is chaotic (Wisdom & Holman 1991), and in our 100 million year integrations of the whole solar system that confirmed that the solar system evolves chaotically (Sussman & Wisdom 1992).

Canonical heliocentric coordinates are canonical extensions of the collection of heliocentric coordinates for the planets, plus the center of mass of the system. Thus,

$$x_i'' = x_i - x_0 \tag{8}$$

for $1 \le i \le n$, plus the center of mass $X$. The conjugate momenta $p_i''$ and the total momentum $P$ satisfy

$$p_i = p_i'' + \frac{m_i}{M} P, \tag{9}$$

with

$$p_0 = -\sum_{i=1}^{n} p_i'' + \frac{m_0}{M} P, \tag{10}$$

where $M$ is the total mass. In canonical heliocentric coordinates the kinetic energy is *not* diagonal in the momenta. Instead,

$$\sum_{i=0}^{n} \frac{p_i^2}{2m_i} = \sum_{i=1}^{n} \frac{(p_i'')^2}{2m_i} + \frac{P^2}{2M} + \frac{1}{2m_0}\left(\sum_{i=1}^{n} p_i''\right)^2. \tag{11}$$

This can also be written

$$\sum_{i=0}^{n} \frac{p_i^2}{2m_i} = \sum_{i=1}^{n} \frac{(p_i'')^2}{2\mu_i} + \frac{P^2}{2M} + \frac{1}{m_0}\left(\sum_{1\le i<j}^{n} p_i'' p_j''\right), \tag{12}$$

with the reduced masses $1/\mu_i = 1/m_i + 1/m_0$.

In canonical heliocentric coordinates the $n$-planet Hamiltonian can be written in the form

$$H = H_{\rm K} + H_C + H_I, \tag{13}$$

a Keplerian Hamiltonian, a quadratic "cross-term" in the momenta, and an interaction Hamiltonian. The center-of-mass component is ignored. For the kinetic energy in equation (12), the Keplerian Hamiltonian is a sum of terms for each planet:

$$H_{\rm K}^i = \frac{(p_i'')^2}{2\mu_i} - \frac{Gm_0 m_i}{r_i''}, \tag{14}$$

with momentum cross-terms

$$H_C = \frac{1}{m_0}\left(\sum_{1\le i<j}^{n} p_i'' p_j''\right), \tag{15}$$

and interaction terms

$$H_I = -\sum_{0<i<j}^{n} \frac{Gm_i m_j}{r_{ij}}. \tag{16}$$

This was the splitting used by Touma & Wisdom (1993) in our discovery of the chaotic evolution of the obliquity of Mars. An advantage of this splitting is that Kepler's period law is satisfied for the individual planets: $n^2 a^3 = G(m_0 + m_i)$ for mean motion $n$ and semimajor axis $a$.

For the kinetic energy in equation (11), the Keplerian Hamiltonian is

$$H_{\rm K}^i = \frac{(p_i'')^2}{2m_i} - \frac{Gm_0 m_i}{r_i}, \tag{17}$$

with momentum cross-terms

$$H_C = \frac{1}{2m_0}\left(\sum_{i=1}^{n} p_i''\right)^2. \tag{18}$$

The interaction Hamiltonian is the same as before. A disadvantage of this splitting is that Kepler's period law is not exactly satisfied, but it has other advantages. This is the splitting used in the algorithms of Duncan et al. (1998) and Chambers (1999).

A second-order map can be made using either splitting. Let $\mathcal{K}(\Delta t)$ be the evolution under the Keplerian Hamiltonian for a time $\Delta t$. Let $\mathcal{C}(\Delta t)$ be the evolution under the cross-momentum Hamiltonian for a time $\Delta t$. And let $\mathcal{I}(\Delta t)$ be the evolution under the interaction Hamiltonian for a time $\Delta t$. Then one evolution step of a second-order symplectic mapping for this problem is

$$\mathcal{E}(\Delta t) = \mathcal{K}(\Delta t/2)\circ\mathcal{C}(\Delta t/2)\circ\mathcal{I}(\Delta t)\circ\mathcal{C}(\Delta t/2)\circ\mathcal{K}(\Delta t/2). \tag{19}$$

But with a map generated in this form, the existing correctors do not apply.

Note that the Poisson bracket of $H_C$, equation (15), and $H_I$, equation (16), is nonzero but that the Poisson bracket of $H_C$,

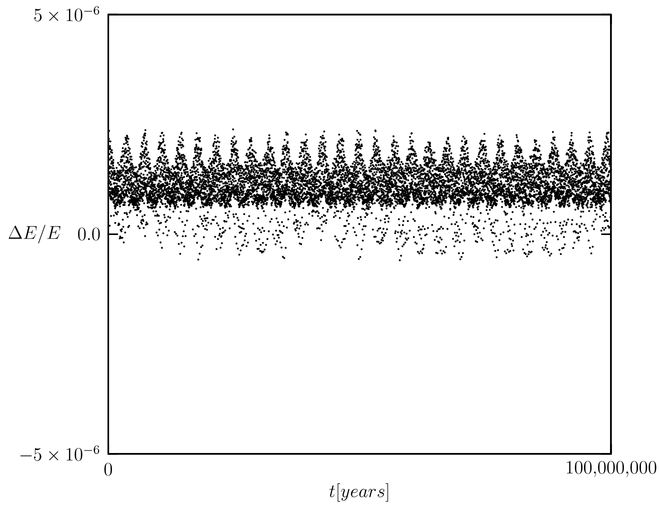


FIG. 1.—Relative energy error of the outer planets using the map, eq. (23), in canonical heliocentric coordinates, without a corrector.

equation (18), and $H_I$, equation (16), is zero. This has important consequences for the applicability of the existing correctors. For the latter splitting, the evolution under $H_C$ commutes with the evolution under $H_I$. This allows the unambiguous definition of the evolution under both:

$$\mathcal{B}(\Delta t) = \mathcal{I}(\Delta t)\circ\mathcal{C}(\Delta t) = \mathcal{C}(\Delta t)\circ\mathcal{I}(\Delta t). \tag{20}$$

The evolution operator for a second-order mapping becomes

$$\mathcal{E}(\Delta t) = \mathcal{K}(\Delta t/2)\circ\mathcal{B}(\Delta t)\circ\mathcal{K}(\Delta t/2), \tag{21}$$

which is the form assumed in Wisdom et al. (1996), so the original correctors apply to maps in canonical heliocentric coordinates with this particular splitting!

## 4. ILLUSTRATION

As an illustration of the various correctors in canonical heliocentric coordinates, the outer planets were integrated for 100 Myr.

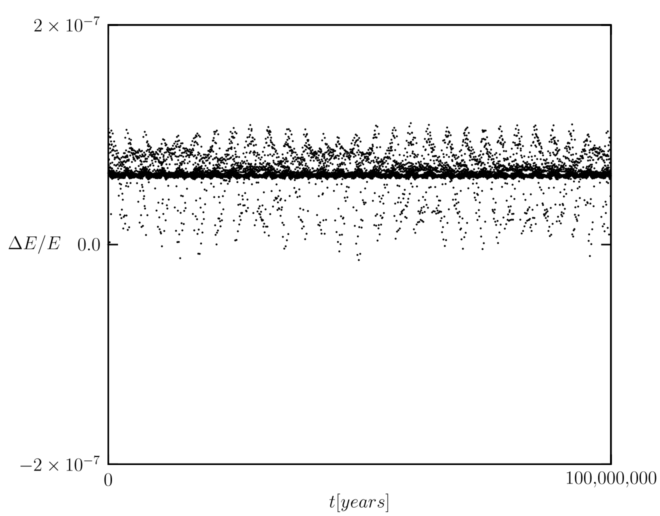


FIG. 2.—Relative energy error of the outer planets using the map, eq. (23), in canonical heliocentric coordinates, with a third-order (two-stage) corrector. The offset from zero is just due to the initial phase of the evolution.

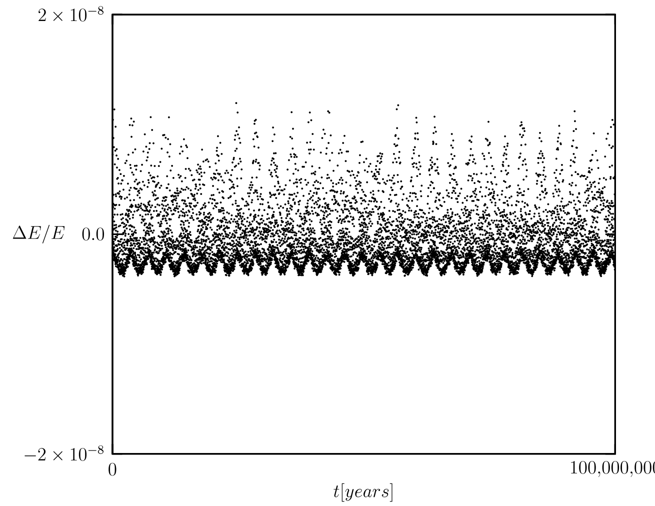


FIG. 3.—Relative energy error of the outer planets using the map, eq. (23), in canonical heliocentric coordinates, with a seventh-order (six-stage) corrector. The corrected error is almost as good, for this problem, as with the 17th-order corrector.

The second splitting, with cross-term equation (18), was used. The relative energy error for the uncorrected integration is shown in Figure 1. It is a few times $10^{-6}$.

Figure 2 shows the relative energy error in the same integration after application of the third-order (two-stage) corrector listed in the Appendix. The error is now of order $10^{-7}$. The Chambers corrector solution works better than this corrector by a factor of about 2–3.

Figure 3 shows the relative energy error in the same integration after application of the seventh-order (six-stage) corrector listed in the Appendix. The error is now of order $10^{-8}$. Application of the 17th-order corrector gives just slightly better results.

Figure 4 shows the relative energy error in a 100 Myr integration using the original Wisdom-Holman mapping in the usual Jacobi coordinates after application of the 17th-order corrector. The error is now of order $10^{-9}$. Evidently, for high-accuracy long-term integrations without close encounters, Jacobi coordinates are preferred. Without a corrector for canonical heliocentric coordinates, a judgment on the relative accuracy of

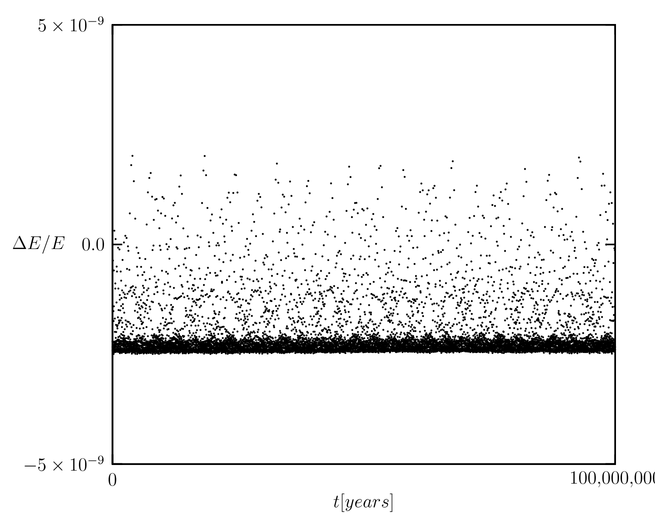


FIG. 4.—Relative energy error of the outer planets using the Wisdom-Holman map in the usual Jacobi coordinates with a 17th-order corrector.

integrations performed in each coordinate system could not be made.

## 5. SUMMARY

Symplectic correctors that were developed for $n$-body maps in Jacobi coordinates (Wisdom et al. 1996; Wisdom & Holman 1991) can also be used for $n$-body maps in canonical heliocentric coordinates, for a particular splitting. In integrations of the outer planets, the seventh-order corrector works better than the third-order (two-stage) corrector, and not much worse than the 17th-order corrector. Integrations of the outer planets performed in Jacobi coordinates using the original Wisdom-Holman map are better corrected, by nearly an order of magnitude. So for high-accuracy integrations with no close encounters, Jacobi coordinates are preferred. The correctors can also be used with schemes developed to handle close encounters using canonical heliocentric coordinates.

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## APPENDIX

## CORRECTOR FORMULAE AND CONSTANTS

Assume a Hamiltonian of the form

$$H = H_A + H_B, \tag{22}$$

where $H_A \gg H_B$. Let $\mathcal{A}(\Delta t)$ be the evolution under $H_A$ for a time of $\Delta t$. Let $\mathcal{B}(\Delta t)$ be the evolution under $H_B$ for a time of $\Delta t$. A second-order map for this Hamiltonian is

$$\mathcal{E}(\Delta t) = \mathcal{A}(\Delta t/2) \circ \mathcal{B}(\Delta t) \circ \mathcal{A}(\Delta t/2). \tag{23}$$

The correctors are defined in terms of some auxiliary quantities. Let

$$\mathcal{X}(a\Delta t,\ b\Delta t) = \mathcal{A}(a\Delta t) \circ \mathcal{B}(b\Delta t) \circ \mathcal{A}(-a\Delta t). \tag{24}$$

Then let

$$\mathcal{Z}(a\Delta t,\ b\Delta t) = \mathcal{X}(a\Delta t,\ b\Delta t) \circ \mathcal{X}(-a\Delta t,\ -b\Delta t). \tag{25}$$

The $n$-stage corrector is

$$\mathcal{C}(\Delta t) = \mathcal{Z}(a_1\Delta t,\ b_1\Delta t) \circ \mathcal{Z}(a_2\Delta t,\ b_2\Delta t) \circ \ldots \circ \mathcal{Z}(a_n\Delta t,\ b_n\Delta t). \tag{26}$$

The inverse corrector is

$$\mathcal{C}^{-1}(\Delta t) = \mathcal{Z}(a_n\Delta t,\ -b_n\Delta t) \circ \ldots \circ \mathcal{Z}(a_2\Delta t,\ -b_2\Delta t) \circ \mathcal{Z}(a_1\Delta t,\ -b_1\Delta t). \tag{27}$$

In terms of these, the corrected evolution is

$$\mathcal{E}'(n\Delta t) = \mathcal{C}^{-1}(\Delta t) \circ \mathcal{E}(\Delta t) \circ \ldots \circ \mathcal{E}(\Delta t) \circ \mathcal{C}(\Delta t). \tag{28}$$

The corrector coefficients presented in Wisdom et al. (1996) are (corrected)

$$\begin{aligned}
a_8 &= 1\alpha, \quad b_8 = \frac{45815578591785473}{24519298961757600}\beta \approx 1.8685517340134143\beta, \\
a_7 &= 2\alpha, \quad b_7 = \frac{-104807478104929387}{80063017017984000}\beta \approx -1.3090623112714728\beta, \\
a_6 &= 3\alpha, \quad b_6 = \frac{422297952838709}{648658702692000}\beta \approx 0.6510325862986641\beta, \\
a_5 &= 4\alpha, \quad b_5 = \frac{-27170077124018711}{112088223825177600}\beta \approx -.24239903351841396\beta, \\
a_4 &= 5\alpha, \quad b_4 = \frac{102433989269}{1539673404192}\beta \approx 0.06652968674402478\beta, \\
a_3 &= 6\alpha, \quad b_3 = \frac{-33737961615779}{2641809989145600}\beta \approx -0.012770775246667285\beta, \\
a_2 &= 7\alpha, \quad b_2 = \frac{26880679644439}{17513784972684000}\beta \approx 0.0015348298318361457\beta, \\
a_1 &= 8\alpha, \quad b_1 = \frac{682938344463443}{7846175667762432000}\beta \approx -0.00008704091947232721\beta,
\end{aligned} \tag{29}$$

with $a_i = -a_{17-i}$ and $b_i = -b_{17-i}$ for $8 < i \leq 16$, and $\alpha = (7/40)^{1/2}$ and $\beta = 1/(48\alpha)$. This corrector has an error term of 17th order.

A third-order (two-stage) corrector is given by the coefficients

$$\begin{aligned}
a_2 &= 1\alpha, \quad b_2 = \frac{1}{2}\beta, \\
a_1 &= -1\alpha, \quad b_1 = \frac{-1}{2}\beta.
\end{aligned} \tag{30}$$

A fifth-order (four-stage) corrector is given by the coefficients

$$\begin{aligned}
a_2 &= 1\alpha, \quad b_2 = \frac{5}{6}\beta, \\
a_1 &= 2\alpha, \quad b_1 = \frac{-1}{6}\beta,
\end{aligned} \tag{31}$$

with $a_i = -a_{5-i}$ and $b_i = -b_{5-i}$ for $2 < i \leq 4$.

A seventh-order (six-stage) corrector is given by the coefficients

$$\begin{aligned}
a_3 &= 1\alpha, \quad b_3 = \frac{53521}{49392}\beta, \\
a_2 &= 2\alpha, \quad b_2 = \frac{-22651}{61740}\beta, \\
a_1 &= 3\alpha, \quad b_1 = \frac{12361}{246960}\beta,
\end{aligned} \tag{32}$$

with $a_i = -a_{7-i}$ and $b_i = -b_{7-i}$ for $3 < i \leq 6$.

An 11th-order (10 stage) corrector is given by the coefficients

$$
\begin{aligned}
a_5 &= 1\alpha, \quad b_5 = \frac{3394141}{2328480}\beta,\\
a_4 &= 2\alpha, \quad b_4 = \frac{-14556229}{19015920}\beta,\\
a_3 &= 3\alpha, \quad b_3 = \frac{895249}{3622080}\beta,\\
a_2 &= 4\alpha, \quad b_2 = \frac{-329447}{6985440}\beta,\\
a_1 &= 5\alpha, \quad b_1 = \frac{2798927}{684573120}\beta,
\end{aligned} \tag{33}
$$

with $a_i = -a_{11-i}$ and $b_i = -b_{11-i}$ for $5 < i \leq 10$.

Instead of the mapping step, equation (23), one can also make a second-order map with the step

$$\mathcal{E}(\Delta t) = \mathcal{B}(\Delta t/2) \circ \mathcal{A}(\Delta t) \circ \mathcal{B}(\Delta t/2). \tag{34}$$

J. Chambers (2005, private communication) has found a third-order corrector solution for this alternate map:

$$
\begin{aligned}
a_1 &= \frac{3}{10}\gamma, \quad b_1 = \frac{-1}{72}\gamma,\\
a_2 &= \frac{1}{5}\gamma, \quad b_2 = \frac{1}{24}\gamma,
\end{aligned} \tag{35}
$$

where $\gamma = 10^{1/2}$. This corrector is available in his MERCURY integrator package.

Note that if all one is interested in is monitoring the energy of an integration, then any of the above correctors can be used. If, for instance, the alternate map, equation (34), is used to perform the integration, then before applying any of the first set of correctors, one should apply $\mathcal{A}(\Delta t/2)\mathcal{B}(\Delta t/2)$ to bring the alternate map output up to the corresponding output of the original map, equation (23).

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