ICARUS 72, 241–275 (1987)

Urey Prize Lecture: Chaotic Dynamics in the Solar System[1]

JACK WISDOM

Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Received March 3, 1987; revised May 27, 1987

Newton's equations have chaotic solutions as well as regular solutions. There are several physical situations in the solar system where chaotic solutions of Newton's equations play an important role. There are examples of both chaotic rotation and chaotic orbital evolution. Hyperion is currently tumbling chaotically. Many of the other irregularly shaped satellites in the solar system have had chaotic rotations in the past. This episode of chaotic tumbling could have had a significant effect on the orbital histories of these satellites. Chaotic orbital evolution seems to be an essential ingredient in the explanation of the Kirkwood gaps in the distribution of asteroids. The phase space boundary of the chaotic zone at the 3/1 mean-motion commensurability with Jupiter is in excellent agreement with the boundary of the observed population of asteroids. Chaotic trajectories at the 3/1 commensurability have the correct properties to provide a dynamical route for the transport of meteoritic material from the asteroid belt to Earth. There is a large chaotic zone at the 2/1 commensurability, where there is a Kirkwood gap, but the phase space near the Hilda group of asteroids at the 3/2 commensurability is dominated by quasiperiodic behavior. Chaotic trajectories in the 2/1 chaotic zone reach very high eccentricities by a path that temporarily takes them to high inclinations. The long-term evolution of Pluto is suspiciously complicated, but objective criteria have not yet indicated that the motion is chaotic.

INTRODUCTION

The solar system is generally perceived as evolving with clockwork regularity. Indeed, it was a search for the principles which underlie the perceived regularities in the motions of the planets which culminated in Newton's formulation of the laws of mechanics and universal gravitation. Newton's mechanics was unquestionably beautiful, encapsulating all that was known about the motions of the planets, yet describing equally well the benign falling of apples and the not-so-benign trajectories of cannon balls. There was no doubt that Newton had snatched a glimpse of Truth. It was appropriate then that this triumph of predictability should be enshrined in the orreries of the 18th century. These finely crafted machines allowed the positions of the planets at any date to be found by turning a dial. In some cases, such as the orrery at the Palace of Versailles, the orreries were also fine clocks, continuously displaying the positions of the planets as well as the time. A chapel is at the center of the old part of the Palace of Versailles; the new wing built by Louis XV has at its center a room solely devoted to the orrery, a new chapel for the enlightenment.

From its very beginning mechanics has been associated, even identified, with predictability. The regularity of the solutions of Newton's equations was so firmly believed it was never questioned. Perhaps the

[1] Text of the Urey Lecture, November 1986, Paris, France.

0019-1035/87 $3.00

extreme difficulty in finding analytic solutions of the three-body problem should have been a clue to the fact that there is more to mechanics than meets the eye. Man can be quite determined in her folly; it is perhaps not surprising that 200 years elapsed before the true nature of the solutions of mechanics began to be appreciated. In his *Méthods nouvelle de la mécanique céleste* (1892) Poincaré showed that the solutions of the *n*-body problem could be formally written as series of purely periodic terms, laying to rest (or at least should have laid to rest) the old controversy about whether there are secular variations of the semimajor axes at any order of perturbation theory. However, he also showed that these series were generally asymptotic series; ultimately the series diverge due to the appearance of uncontrolled small divisors. Poincaré pioneered a new approach to mechanics. Rather than constructing solutions of an assumed form by successive approximation, he investigated the qualitative nature of the motion. He made two startling discoveries: The assumption of regularity of the motion implies the existence of a complete set of independent integrals of the motion. Poincaré showed that integrals of the motion generally do not persist under perturbation. Thus most Hamiltonian systems do not possess the integrals required to be reducible to quadratures. Poincaré also found that the motion near unstable periodic orbits possesses almost unimaginable complexity. Poincaré said, "On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problème des trois corps et en général de tous problemes de Dynamique où il n'y a pas d'intégrale uniforme. . . ." Poincaré's discoveries were not fully appreciated and in fact were misinterpreted by some. Fermi (1923) tried to prove, on the basis of Poincaré's proof of the nonexistence of analytic integrals, that Hamiltonian systems were generally ergodic, that most trajectories explore the full space to which they are constrained by the classical integrals such as energy and angular momentum. The solar system seems to stand out as a counterexample to his proof; indeed, it is fortunate for our solar system that his proof was erroneous.

The true nature of dynamical systems began to emerge in the 1960s (for more background see, for example, Arnold and Avez 1968, Chirikov 1979). On the analytical front, the theorem on the persistence of quasiperiodic motions which was outlined by Kolmogorov in 1954 was proven independently by Arnold and Moser in 1961. The KAM theorem, as it is now known, demonstrates that under perturbations which are sufficiently small and smooth, integrable motion remains in large measure quasiperiodic, i.e., remains regular. While the global integrals no longer exist by Poincaré's theorem, a large proportion of the solutions possess the same regularity as the solutions of the unperturbed problem. The set of regular motions is extremely complicated; the regular solutions disappear or change character whenever the ratios of the frequencies of the solution are sufficiently well approximated by rationals, and the rationals are dense. The KAM theorem is of no practical importance; the magnitude of the perturbations for which the conditions of the theorem are satisfied is extremely small. Hénon (1966b) gives an estimate of 10^{-48} for the relative magnitude of the perturbation for applicability of the KAM theorem. Nevertheless, the KAM theorem is a fundamental contribution to the understanding of dynamical systems: quasiperiodic motion can indeed persist under perturbation. Arnold (1961) has extended the proof to show that solar systems with planetary masses, eccentricities, and inclinations sufficiently small are in large measure quasiperiodic. However, the actual solar system falls outside the range of validity of the theorem.

The computer has been enormously important in developing the qualitative understanding of dynamical systems. There were two seminal numerical studies: Lorenz (1963) and Hénon and Heiles (1964). Both works show that complicated behavior can result from extremely simple dynamical systems. It is not necessary that a system be complicated for it to exhibit complicated behavior. The Lorenz system is now the archetype of dissipative dynamical systems which show irregular behavior; the Hénon–Heiles system is the archetype for conservative systems. The Lorenz system was derived as a highly truncated model for Earth's atmosphere; Hénon and Heiles were motivated by the observation that nearby stars in the Galaxy seemed to possess an extra integral beyond energy and angular momentum. Irregular behavior in both conservative and dissipative systems is termed "chaotic." Chaotic behavior has several distinguishing characteristics: First, it simply looks markedly more irregular than regular behavior. With some experience, it is often possible to determine "by eye" whether a trajectory is chaotic long before any objective criterion can confirm the diagnosis. Most important among the objective criteria, chaotic trajectories show an average exponential divergence of initially nearby trajectories. Another way of saying this is that chaotic trajectories show sensitive dependence on initial conditions. The rates of exponential divergence of nearby trajectories are quantified by the Lyapunov characteristic exponents. Finally, for conservative systems with two degrees of freedom chaotic behavior may be distinguished from quasiperiodic behavior by the appearance of the set of points generated by the trajectory on a surface of section. Basically, a surface of section is generated by plotting the successive intersections of a trajectory with some particular plane through the phase space. The successive points of a regular trajectory form smooth curves, while the points generated by a chaotic trajectory seem to fill an area in an irregular manner. It is a remarkable fact that these two types of motion are generally easily distinguishable.

The Hénon–Heiles system is more relevant here since the solar system is by and large conservative. Hénon and Heiles chose to study a simple dynamical system which was easily computed to gain a qualitative understanding of the nature of the solutions of dynamical systems. The Hamiltonian chosen was

$$H = \tfrac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{y^3}{3}.$$

This is simply the problem of two harmonic oscillators with nonlinear coupling. This problem has two degrees of freedom, the minimum required for complicated behavior. Hénon and Heiles found that the surfaces of sections for their model problem showed a complicated interweaving of regular and irregular trajectories, and that the proportion of irregular trajectories increased as the energy increased. Considerable experience with other systems has shown that the behavior found in the Hénon–Heiles problem was not unusual, and in fact almost all two-degree-of-freedom Hamiltonian systems behave, qualitatively at least, in the same way. The perturbation theories of the 19th century failed because the nature of the motion they were intended to approximate was not understood. Generally speaking, the types of motion are now known, though little has been proven about the chaotic component of phase space. The discovery that almost all Hamiltonian systems share a certain comradery has had a profound impact. A wide range of disciplines now have a unifying thread. Results from one field can have important consequences in another. Chaotic behavior of conservative systems is important in galactic dynamics, plasma physics, particle accelerators, and mechanical systems, to list only a few applications, and of course the solar system.

The solar system is the birthplace of mechanics; the preeminent dynamical system is not untouched by the discoveries in nonlinear dynamics. Solar system dynamics encompasses the orbital and rotational dynamics of the planets and their natural satellites, the coupling between them, and the slow evolution of the orbits and spins due to tidal friction. It is primarily the dynamics of resonances and resonances are almost always associated with chaotic zones. Chaotic behavior must be considdered a possibility in almost any dynamical situation in the solar system. In this paper a number of applications of modern dynamics to the solar system are reviewed. Applications to rotational dynamics are considered first, followed by applications to orbital dynamics.

TUMBLING OF HYPERION

The chaotic rotation of Saturn's satellite Hyperion still offers one of the most dramatic examples of chaotic behavior in the solar system (Wisdom *et al.*, 1984). Hyperion's spin rate and spin axis orientation are predicted to be undergoing significant variations in only a few orbit periods, where the orbit period is about 21 days. The chaotic tumbling of Hyperion is a consequence of several factors. Foremost among them is that Hyperion has a highly aspherical shape, which has been determined from Voyager 2 images to have principal radii of $190 \times 145 \times 114$ km $\pm$ 15 km (Smith *et al.* 1982). Hyperion is nearly twice as long as it is across. Hyperion's orbit is unusually eccentric, with an eccentricity of approximately 0.1. This eccentricity is primarily a forced eccentricity which is a consequence of the 4/3 mean-motion resonance between Titan and Hyperion. Finally, the timescale for the spin of Hyperion to be slowed by tidal friction to synchronous rotation is on the order of the age of the solar system. Thus, the magnitude of Hyperion's rate of rotation is near that which Hyperion would need to always point one face toward Saturn as Earth's Moon always points one face to Earth. Hyperion's rotation would not be chaotic if it were not tidally evolved. At the same time if the timescale for tidal despinning were much shorter than the age of the solar system Hyperion might have already found its way into a stable commensurate rotation state.

The chaotic rotation of Hyperion is best illustrated in a simplified model. In this model the orbit of Hyperion is taken to be a fixed ellipse, the spin axis is aligned with the axis of largest principal moment of inertia (generally the shortest physical axis of a symmetrical body), and the spin axis is taken to be perpendicular to the orbit plane. The assumption of a fixed orbit is acceptable because the timescale for significant variations in the spin rate is much shorter than the timescale for significant variations in Hyperion's orbit, which are in any case not large. The spin axis configuration is the natural outcome of tidal evolution. For a freely rotating body, the rotation vector moves through the body, unless the rotation vector is aligned with one of the principal axes. A physical body is distorted both due to the solid-body tides raised by the centrifugal potential, which changes as the rotation vector moves through the body, and due to the bending which results directly from the changing rotation vector (Burns and Safronov 1973). The continual bending and stretching of the body dissipates energy, but for a free body with no external torques angular momentum is conserved. The rotation evolves to the state which minimizes energy for a given angular momentum. This state is rotation about the axis with largest principal moment of inertia, as can be seen with the following geometrical construction. The magnitude of the angular momentum is given by the relation

$$L^2 = L_a^2 + L_b^2 + L_c^2,$$

where L_i are the components of the angular momentum on the principal axes. The energy is simply

$$E = \frac{L_a^2}{2A} + \frac{L_b^2}{2B} + \frac{L_c^2}{2C},$$

where A, B, and C are the principal moments of inertia about the principal axes a,

b, and c, respectively. The first is the equation for a sphere; the latter is the equation for a triaxial ellipsoid. The angular momentum is constrained to lie on the intersection of these two surfaces. As energy is dissipated the size of the ellipsoid decreases, and the angular momentum is constrained to move toward the longest axis of the ellipsoid, which corresponds to the largest principal moment of inertia. This process is nicely illustrated by tossing a partially filled bottle of Liquid Paper which is initially spinning about the longest axis. For a body in orbit the tides raised by the planet on the satellite cause further evolution of the rotation of the satellite. Darwin (1879) has shown that these tides may be modeled by a tidal bulge which is slightly delayed relative to the potential which raises them. This is directly analogous to the motion of a forced, damped harmonic oscillator where the response lags in time the periodic forcing if the forcing frequency is smaller than the natural frequency of the oscillator. For a rotating body the tidal bulge is then carried slightly along in the direction of rotation. The gravitational attraction on this now asymmetrically oriented tidal bulge leads to a torque which tends to slow the rotation of the body to synchronous rotation (if the orbit is circular), and at the same time drives the obliquity, the angle between the spin axis and the orbit normal, to zero (neglecting the variations in the orbit). The obliquity reaches zero, when the spin rate is near twice the mean orbital motion (see, e.g., Peale 1977 for a more detailed discussion). It is thus natural to study the simplified problem of the rotational dynamics of a satellite spinning about its largest principal moment of inertia, with that spin axis perpendicular to the orbit plane. This dynamical model is the same model used in the theory of spin–orbit coupling (see Goldreich and Peale 1966) developed after the discovery that Mercury's spin period is commensurate with its orbital period (Pettengill and Dyce 1965).

The differential gravitational force across an out-of-round body gives rise to a torque on that body. The equation of motion for the orientation of the body in the simplified problem just described is quite simple:

$$C \frac{d^2\theta}{dt^2} = -n^2(B - A)\frac{3}{2}\left(\frac{a}{r}\right)^3 \sin 2(\theta - f).$$

Figure 1 presents a sketch of the spin–orbit geometry. With the spin axis fixed perpendicular to the orbit plane, the orientation of the satellite is specified by a single angle, θ, which is taken to be the angle between the axis of smallest principal moment of inertia (the longest axis of a triaxial ellipsoid) and the inertially fixed line of periapse (the line joining the planet and the point in the orbit closest to the planet). The angular position of the satellite in its orbit is also measured from the periapse of the orbit. This angle is the true anomaly, denoted here by the symbol f. The principal moments of inertia are $A < B < C$, and thus C is the moment of inertia about the spin axis. The mean angular motion of the satellite in its orbit is n, r is the instantaneous distance from the planet to the satellite, and a is the semimajor axis of the orbit. The equation of motion equates the external torque to the rate of change of the angular momentum, or equiv-

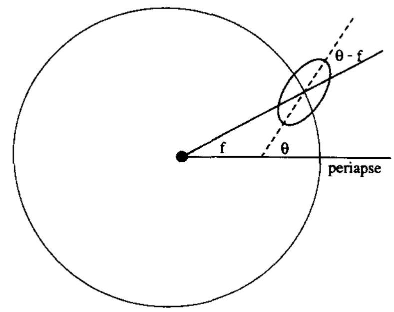

FIG. 1. Sketch of the geometry of the reduced spin–orbit coupling problem. The orientation of the satellite is specified by the angle θ, and the orbital longitude is specified by the true anomaly f. The angle $\theta - f$ measures the angle between the axis of least moment of inertia of the satellite and the planet-to-satellite line.

alently, to the product of the moment of inertia about the spin axis and the acceleration of the orientation. Since the gravitational force depends on the inverse square of the distance the attractive force on the permanent bulge on the side of the satellite near the planet is greater than the attractive force on the opposite bulge (treating the body as a triaxial ellipsoid). Consequently the torques arising from these forces do not balance. This torque is often called the "gravity gradient torque." The inverse cube dependence of the torque on the distance from the planet reflects the origin of the torque as a gravity gradient. Likewise, there would be no net torque if the body were axisymmetric about the spin axis; the asymmetry of the body enters the equation of motion through the difference of the principal moments of inertia in the plane of the orbit, $B - A$. Instantaneously, the torque always tends to try to align the long axis of the satellite with the line between the satellite and the planet. The angle $\theta - f$ is the angle between the long axis and the planet–satellite line. This equation keeps only the lowest moments of the mass distribution in the orientation-dependent part of the potential energy. The contributions which are ignored are of one higher order in the small ratio of the radius of the satellite to the orbital radius. In this approximation all bodies have a symmetry under which a rotation by 180° gives a dynamically equivalent configuration. The factor of two multiplying the difference of angles reflects this symmetry.

A problem is said to be "integrable" if it possesses an independent integral of the motion for every degree of freedom. A theorem of Liouville then guarantees that the problem may be reduced to quadratures. The solution will most likely not be expressible in terms of standard functions, but the essence of the solution has been found. A problem of n degrees of freedom with n independent integrals is guaranteed to have only regular quasiperiodic trajectories with at most n independent frequencies. All problems with one degree of freedom are integrable, though again the solutions may not be expressible in terms of previously known functions. For Hamiltonian systems with one degree of freedom, the solutions are simply contours of the Hamiltonian on the phase plane. Hamiltonian problems with more than one degree of freedom or problems with explicit time dependence are generally not reducible to quadratures. Generically, such systems show the mixed phase space: some initial conditions lead to chaotic behavior and others lead to quasiperiodic motion.

The equation of motion for spin–orbit coupling has only a single degree of freedom, the orientation angle θ, but depends explicitly on the time through the distance to the planet, r, and the nonuniform Keplerian motion of the true anomaly, f. The spin–orbit problem thus falls outside the class of problems which are guaranteed to be integrable. It is worth emphasizing that it is the nonzero eccentricity of the orbit which spoils the integrability of the problem. If the eccentricity is set to zero then the planet-to-satellite distance remains equal to the semimajor axis, and the true anomaly becomes simply the mean motion times the time. The equation of motion for the angle $\theta' = \theta - nt$ is then

$$C \frac{d^2\theta'}{dt^2} = - \frac{3n^2(B - A)}{2} \sin 2\theta'.$$

Except for the factor of two which could easily be removed by a further change of variables this is the equation of motion for a pendulum, which of course can be explicitly integrated in terms of elliptic functions. The exact solution, however, is not important; the important feature which changes the nature of the solutions is that the problem now has an integral

$$E = \frac{1}{2} C \left(\frac{d\theta'}{dt}\right)^2 - \frac{3n^2(B - A)}{4} \cos 2\theta'.$$

When the eccentricity is nonzero the explicit time dependence cannot be eliminated from the equations of motion. The spin–orbit problem is sufficiently complex that it may be expected to display the generic mixed phase space. The structure of the phase space is most easily understood by computing surfaces of section. For the simplified spin–orbit problem surfaces of section are generated by looking at the rotation state stroboscopically, i.e., once per orbit. The equation of motion is numerically integrated, and every time the satellite goes through periapse the rate of change of the orientation, $d\theta/dt$, is plotted versus the orientation, θ. The surface of section for Hyperion ($\sqrt{3(B - A)/C} \approx 0.89$ and orbital eccentricity $e \approx 0.1$) is shown in Fig. 2. A number of trajectories have been used in constructing this section, to illustrate the principal types of motion which are possible. If the points seem to generate a one-dimensional curve the motion is quasiperiodic; if the points seem to fill an area the motion is chaotic. In practice, these two types of motion are readily distinguishable, though sometimes upon magnification a trajectory which was originally thought to be quasiperiodic is found to be in a very narrow chaotic zone. This is not surprising since the regions of regular behavior actually have an incredibly rich structure; with sufficient magnification a chaotic zone can be found arbitrarily close to every point in the phase space. Practically speaking, this rich structure is generally irrelevant; the grosser division between regular regions and chaotic zones plays a much more important role in physical problems. Three of the trajectories used to generate Fig. 2 were chaotic. All of the scattered points in the center of the section belong to the same trajectory. This large chaotic zone extends from no rotation at all in the inertial frame to a rotation rate of about 2.5 times the mean angular orbital motion. The two trajectories that generate an × in the upper center part of the section are also chaotic. The other trajectories appear, without close examination, to be quasiperiodic, and certainly identify the main

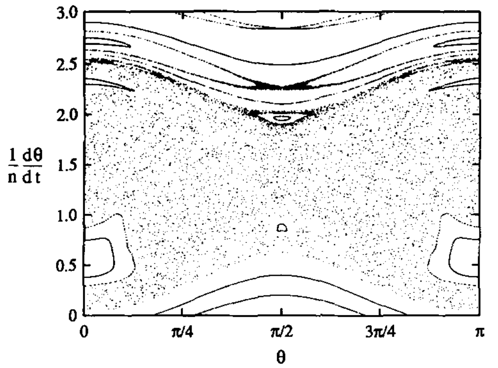

FIG. 2. Surface of section for Hyperion (using $\alpha = 0.89$ and $e = 0.1$). The rate of change of the orientation is plotted versus the orientation every pericenter passage. The spin axis is constrained to be perpendicular to the orbit plane.

regions of quasiperiodic motion. The islands in the chaotic sea correspond to various commensurate spin–orbit states. The island in the lower part of the section near $\theta = 0$ is the synchronous island, where Hyperion would on the average always point one face toward Saturn (i.e., never make a complete relative rotation). The island in the upper part of the large chaotic zone is the 2 state, where Hyperion would on the average rotate twice every orbit period. A number of other islands are shown. The curves in the bottom of the section near $\theta = \pi/2$ represent a noncommensurate quasiperiodic rotation. If the range of the ordinate were greater it would be seen that they stretch all the way across the figure, as do other noncommensurate quasiperiodic curves near the top of the section. Only the portion of the section between 0 and π is shown since the addition of π to θ gives a dynamically equivalent state. Note, however, that the synchronous states on the section at $\theta = 0$ and $\theta = \pi$ differ in that they present opposite faces to Saturn.

While this reduced problem gives some appreciation of the nonlinear spin–orbit coupling problem, it is necessary to reexamine the approximations which went into it. Certainly, the analysis which led to the standard picture of the tidal evolution of rotations in which the obliquity diminishes while the spin slows to synchronous does not take into account the complicated structure of the phase space for large asphericity and eccentricity which is observed in Fig. 2. In particular, it is necessary to reexamine the stability of the spin axis orientation perpendicular to the orbit plane. Without giving the details of the methods used (see Wisdom *et al.* 1984), it turns out that the chaotic zone is attitude unstable. This means that if Hyperion were placed in the chaotic zone with the slightest deviation of its spin axis from the orbit normal this deviation would grow exponentially, on a timescale of just a few orbit periods. This is also true of the synchronous state; that state in which all other tidally evolved satellites in the solar system are found is attitude unstable for Hyperion! The attitude stability of the other commensurate islands is mixed; some are stable while others are unstable. The equations which govern the three-dimensional tumbling motion are Euler's equations with the full three-dimensional gravity–gradient torque. These equations have three degrees of freedom, through, say, the three Euler angles, plus the explicit time dependence from the nonuniform Keplerian motion in an orbit with nonzero eccentricity. The three-dimensional tumbling state which is entered as the spin axis falls away from the orbit normal is a fully chaotic state. While it is no longer possible to plot a surface of section for such a problem with so many degrees of freedom, it is possible to show that trajectories which are initially nearby separate exponentially from one another. There are no hidden integrals of the motion; the chaotic tumbling motion has three positive Lyapunov exponents.

When the evolution due to tidal friction is included the problem is no longer strictly Hamiltonian. However, there is a tremendous disparity between the dynamical timescale and the timescale over which the tides are important. The tides are important only over the age of the solar system; they are especially weak for Hyperion because of its large distance from Saturn. The dynamical timescale is set by the orbital period. The tidal evolution is consequently viewed as a slow evolution through the phase space of the Hamiltonian system. Most likely Hyperion at one time had a rotation period much shorter than its orbital period and began its evolution high above the top of the section in Fig. 2. Over the age of the solar system its spin gradually slowed, while the obliquity damped nearly to zero. As it damped to zero the assumptions made in computing Fig. 2 come closer to being realized. By the time Hyperion reached the large chaotic zone its spin axis was nearly normal to the orbit plane. Once the large chaotic zone was entered, however, the work of the tides over aeons to stand Hyperion up was undone in a matter of days. Since the large chaotic zone is attitude unstable, in only a couple of orbit periods Hyperion began to tumble through all orientations. Ultimately, Hyperion may be captured by one of the small attitude-stable islands. However, it can never be captured by the synchronous island since the synchronous island is attitude unstable.

Is there observational support for the conclusion that Hyperion is currently tumbling chaotically? Unfortunately, the data are inconclusive. As yet it has not been possible to extract from the Voyager images the magnitude and direction of the spin vector at the time of the Voyager 2 encounter. A conventional fit of the lightcurve determined from the Voyager images to a sine curve gave a period near 13 days (Thomas *et al.* 1984, Thomas and Veverka 1985). This is adequate to rule out the known commensurate spin–orbit states but has no bearing on the question of whether the rotation of Hyperion is chaotic or not. The Voyager images are too sparse to get a clear picture of the nature of the motion. Similar analysis applied to simulated lightcurves of chaotic trajectories with data taken at the same intervals as the Voyager images often gave periods near 13 days with comparable goodness of fit (Wisdom and Peale 1984). It is simply too easy to fold sparse irregular data into a regular sine curve with arbitrary period. It turned out to be an artifact of poor statistics, but in the beginning of the simulations there was even a preponderance of periods near 13 days! The autocorrelation function of a numerically simulated lightcurve first crosses zero near 1.5 days. This indicates that if aliasing problems are to be avoided, data taken on the average of once per day will be required. There have been several ground-based attempts to pin down Hyperion's rotation state (Andersson 1974, Goguen 1983, Conner 1984, Binzel *et al.* 1986). Each of these sets of observations is consistent with the lightcurve of an irregularly shaped satellite with roughly 0.6-magnitude variations in brightness with an irregular rotation period. The amplitude of the lightcurve is consistent with that expected from the shape determined from Voyager images. All of these data sets are rather incomplete, and unfortunately rather sparse. There is apparently a need in this problem for a more extensive observing program. The most convincing evidence that Hyperion is in fact tumbling chaotically comes from the Voyager images themselves. They show that the long axis of Hyperion is out of the orbit plane, and the best determination of the spin axis orientation shows it to be nearly in the orbit plane. Such a configuration is consistent with chaotic tumbling, but inconsistent with the known regular rotation states. The dynamical state which is most consistent with the available data is the chaotic tumbling state. While the existence of a natural satellite executing a chaotic tumbling motion is perhaps unexpected and surprising, the Newtonian mechanics is quite secure. There is no question that mechanical systems can behave irregularly. Hyperion is just another dynamical system. The principal uncertainty on the theoretical side is whether the shape of Hyperion is representative of the mass distribution, as must be assumed to estimate the principal moments of inertia.

Although it would be nice to unambiguously verify that Hyperion is chaotically tumbling, there is another motivation for a new observing program of Hyperion. It is the essence of chaotic behavior that trajectories separate from one another exponentially with time. A consequence of this is that it is very difficult to predict the future evolution from real measurements. Any uncertainty at all in the knowledge of the initial conditions will quickly grow (exponentially) to yield complete ignorance. As an illustration, even if it had been possible to determine the orientation and the spin magnitude and direction of Hyperion to 10 significant figures at the time of the Voyager 1 encounter, it would not have been possible to predict the orientation of Hyperion at the time of Voyager 2! It appears to be possible though to turn Hyperion's chaotic rotation to our benefit. Recent numerical simulations (Chakrabarty, Klavetter, and Wisdom, unpublished) indicate that it is possible to invert a well-sampled lightcurve to obtain both the initial conditions of the trajectory which generated the lightcurve and an improved estimate of the principal moments. In fact because of the exponential variety of trajectories which exist, the rotation state at the midpoint of the interval covered by the observations and the principal moments of inertia are determined with exponential accuracy. Thus the knowledge gained from measurements on a chaotic dynamical system grows exponentially with the timespan covered by the observations. This may be contrasted with the statistical $\sqrt{N}$ increase in knowledge usually expected. The sensitivity introduced by the chaos seems to overcome not only error in measurement, but also error in the knowledge of the shape (departure from triaxiallity), as long as neither is too large. The simulations indicate that 60 continuous days of accurate observation could increase the knowledge of Hyperion's moments by a factor of 10–100.

This could be scientifically interesting. The only other small body for which both shape and principal moments are known is Phobos. It appears that in this case the shape is *not* representative of the moments. Estimates of $(B - A)/C$ from observations of the forced libration are roughly half the value determined from estimates of the shape (Duxbury and Callahan 1982).

IRREGULARLY SHAPED SATELLITES

Is the chaotic tumbling of Hyperion a result of a unique combination of factors which are nowhere else realized in the solar system? This appears to be the case. Just as Mercury, the commensurate rotation of which led to the development of the theory of spin–orbit coupling, is the only example of a nonsynchronous yet commensurate rotation, Hyperion appears to be the only example of chaotic tumbling today. One possibility which is often suggested is Nereid, with its extremely large orbital eccentricity ($e \approx 0.75$) and size comparable to that of Hyperion. However, the orbital period of Nereid is very long, and the timescale to reach a near-synchronous rotation rate is very much longer than the age of the solar system.

While Hyperion is the only example of chaotic tumbling today, it turns out that many satellites tumbled chaotically in the past. In fact, *all* irregularly shaped satellites in the solar system must tumble chaotically just at the point where the spin is about to be captured into synchronous rotation (Wisdom 1987). The commensurate spin–orbit states are examples of resonances. Resonances appear as islands on a surface of section. Almost all resonances are surrounded by chaotic zones, though in some cases these chaotic zones may be very narrow. Two moderately narrow chaotic zones are illustrated in the upper part of Fig. 2, those surrounding the 5/2 and 3/1 resonance states. There exist approximate methods of estimating the size of these chaotic zones (see Chirikov 1979). The width of the chaotic zone surrounding the synchronous island may be specified in terms of the magnitude of the chaotic variations of the integral E of the zero eccentricity problem,

$$\frac{\Delta E}{E} \approx \frac{14\pi e}{\alpha^3} e^{-\pi/2\alpha},$$

where α, the asphericity parameter, is $\sqrt{3(B - A)/C}$. (Strictly speaking the E in this expression refers to the energy in the pendulum approximation of the resonance in the full nonzero eccentricity problem. This differs from the energy of the zero eccentricity problem only by terms of order e^2.) This estimate is remarkably good; it has been verified to be accurate to within a factor of two over about eight orders of magnitude. Note the dependence on the orbital eccentricity and the asphericity parameter. While the width of the chaotic zone depends exponentially on the asphericity parameter, it depends only linearly on the orbital eccentricity. Thus satellites with large deviations from spherical symmetry, yet small eccentricities, may nevertheless have significant chaotic zones. For example, the estimate for Phobos yields $\Delta E/E \approx 0.2$ even with its rather low orbital eccentricity of 0.015. For Phobos α is approximately 0.83 on the basis of its shape whereas the forced libration requires $\alpha \approx 0.56$. A surface of section for Phobos is shown in Fig. 3 ($\alpha = 0.83$). The extent of the chaotic zone is roughly that predicted by the estimate, even though it is really only intended to be applied to much narrower chaotic zones. Thus the spin–orbit phase space of Phobos has a sizable chaotic zone. For both estimates of α the chaotic zone engulfs the 3/2 state; for the larger value it engulfs the 1/2 state as well. That the 3/2 state would be engulfed could have been expected on the basis of resonance overlap arguments, since in the pendulum approximation the synchronous island extends beyond the nominal center of the 3/2 state for $\alpha > 0.5$ (see Wisdom *et al.* 1984, Chirikov 1979). Even for Deimos, where

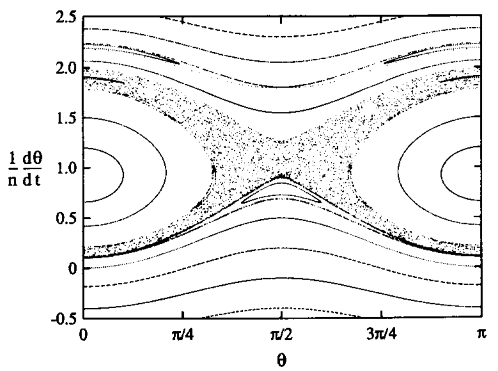

FIG. 3. Surface of section for Phobos (using $\alpha = 0.83$ and $e = 0.015$). The chaotic zone is a significant feature on the section.

the orbital eccentricity is considered to be anomalously small ($e \approx 0.0005$), the chaotic zone is not microscopic (see Fig. 4). Since the timescale for despinning Deimos is on the order of 100 my, while the timescale for Phobos is only 10 my, it may be that Deimos actually spent more time in its chaotic zone than did Phobos. Surfaces of section for several other irregularly shaped satellites with α near unity confirm the existence of significant chaotic zones surrounding the synchronous island.

Considering the attitude instabilities that were found for Hyperion, it is necessary to examine the attitude stability of these chaotic zones. In every case examined, the chaotic zone is attitude unstable. A slight displacement of the spin axis from the orbit normals grows exponentially, leading to chaotic tumbling. The surprising result is the strength of this attitude instability. In every case the timescale for the exponential growth of obliquity is only a few orbit periods, just as it was for Hyperion. This is true even for Deimos, with its low orbital eccentricity. The magnitude of the out-of-plane motion of the long axis is also roughly independent of the eccentricity and precise shape, being in all cases near 1 radian. The fact that Hyperion's long axis seems to make larger excursions, in fact going through all orientations, is a consequence of Hyperion's especially large orbital eccentricity.

Since the characteristics of the attitude instability are relatively insensitive to the orbital eccentricity, the eccentricity evidently does not play a crucial role. This suggests that the spin–orbit problem with a circular orbit be studied. When the orbit is circular the time dependence can be eliminated through a transformation to a rotating frame of reference, leaving an autonomous problem with three degrees of freedom. A new integral analogous to the Jacobi integral of the circular restricted problem is obtained. Unfortunately, the usual surfaces of section cannot be generated when there are three degrees of freedom. To further reduce the problem to two degrees of freedom, the body may be taken to be prolate axisymmetric. The Hamiltonian is then cyclic in the angle of rotation about the symmetry axis, and the momentum conjugate to this angle is a new integral. This highly reduced problem now has two degrees of freedom, and the geometry of the space of three-dimensional motions can now be studied by making surfaces of section. The sections reveal that even a prolate axisymmetric body in a *circular* orbit has a sizable chaotic zone. Furthermore, when the integrals are chosen to be those of a

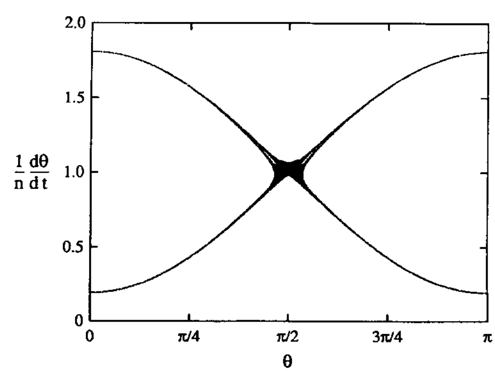

FIG. 4. Chaotic separatrix for Deimos (using $\alpha = 0.81$ and $e = 0.0005$). The chaotic zone is sizable considering the very low orbital eccentricity.

body just at the point of entry into synchronous rotation (on the separatrix), the chaotic zone of the three-dimensional tumbling extends to zero obliquity. In other words even in the zero-eccentricity case, the separatrix is attitude unstable.

It is not possible to tidally evolve into the synchronous state without passing through a region which is attitude unstable. The resulting tumbling motion is always chaotic. All synchronously rotating satellites with irregular shapes must have spent a period of time tumbling chaotically. The length of time spent in this state is probably comparable to or somewhat greater than the despinning timescale; it is not yet possible to make a more rigorous estimate. Thus Deimos probably spent on the order of 100 my tumbling chaotically.

This new episode in the adolescence of the irregularly shaped satellites is fascinating in itself. The world, and Newtonian mechanics in particular, works in a surprising way. One may ask, though, if there are any observable consequences of the episodic of chaotic tumbling (see Wisdom 1987 for a more thorough discussion). The first possibility which came to mind was Miranda. Is it possible that its exotic surface features are a result of a period of chaotic tumbling? This appears unlikely. While the enhanced tidal dissipation during the period of chaotic tumbling could have provided a significant heat source, the attitude instability described above occurs only as the synchronous rotation state is entered. For Miranda this must have occurred very early after its formation since the timescale for tidal despinning to synchronous rotation is only 300,000 years (Peale 1977). Thus this episode of chaotic tumbling would have occurred too close to the time of formation to account for the disparate ages of the features on Miranda. Another possibility is that the stretch marks on Phobos are a result of chaotic tumbling. However, the association of the marks with the large crater Stickney seems to argue in favor of the impact-fracturing hypothesis. Perhaps one could appeal (very weakly) to an episode of tumbling coupled with weaknesses induced by the impact.

A plausible physical application concerns the anomalously low eccentricity of Deimos. Yoder (1982) has shown that passage through a 2/1 mean-motion resonance between Phobos and Deimos should have left Deimos with an eccentricity of 0.002. Instead Deimos is observed to have an eccentricity of 0.0005. Could an episode of chaotic tumbling account for this low eccentricity? The long-term effects of chaotic tumbling on the orbit of a satellite have not yet been worked out, but some conjectures may be offered. Dissipation within a synchronously rotating satellite in an eccentric orbit tends to damp the orbital eccentricity (Urey *et al.* 1959, Goldreich 1963). In this case the tides dissipate energy but conserve angular momentum, since the spin is locked in the synchronous rotation state. Dissipation in a tumbling satellite is significantly enhanced over that in a synchronously rotating satellite. For a synchronously rotating satellite there is no time-dependent tide at all unless the orbit is eccentric; the magnitudes of the time-dependent variations are all proportional to the orbital eccentricity. In a tumbling or nonsynchronously rotating satellite the whole magnitude of the tide is time dependent. The effect of the tides on the orbit of a nonsynchronously rotating satellite may be calculated in the context of a particular tidal model. The tides on the satellite slow the rotation and generally tend to increase the orbital eccentricity, though the sign of the eccentricity variation may change as the rotation approaches synchronous rotation. Both the orbital energy and angular momentum change as a result of the tides on the satellite. For a chaotically tumbling satellite the rate of energy dissipation must be comparable to and is probably somewhat greater than that in a nonsynchronous satellite with a regular rotation. The difficulty is in determining whether there is a secular change in the orbital angular momentum. In the chaotic tumbling state, the angular momentum of the satellite is constantly changing. In the "restricted" problem where the orbit is considered to be fixed, the angular momentum of the satellite is not a conserved quantity. Of course, the total angular momentum of the system is conserved; the orbital angular momentum simply acts as a large reservoir. Qualitatively, it is plausible that the tides cannot accomplish a secular change in the angular momentum of the satellite since any small change which occurs as a result of the tidal torque is quickly swamped by a large, effectively random chaotic variation in the angular momentum. If indeed it is the case that there is no secular change in the orbital angular momentum then the eccentricity will be damped on the timescale:

$$T_e \approx \frac{e^2 \mu Q a^2}{\rho \omega^3 R^4}.$$

This timescale is two factors of the orbital eccentricity smaller than the timescale for damping the orbital eccentricity when the rotation is synchronous. For Deimos, the timescale to damp its eccentricity during the chaotic tumbling phase is only 300,000,000 years; the timescale to damp its eccentricity while in synchronous rotation is much longer than the age of the solar system. The time spent in the chaotic zone must be comparable to, but probably somewhat longer than, the timescale for despinning to synchronous rotation. For Deimos this timescale is also of order 100,000,000 years. Consequently, enough time might have elapsed to damp the eccentricity. The catch is that Deimos must have already passed through the resonance with Phobos before the chaotic phase ended, and the time of resonance passage is highly uncertain. For this mechanism to explain the low eccentricity of Deimos the resonance with Phobos must have occurred in the first several hundred million years after the introduction of Deimos to the Mars system.

For an irregularly shaped satellite a chaotic tumbling episode always occurs as the synchronous spin–orbit state is entered, regardless of the orbital eccentricity. It may also occur if the eccentricity becomes so large that the synchronous state becomes attitude unstable. Yoder (1982) pointed out that the current eccentricity of Phobos can be explained as a remnant of the passage through another orbital resonance. This calls into question the numerous evolutionary histories of Phobos which give it a large orbital eccentricity in the past. It is still worth pointing out, however, that if Phobos ever did have an orbital eccentricity greater than about 0.1 then the synchronous rotation state would have been attitude unstable, and Phobos would have tumbled chaotically. The enhanced dissipation in the tumbling satellite would surely invalidate any history which had not taken the chaotic tumbling into account. The possibility of a chaotic tumbling episode must be kept in mind in the determination of the orbital history of any irregularly shaped satellite.

CHAOTIC ORBITS AND THE 3/1 KIRKWOOD GAP

After the discovery of the divided phase space, no time was lost before the same techniques were applied to the classical orbital problems in dynamical astronomy; the phase spaces of the circular restricted three-body problem (Hénon 1966a,b, Jefferys 1966) and Hill's problem (Hénon 1970) were shown to exhibit a similar division into regular and irregular trajectories. The study of Hill's problem in particular led to a rather nice physical application of the divided phase space. It turns out that for satellites beyond a certain distance from a planet, only retrograde orbits are quasiperiodic. At the same distance all prograde orbits lead to escape or chaotic behavior. Thus the orbits of the outermost group of Jupiter satellites would not be stable at the same distance from Jupiter if their orbits were prograde rather than retrograde.

One of the most famous problems in dynamical astronomy concerns the distribution of asteroids. The distribution of the semimajor axes of the asteroids is not uniform; it shows several gaps as well as several enhancements. The origin of these gaps has been the object of a great deal of speculation. One major clue to the cause of these nonuniformities, which was noted at the time of their discovery by D. Kirkwood, is that they occur near mean-motion commensurabilities with Jupiter. That is, a small integer times the mean motion of an asteroid in a gap will nearly equal the product of another small integer times the mean motion of Jupiter. Kirkwood suggested that the repeated tugs from Jupiter somehow caused the asteroids to leave the vicinity of the resonance. This explanation certainly appears naive today, but it may actually be closer to the truth than many of the other hypotheses which have been proposed since the discovery of the gaps. By itself, however, such a simple explanation is doomed to failure. It is difficult for a qualitative argument about the stability of resonant motion to account for the fact that there are enhancements near some resonances, but gaps at others. Clearly there can be no hope of understanding how the gaps came about without first understanding the dynamics of the resonances with which they are associated.

Until recently most investigations of the dynamical behavior of asteroids near resonances relied in one way or another on the principle of averaging. The idea of averaging a system to isolate the long-period variations goes back to Gauss and Lagrange, who investigated the long-term evolution of the solar system by averaging the equations of motion over the mean longitudes. The procedure makes sense intuitively; the most rapid variations give rise to periodic effects which tend to average to zero, while the more slowly varying contributions can add up to give a secular variation. Averaging plays a central role in the classical perturbation theory a la Linstedt, Poincaré, and von Zeipel. The error at any order can be estimated with the usual conclusion that it is less than a certain amount for a specified length of time. However, Poincaré showed that the classical perturbation expansions were asymptotic series and generally diverge. The relationship between solutions derived with classical perturbation theory and the actual trajectories is thus unclear. In the long term the actual trajectory may have nothing to do with the solution given by perturbation theory, though in the beginning the error is calculable. The basic assumption of classical perturbation theory is that all problems are integrable, and the nonintegrable part of the Hamiltonian is successively pushed to higher and higher orders. After an infinity of such transformations the solutions are obtained by solving the integrable problem which remains and then undoing the transformations which led to it. Since all problems are assumed to be integrable, only quasiperiodic solutions can arise from classical perturbation theory. A classical perturbation expansion for a chaotic trajectory will simply have the wrong qualitative behavior. What is to be made of averaging then? The proof of the KAM theorem is a constructive proof. Quasiperiodic motions are proven to exist by actually constructing them with a new convergent perturbation scheme. Averaging still plays a role in the new perturbation scheme. Thus, when the conditions of the proof are satisfied the validity of that averaging step is beyond question, since the perturbation expansion converges. In Arnold's extension of the proof to the stability of solar systems with sufficiently small masses, eccentricities, and inclinations the construction of the quasiperiodic solutions begins with the Lagrange solution. Thus under the restrictive assumptions of the proof the relationship of the Lagrange solutions to the actual solutions is completely known. The averaging method used by Lagrange is rigorously justified when the conditions of Arnold's proof are met. While the averaging method may now claim a certain respect, in most cases the validity of the averaging principle remains an unproven conjecture. Arnold (1974) has said, "We note that this principle is neither a theorem, an axiom, nor a definition, but rather a physical proposition, i.e. a vaguely formulated and, strictly speaking untrue assertion. Such assertions are often fruitful sources of mathematical theorems." It is an observed fact that the Lagrange solution is a good first approximation to the motions of the planets, though the long-term stability of the solar system is still an open question since the conditions of Arnold's proof are not met. The averaging method has been used successfully in a wide variety of situations. It is of unquestionable usefulness, even though its validity is often questionable.

The first analytic investigation of asteroid motion near mean-motion resonances was undertaken by Poincaré (1902). Poincaré studied the motion of asteroids near the 2/1 commensurability in the context of the circular restricted three-body problem. (At the 2/1 commensurability the mean motion of the asteroid is approximately twice the mean motion of Jupiter.) Poincaré made use of the averaging principle. He analytically averaged the equations of motion to isolate the long-period resonance effects, keeping only the lowest-order terms in the eccentricity. In effect he approximated the motion near the 2/1 commensurability by the single largest term in the disturbing function. Poincaré's treatment is easily generalized to other resonances and can also accommodate arbitrarily many higher-order resonant terms. After averaging, the Hamiltonian for motion near the 2/1 commensurability in the circular restricted problem has only a single degree of freedom, the resonant combination of longitudes $l - 2l_J + \tilde{\omega}$, and consequently the averaged problem is integrable. (l and l_J are the mean longitudes of the asteroid and Jupiter, respectively, and $\tilde{\omega}$ is the longitude of perihelion of the asteroid.)

To some extent "intuition" is the process whereby a new problem is understood in terms of those problems which are so well understood, or so often seen, that they have become a part of the way we think about the world. Most people's intuition about orbital resonances is based on Poincaré's problem. The solutions or level curves of Poincaré's averaged Hamiltonian have been published innumerable times. The examination of the various hypotheses concerning the origin of the Kirkwood gaps has largely been pursued in the context of Poincaré's problem. However, the intuition derived from Poincaré's problem has not led to an explanation of the distribution of asteroids. The lack of a satisfactory explanation has in recent years led more and more people to consider cosmogonic hypotheses, those which invoke solar system formation processes, for the formation of the gaps. Again these are generally pursued in the context of Poincaré's problem. Unfortunately, while Poincaré's integrable model qualitatively represents the behavior of an asteroid in resonance over a few hundreds of years, it is a rather poor representation of the long-term evolution. Even the qualitative picture has serious deficiencies; there is a need for a new "intuition."

A new paradigm is provided by the study of dynamical systems. The experience gained in the study of the already classic problems, such as the Hénon–Heiles problem, provides this new injection of "intuition." The major, but by no means only, elements in this intuition are the following. Hamiltonian systems with more than one degree of freedom almost always give rise to chaotic behavior for some initial conditions. The phase space is generally divided into chaotic regions and regular regions. And most importantly for the problem of the Kirkwood gaps, resonances are almost always accompanied by chaotic zones. It is clearly important to understand the qualitative nature of the motion near the mean-motion commensurabilities before considering any further hypotheses for the origin of the gaps. What is the extent of the chaotic region near each of the commensurabilities? What are the characteristics of both the regular and chaotic trajectories?

In trying to answer these questions one immediately runs into difficulties. First, it is rather difficult to predict analytically the extent of a chaotic region. The only fully reliable way to determine the extent is to perform a numerical exploration. This leads immediately to the next difficulty. The numerical integration of orbits is extremely time consuming. This is basically a consequence of the great disparity of timescales involved in problems of celestial mechanics, which in turn is a reflection of the basic degeneracy of the Kepler problem. The Kepler problem has three degrees of freedom, but only a single frequency, the orbital frequency. Without external perturbation Keplerian orbits are closed; there is no motion of the node or the pericenter. The degeneracy is broken only when perturbations are taken into account; i.e., the nodes and pericenters move only due to external perturbations, such as those of other planets. The frequencies acquired by the nodes and pericenters depend on the distance to the perturber but are roughly of order the ratio of the mass of the perturber to the central mass times the orbital frequency. The ratio of the mass of Jupiter to the mass of the Sun is about 1/1047. Thus the nodes and pericenters of the planets move with periods on the order of a thousand times the period of Jupiter, or tens of thousands of years. How long must the integration be before the qualitative structure of the solutions becomes apparent? This is difficult to estimate and has really been the stumbling block for a number of numerical investigations. In hindsight, one might have expected that it would be necessary to integrate for several times the longest "fundamental" period in the problem before one could begin to determine the qualitative character of the motion. For asteroid problems this time interval is in the neighborhood of a hundred thousand years. Numerical explorations which covered 10,000 years or less had no hope of success in really determining the structure of the phase space near the mean-motion commensurabilities, since this is shorter than a single precession period.

The reason for the short intervals of the numerical explorations is clear, however, since a full integration of the equations of motion must necessarily take a number of basic integration steps to cover the shortest period in the problem, which is here the shortest orbital period. It takes roughly an hour on a VAX to integrate an asteroid for 10,000 years. Thus even short systematic explorations would have been prohibitively expensive if full integrations had been performed. The compromise, which has almost universally been used in long integrations of asteroid trajectories, is to use a numerical procedure devised by Schubart (1964). Based again on the nonrigorous principle of averaging, Schubart's procedure numerically averages the equations of motion to remove the most rapidly varying contributions, those of the orbital period, while retaining the resonant contributions and those of longer period. A numerical averaging is preferred over an analytical averaging because the analytical description of the disturbing function in regions of interest is often difficult to obtain or is of questionable validity. Using Schubart's method, Giffen (1973) investigated the motion near the 2/1 commensurability in the planar *elliptic* problem. He found a zone of chaotic behavior and suggested that this might be related to the formation of the gaps. Subsequent investigation by Froeschlé and Scholl (1976, 1981) indicated that while there was indeed a chaotic zone, it seemed to be a very mild chaos, and the region of chaotic behavior was bound to small eccentricities. Their systematic investigations, primarily using Schubart's program, found further chaotic orbits, but not very many. They concluded that chaotic behavior was not a significant factor in the creation of the gaps. Unfortunately, their integrations typically spanned only 10,000 years.

Hénon and Heiles noted in their original paper that an important feature of a surface of section generated by a Hamiltonian system is that it is area preserving. They suggested that essentially the same structure might be more rapidly explored by studying an iterated area-preserving map of the plane onto itself. They did indeed find similar behavior for an arbitrarily chosen area-preserving map, but with a *thousand-fold* increase in the speed of computation. It is clear that since the map generated by a Hamiltonian system is area preserving, anything which is true of area-preserving maps in general must also be true for Hamiltonian systems. In fact, most of what is known about Hamiltonian systems has come from the study of area-preserving maps. To explore the general properties of Hamiltonian systems why encumber oneself with a set of differential equations? On the other hand, if one is interested in a particular dynamical system, the equivalence is of no practical use since an explicit form for the map is rarely known. To compute the surface of section it is still necessary to integrate the orbit between the intersections.

Chirikov (1979) has shown how to make an approximate map for some problems. Chirikov emphasized that a large class of resonance problems can be approximated by a pendulum-like Hamiltonian. This was, for example, demonstrated above for the spin–orbit problem near synchronous rotation. Chirikov argued that if neighboring resonances are of comparable strengths then the qualitative behavior in the resonance region of the phase space can be expected to be similar to that of the "standard map" of the plane onto itself,

$$I' = I + K \sin \phi$$

$$\phi' = \phi + I',$$

where K is a parameter. The standard map may be derived by first removing nonresonant terms by averaging and then adding new nonresonant terms so that the perturbation forms a periodic sequence of Dirac delta functions. The new problem can be integrated between, as well as across, the delta functions, giving an explicit algebraic expression for the map. The averaging principle is invoked to argue that it is the nearby resonances of the problem which are most important, and the precise form of the more distant resonances, which give rise to rapid variations, is to a large extent irrelevant. A very general application of the standard map is to the motion near the separatrix of a resonance which is well approximated as a pendulum. It is this application which leads to the approximation given above for the width of the narrow chaotic zone near the synchronous state in the spin–orbit problem. As another application, the standard map could be considered a first approximation to the spin–orbit problem itself, where ϕ could be the angle $\theta - nt$ and I would be the angular momentum. This approximation is not particularly good, since the strengths of the spin–orbit resonances depend on various powers of the orbital eccentricity and are consequently not uniform in magnitude. Better approximate maps could be made if so desired, but computing time is not particularly a problem in the spin–orbit problem.

Motion near commensurabilities in the elliptic restricted problem does not fall into the category of problems well approximated by a pendulum, so the standard map is not relevant. However, the same principles can be used to construct an approximate map for motion near a mean-motion commensurability (Wisdom 1982, 1983). The terms with highest frequency, the orbital frequency, are first removed by averaging, leaving the resonant terms and the secular terms. New high-frequency terms are added in such a way that periodic delta functions are formed. The new equations can be integrated across the delta functions and between them, giving a map of the phase space onto itself. The new map is four dimensional (for the planar elliptic problem), and successive iterations of it advance the dynamical state by one orbital period of Jupiter. The map approximation is considerably faster than a conventional numerical integration. Comparing timings provided by Scholl (personal letter, 1982), the map is several hundred times faster than even Schubart's averaging program. Consequently, considerably longer and more thorough systematic investigations can be made, presuming the map accurately represents the behavior of the differential equations. It should be emphasized that the derivation of the map relies on the averaging principle. Consequently, it bears the same relationship to the unaveraged differential equations as the analytical models of Poincaré and others as well as the numerical explorations utilizing Schubart's numerical averaging program. They all rely on the averaging principle. The map is perhaps only slightly more suspect since the introduction of delta functions makes the perturbation rather singular. However, experience with the standard map indicates this is no particular cause for alarm.

Figure 5 shows the orbital eccentricity as a function of time for a chaotic trajectory

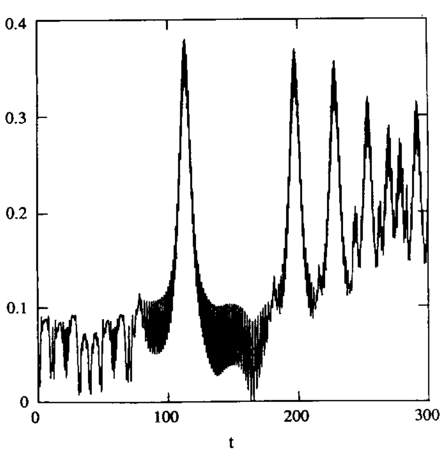

FIG. 5. Eccentricity versus time for a chaotic trajectory near the 3/1 commensurability. Time is measured in millennia.

near the 3/1 commensurability computed with the planar elliptic map. While excursions in eccentricity of this magnitude were previously known (Scholl and Froeschlé 1974), the possibility that an orbit could spend a hundred thousand years or longer at low eccentricity and then "suddenly" take large excursions was quite unexpected. This was not a part of the "intuition" of Poincaré's problem; such trajectories cannot be described in that context. A natural conclusion is that the behavior must be an artifact of the method. However, subsequent numerical integrations of the full, unaveraged, differential equations verified that the behavior actually occurs (Wisdom 1983, Murray and Fox 1984). A minor annoyance is that there is no direct correspondence between such trajectories of the map and the differential equations. This is a consequence of two facts: the map is only an approximation and such trajectories are chaotic. The chaotic character of the trajectories means that the most minor difference in the initial conditions or in the specification of the dynamics quickly leads to a completely different evolution. This is not important for the physical applications since there it is only the extent of the chaotic zone which is relevant. Trajectories of the map do the same sort of things as the trajectories of the differential equations; it is just not possible to predict one from the other very far in the future.

Figure 6 illustrates the behavior of the semimajor axis for this trajectory. Note what a poor idea of the trajectory might be gained from a 10,000-year integration. Between about 30,000 and 50,000 years the trajectory even stays on one side of the resonance. If that behavior was observed in a numerical survey the orbit would probably have been classified as a nonresonant, quasiperiodic circulator (since the resonant combination of longitudes is then circulating). Viewed over 300,000 years the character of the motion is quite different. But it is only when the trajectory is computed over millions of years that one begins

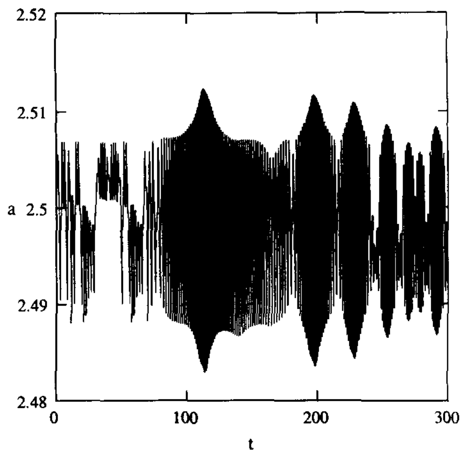

FIG. 6. Semimajor axis versus time for the same trajectory as in Fig. 5. Time is again measured in millennia. Note that between 30,000 and 50,000 years the semimajor axis remains on one side of the resonance. A short 10,000-year integration could give a very poor idea of the nature of this trajectory.

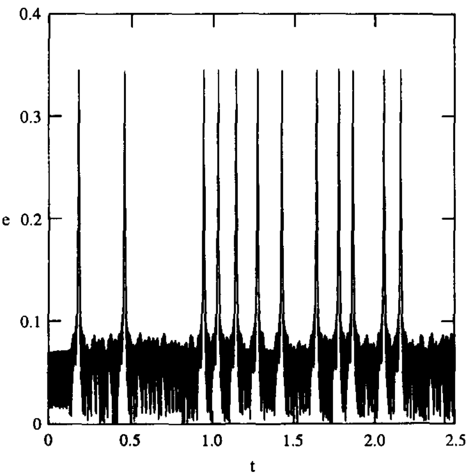

FIG. 7. Eccentricity of a chaotic trajectory over a longer time interval. The time is now measured in millions of years. This rather interesting behavior turned out to be relatively rare.

to feel as though the true nature of the motion is now represented. Figure 7 shows a very interesting, though relatively rare, behavior, while Fig. 8 shows the more typical behavior of the eccentricity of a chaotic trajectory near the 3/1 resonance in the planar elliptic problem. Both of these trajectories were computed with the map. The eccentricity jumps shown in Fig. 7 all reach the same eccentricity but seem to occur at irregular intervals. A most surprising result is that if the plot is expanded near two different jumps and then superimposed the eccentricity jumps are practically identical. The more typical behavior shown in Fig. 8 shows bursts of irregular high-eccentricity behavior interspersed with intervals of irregular low-eccentricity behavior, with an occasional eccentricity spike.

At this point a few things should be mentioned about numerical error. First, long numerical integrations of chaotic trajectories are rarely reversible. Sixty-four bit integrations of chaotic asteroid trajectories beyond a couple hundred thousand years cannot be reversed to recover the initial conditions. This is a consequence of the exponential divergence of neighboring trajectories which is characteristic of chaotic trajectories. Exponential divergence of nearby trajectories leads to an exponential growth of roundoff error. To be reversible a

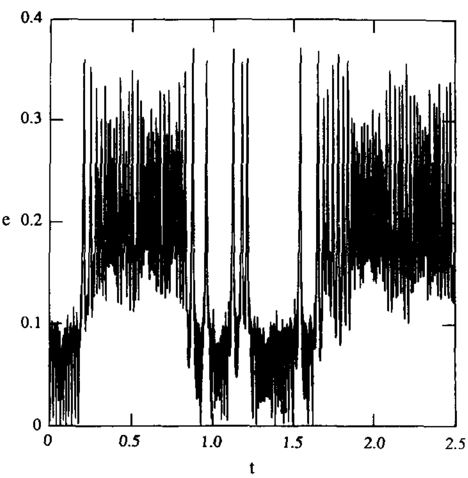

FIG. 8. Eccentricity of a more typical chaotic trajectory over a longer time interval. The time is measured in millions of years. Bursts of high-eccentricity behavior are interspersed with intervals of irregular low-eccentricity behavior, broken by occasional spikes.

million-year integration would have to carry 200 digits; 100 would be lost in going forward while the remaining 100 would just barely enable the prodigal trajectory to find the way back home! While the basic jumping phenomenon in the eccentricity is beyond question because it happens so quickly, the extent to which the longer integrations should be trusted is not clear. Not only is the map an unconventional method; it is being used beyond the point at which it can be shown to be numerically self-consistent! Can this be justified? No, not rigorously. The justification comes from that alternate "intuition" derived from the broader experience of the general study of dynamical systems. Whether the results of a long integration should be trusted depends in fact on what one would like to get out of them. The traditional celestial mechanic shudders at the thought of using a nonreversible trajectory; a modern nonlinear dynamicist says, "Sure, what do you want to do with it." For example, by now there must have been thousands of surfaces of section published. Yet whenever a chaotic zone shows more than a couple of hundred points the trajectory used in the computation was almost certainly not numerically reversible. Nevertheless, there is usually no particular sign that the trajectory is behaving in a non-Hamiltonian manner, and most nonlinear dynamicists believe such a trajectory continues to do essentially the correct things. Thus a true trajectory is believed to fill very nearly the same region filled by the computed trajectory, as long as a certain minimal care is used in computing the trajectory. Other properties are independent of numerical error. For instance, it has been shown that the rates of exponential divergence are independent of the accuracy of the integration and agree with theoretical determinations when they are available, again as long as sufficient care is exercised in carrying out the integration (Benettin *et al.* 1978). There is also a remarkable theorem which is often appealed to to justify this belief in the validity of numerical calculations. Roughly stated, it has been proven that for systems where the rate of divergence of nearby trajectories is bounded from below, there is a true trajectory which shadows the computed trajectory for all time; i.e., there is always a real trajectory which does essentially the same thing as the computed trajectory however long the interval of integration! If the system under study falls in this category, the nonlinear dynamicist may compute with impunity. Unfortunately, that idyllic world where all computations are correct still lies beyond the rainbow; most Hamiltonian systems do not fall in this category, and it appears unlikely that a general theorem of this sort can be proven for Hamiltonian systems. Nevertheless, there is a common belief that it is almost true, or perhaps practically true. Still, in determining long-term correlations there is no alternative but to use a lot of digits.

As mentioned before, the basic phenomenon of irregular jumps in eccentricity has been amply verified in conventional numerical integrations. Any particular segment of a trajectory less than about 100,000 years long is numerically self-consistent, and this interval is long enough to span a typical jump in eccentricity. To accept aspects of the trajectories over longer times, whatever the method used to compute them, it is necessary to some extent to appeal to that body of "intuition" developed from nonlinear dynamics. Fortunately, this does not (just) mean that the hands are clasped in a fervent prayer to the god of dynamics that the results are correct. Rather, one must as far as possible bring the phenomenon into the broader context of dynamical systems and show that the behavior is in fact not extraordinary, but just like that found in other dynamical systems. One way of "understanding" the strange trajectories of the elliptic problem would be to place them in context on a surface of section, where the geometric structure of the phase space could be visualized. However, at first sight it is not possible to make surfaces of section for the elliptic problem. There are too many degrees of freedom. The elliptic problem has two and a half degrees of freedom. It has two degrees of freedom through, say, the ordinary cartesian x and y coordinates, and depends explicitly on the time through the Keplerian motion of Jupiter in its orbit. A stroboscopic surface of section such as the one used in the reduced spin–orbit problem to compute the sections shown in Figs. 2–4 would be four dimensional. Such a section would give, for instance, x and y and their velocities every time Jupiter passes through perihelion. As yet nonlinear dynamicists have little experience with four-dimensional maps; it would be difficult to gain "intuition" about the trajectory from such a section. However, the averaged planar elliptic system has only two degrees of freedom and surfaces of section can indeed be generated. The only difficulty is in choosing a section which captures all the variety of motions which can occur. A good choice for the section was clear only after the perturbative treatment of the motion which is described below was accomplished (Wisdom 1985b). The section is generated by plotting the Poincaré variables $y = e \sin(\tilde{\omega} - \tilde{\omega}_J)$ versus $x = e \cos(\tilde{\omega} - \tilde{\omega}_J)$ whenever the resonant combination of mean longitudes $l - 3l_J$ goes through a particular value which in turn depends on the asteroid eccentricity, e, and longitudes of perihelion, $\tilde{\omega}$ and $\tilde{\omega}_J$. On this section the eccentricity is just the radial distance from the origin. This section is quite closely related to a section briefly used by Giffen, who plotted these same variables whenever the semimajor axis passed through a maximum, but who abandoned the section because it seemed to require excessively long integrations to accumulate an adequate number of points! Surfaces of section for trajectories similar to the ones shown in Figs. 7 and 8 are shown in Figs. 9 and 10, respectively. They were computed with the map. The origin of the strange intermittent behavior is now clear. There exists a chaotic zone which surrounds the origin, and the chaotic zone has a very narrow branch which extends to eccentricities near 0.3. A trajectory wanders in the chaotic zone and every now and then it enters the narrow bridge to large eccentricity; as would be expected there are unstable periodic orbits which stand at the gateways ushering the

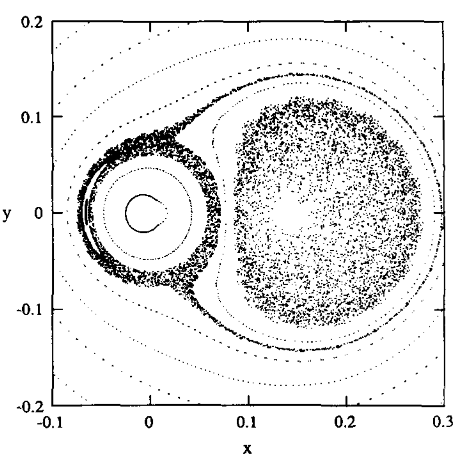

FIG. 9. Surface of section corresponding to the trajectory of Fig. 7. The coordinates are $x = e \cos(\tilde{\omega} - \tilde{\omega}_J)$, and $y = e \sin(\tilde{\omega} - \tilde{\omega}_J)$. The orbital eccentricity is the radius from the origin. The narrow branch of the chaotic zone explains the irregularly appearing, but nearly identical jumps in eccentricity.

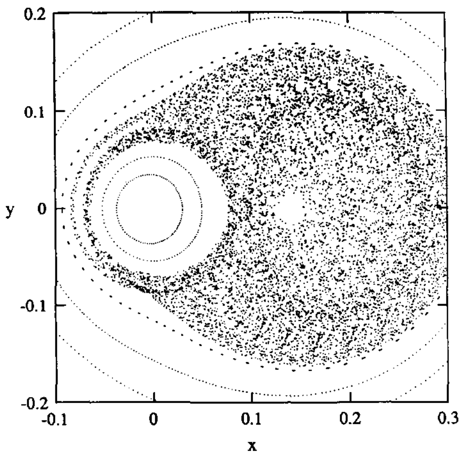

FIG. 10. Surface of section for a trajectory similar to the one used to generate Fig. 8. The trajectory is free to explore a rather large chaotic zone.

trajectories one way or the other. Once the section has been plotted there is no more mystery. The irregular evolution of the eccentricity as the trajectory wanders in the chaotic zone is just like the irregular wandering of chaotic trajectories in any other chaotic zone. The remarkable similarity in the jumps simply reflects the narrowness of that part of the chaotic zone which extends to high eccentricity. The time from one jump to the next is just as difficult to predict as it would be in any other dynamical system (see, e.g., Channon and Lebowitz 1980), but the range of the motion is clearly defined by the section. The characteristics of the more general behavior seen in Fig. 8 may be traced to the characteristics of the surface of section shown in Fig. 10. The eccentricity varies irregularly because the chaotic zone is large.

Another way to "understand" the basic phenomenon is through analytic or semianalytic approximation of particular features of the trajectories. Following a suggestion of Goldreich, the jumps in eccentricity can be approximated semianalytically (Wisdom 1985b). It is often the case that the variations in eccentricity seem to have two timescales, a fast oscillation superimposed on the more long term jump itself (cf. Fig. 5). This suggests that the jumps might be approximated by averaging over the resonant timescale, leaving only variations on the secular timescale. The averaging is accomplished by transforming the problem to variables in which the fast resonance variable obeys the equations of motion of a pendulum with slowly varying constants. The constants vary as the system evolves on the secular timescale. As long as the frequency of the pendulum is sufficiently distinct from the frequency of the long-period motion, the amplitude of the pendulum motion varies in such a way as to conserve the pendulum action, which is an adiabatic invariant. The equations governing the long-period evolution are obtained by averaging the equations of motion over the timescale for pendulum librations. Since the solutions of the pendulum are known in terms of elliptic functions the average over the resonant timescale can be accomplished analytically. This procedure yields a problem with one degree of freedom which can then be integrated numerically. The approximation breaks down when the amplitude of the pendulum variable is forced by conservation of action to be near π, for then the pendulum period diverges and the resonant period is no longer much shorter than the secular timescale. As long as the pendulum amplitude remains near π the trajectory cannot be approximated by this method. After some time the trajectory emerges from this region, where it is unpredictable, into the predictable region with a new value of the conserved pendulum action. Curiously, the regions where the motion is unpredictable are relatively small. Thus the irregular motion of chaotic asteroids is to a large extent predictable! The guiding trajectories corresponding to the section of Fig. 10 are shown in Fig. 11. The shaded regions are

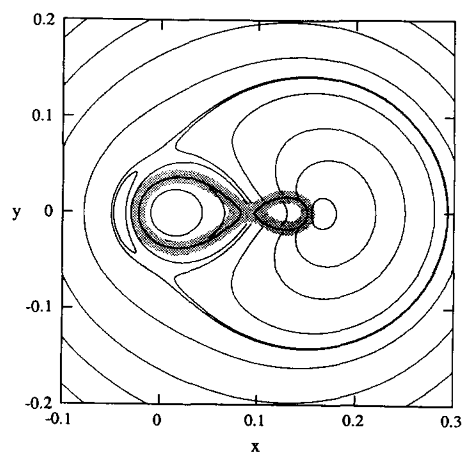

FIG. 11. Guiding trajectories of the long-period motion, corresponding to the trajectory of Fig. 10. The coordinates are essentially the same as those which are plotted on the sections, though a small correction must be applied to find one from the other. The regions where the two timescales are not sufficiently distinct to approximate the motion are shaded. The two-timescale method allows the motion in a large portion of the phase plane to be approximated.

the regions where the approximation breaks down. This picture of the motion successfully describes fragments of trajectories in a large portion of the phase space. This picture also suggests a new method for determining the extent of the chaotic zone. If a guiding trajectory does not enter the unpredictable zone the trajectory should remain well described by the two-timescale method. Only trajectories which enter the zone of unpredictability are candidates for chaotic behavior. Of course the breakdown of a perturbation method is by itself no reason to believe the trajectories should be chaotic. After all, where classical perturbation theory diverges, KAM perturbation theory converges to quasiperiodic solutions. However, in this case the entry into the zone of unpredictability does seem to be correlated with the onset of chaos. Simply shading the region filled with trajectories which enter these zones gives a quite good correspondence with the computed chaotic zone (Fig. 12; for more details see Wisdom 1985b).

The unusual behavior of chaotic asteroids is thus believable on several counts.

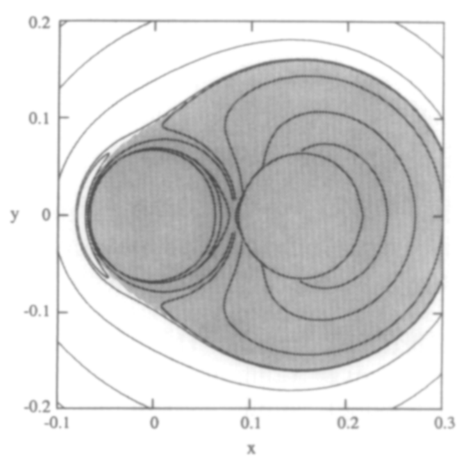

FIG. 12. Shading those regions where the guiding trajectories eventually enter the unpredictable region gives quite a good correspondence with the chaotic zone on the surface of section. This figure corresponds to the section in Fig. 10.

The basic phenomenon has now been repeatedly reproduced in conventional numerical integrations. Second, surfaces of section can be generated which bring the unusual behavior into the well-studied realm of two Hamiltonian systems with degrees of freedom. The behavior of the eccentricity is simply a consequence of the shape of the chaotic zones on those sections. Finally, the behavior is believable because it can be well approximated in a semianalytic treatment, which even allows the shape of the chaotic zone to be reproduced. It remains numerically nonreversible.

Now that the validity of the phenomenon has been established, is it relevant to the problem of the formation of the 3/1 Kirkwood gap? The large eccentricity increases are important for the formation of the gap because at the location of the gap eccentricities above 0.3 are Mars crossing. It turns out that all of the chaotic trajectories cross the orbit of Mars. This is also true of the quasiperiodic librators. Thus asteroids near the 3/1 commensurability can be removed by close encounters or collisions with Mars. A question which remains is the lifetime of the chaotic asteroids to removal from the gap by Mars. A crude estimate (Wisdom 1983) shows that Mars alone could clear the 3/1 Kirkwood gap in the age of the solar system. An element of this calculation is the average time spent by the asteroid as a Mars crosser. This is precisely the sort of question which is dangerous to ask of a numerically computed chaotic trajectory, since the long-term correlations may not be well represented. However, there is as yet no other way to estimate it. Comparison of the outer boundary of the chaotic zone with the actual distribution of TRIAD asteroids (circles) and Palomar–Leiden asteroids of qualities 1 and 2 (plus signs) shows remarkably good agreement (Fig. 13). This figure gives very strong evidence that chaotic behavior has played an important role in the formation of the gap.

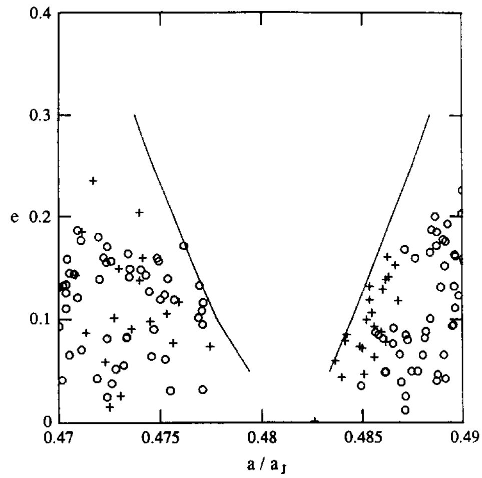

FIG. 13. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and a quasi-periodic region in the gap, but trajectories of both types are planet crossing.

DELIVERY OF METEORITES

Actually the lifetime for removal by Mars is not a crucial point since it turns out that chaotic trajectories reach much higher eccentricities when the numerical integration is made more realistic, by successively including the three-dimensional motions and then including the perturbations of the other planets, which primarily affect the asteroid through the variations in Jupiter's orbit. Computed with the map the eccentricity reaches values near unity, which is certainly beyond the range of validity of the map, since only second-order terms in eccentricity were retained in the disturbing function. However, the success of the map even at eccentricities as large as 0.4 makes one take these extreme eccentricities seriously, even though the actual extent of the variations can be determined only in a direct numerical integration. The very large eccentricities are important since for eccentricities above 0.6 the orbits are Earth crossing. This not only gives a much stronger mechanism for removing the asteroids to create the gaps, but also provides a mechanism for the continual transport of asteroidal debris to Earth. While it has long been suspected that the meteorites come from the asteroid belt, a transport mechanism consistent with the meteorite data had not previously been found. On the basis of Monte Carlo calculations of the evolution of a population of planet-crossing debris, Wetherill (1968) concluded that the meteorite data seemed to require the existence of an unknown source of asteroidal debris with orbits which have semimajor axes in the middle of the asteroid belt, aphelia near Jupiter, and perihelia just crossing Earth's orbit. The 3/1 chaotic zone seems to provide that previously unknown source of debris with Earth-crossing orbits (Wisdom 1985a).

The essential step is to determine whether chaotic trajectories near the 3/1 commensurability actually reach Earth-crossing eccentricities. This can be determined only by carrying out full numerical integrations. A number of numerical integrations of the equations of motion for a test particle perturbed by the four Jovian planets, each evolving under their mutual interactions, were carried out. Each integration spanned about 500,000 years and consumed the equivalent of about 200 VAX hours. The integrations were performed on a variety of machines. After four unsuccessful but encouraging attempts, an asteroidal trajectory which reached Earth-crossing eccentricities was found. The eccentricity as a function of time is shown in Fig. 14.

The numerical reliability of these trajectories is more difficult to justify than that of the trajectories in the planar elliptic problem, since it is no longer possible to compute surfaces of section. The trajectories do however cover a time interval comparable to the planar elliptic integrations. Since there was no indication of non-Hamiltonian behavior in that case, it seems likely that these integrations are still representative of true trajectories of the system. The essential characteristic for the delivery of meteorites is the range of eccentricity variation.

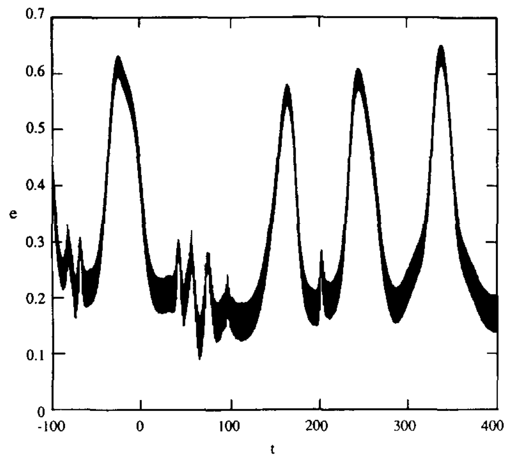

FIG. 14. Eccentricity versus time in millennia for a test particle perturbed by the four Jovian planets. At times the eccentricity is typical of asteroids; at other times it is large enough for the orbit to cross the orbit of Earth. This integration shows that meteoritic material may be directly transported to Earth from the asteroid belt by way of the 3/1 chaotic zone.

Note that the eccentricity rises rapidly enough that the numerical self-consistency of the integration is not a problem in determining the range of variation.

Wetherill (1985) has repeated the Monte Carlo calculation for this new source and has found that it does indeed appear to be consistent with the meteorite data, and partially accounts for the observed population of Earth-crossing asteroids. While asteroids near the 3/1 Kirkwood gap apparently do not show the spectral contrast seen in laboratory measurements of ordinary chondrites (McFadden and Vilas 1987), the dynamical evidence points to the region near the 3/1 Kirkwood gap as the source of the ordinary chondrites.

There is still the question of flux. Wetherill (1985) concluded that the flux from this source was roughly consistent with the observed source. However, a factor which enters the flux calculation is the fraction of chaotic debris which reaches Earth-crossing eccentricities within the first few million years after its injection into the chaotic zone. Unfortunately, this is again just the sort of problem which should properly be avoided because of the chaotic character of the trajectories. It may be possible to justify the validity of the range of eccentricities determined from such long integrations, but it is difficult to determine whether the typical time for a computed trajectory to reach Earth-crossing eccentricities is the same as the typical time for actual asteroidal debris to do the same. Frankly, I believe they are probably not much different, and in any case there is not yet any way to estimate the flux other than through numerical simulations. Of course the interpretation of the numerical results should always be tempered with an understanding of the problem of integrating chaotic trajectories. Actually the real difficulty is that it takes a tremendous amount of computer time to determine a statistic like that. It is out of the question without a dedicated computer. The Digital Orrery is being applied to this problem.

The Digital Orrery (Applegate *et al.* 1985) is a special-purpose computer specifically designed to study problems in celestial mechanics. The design and construction of the Orrery were led by Gerald J. Sussman, from the Artificial Intelligence Laboratory and the Department of Electrical Engineering at MIT. The construction of the Orrery is an extremely important advance for planetary dynamics, which evidently is in need of a dedicated supercomputer. The Orrery runs at about 60 times the speed of a VAX for celestial mechanics problems, or about a third the speed of a Cray.

2/1 KIRKWOOD GAP AND THE HILDA ASTEROIDS

The fact that there is a gap in the distribution of asteroid semimajor axes near the 2/1 commensurability and an enhancement in the distribution near the 3/2 commensurability needs an explanation. The problem is especially acute when approached with the intuition gained from Poincaré's problem. Both of the resonances are first-order resonances; i.e., the largest terms in the disturbing function are in both cases proportional to a single power of the orbital eccentricity. While the numerical coefficients are somewhat different, the analytic forms of the terms in the equations of motion are identical. The qualitative natures of the solutions of Poincaré's problem in the two cases are identical. The intuition gained from Poincaré's problem simply leads to paradox, or again to cosmogonic hypotheses.

The problem of the 2/1 Kirkwood gap and the abundance of the Hilda asteroids is associated with another puzzling feature in the distribution of asteroids. An explanation of the marked decline in the number of asteroids beyond the 2/1 resonance is not known. The most obvious explanation, that Jupiter perturbations are too strong and that these regions are dynamically unstable, has not held up under close examination. Numerical studies by Lecar and Franklin (1973) and Froeschlé and Scholl (1979) did not find that the region between the 2/1 and 3/2 resonances was particularly unstable or chaotic. It may still be the case that sufficiently longer, perhaps more realistic integrations would show this region to be unstable. The instability of the region beyond the 3/2 resonance may be understood in terms of the resonance overlap criterion (Wisdom 1980). The largest Kirkwood gap, that associated with the 2/1 commensurability, is near the outer boundary of the main asteroid belt. It is not clear whether the two problems of the formation of the 2/1 gap and the clearing of the outer belt are related.

The necessary first step is again to explore the dynamics of the resonances. Unfortunately, the dynamics of the 2/1 and 3/2 resonances are considerably more complicated than the dynamics of the 3/1 resonance. The relative simplicity of the 3/1 resonance arises from several factors. As a second-order resonance, the largest terms in the disturbing function are proportional to two powers of the eccentricities, either Jupiter's eccentricity or the asteroid's eccentricity, or a combination of the two. The next terms which contribute are fourth order in the eccentricities. The disturbing function is also second order in the inclinations. A purely combinatorial "accident" is that the 3/1 resonance is far from other resonances which could complicate the dynamics. The relative analytic simplicity of the 3/1 resonance is what is responsible for the success of the map and the perturbative treatments of the motion. At first sight, the dynamics of the first-order resonances is even simpler. After all, Poincaré's representation of the motion near the 2/1 resonance includes only a single term in the disturbing function. However, the real situation is much more complicated. Poincaré's problem is integrable; yet from Giffen's work it is already known that in the planar elliptic problem there is chaotic behavior near the 2/1 commensurability. It might have been hoped that the essence of the 2/1 dynamics would be captured in the only slightly more complicated problem which includes both terms in the disturbing function proportional to a single power of the eccentricity. It turns out though that this problem is also integrable (Sessin and Ferraz-Mello 1984, Wisdom 1986, Henrard *et al.* 1987). To achieve a representation of the resonance which encompasses the observed chaotic behavior one must evidently include higher-order terms.

Chaotic behavior begins to appear when those terms which are second order in eccentricity terms are included. Murray (1986) derived a map for the 2/1 and 3/2 resonances which included the terms through second order, following the earlier derivation of the 3/1 map. A systematic exploration of the 2/1 and 3/2 resonances with this map revealed large chaotic zones at both resonances. The second-order map for the 2/1 resonance, however, gave unrealistically large eccentricity jumps ($e > 0.8$) for initial eccentricities above 0.1, and the large chaotic zone at the 3/2 resonance is questionable purely on the physical grounds that the Hildas are there. Indeed,

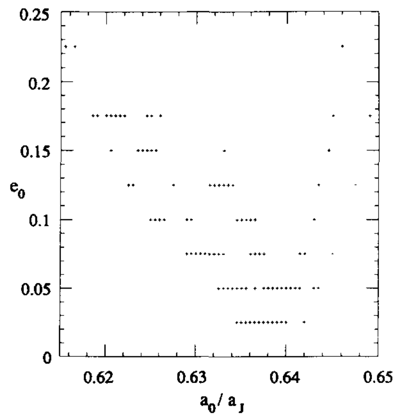

FIG. 15. Chaotic trajectories near the 2/1 commensurability in the planar elliptic approximation. A + marks those initial conditions (eccentricity e, and semimajor axis a, referred to Jupiter's semimajor axis) which lead to chaotic behavior. The survey is not complete, but the basic extent of the chaotic zone is apparent. There is a significant chaotic zone.

the large eccentricity increases and the 3/2 chaotic zone were an artifact of the truncation of the disturbing function to terms of second order. It must be emphasized that these artifacts were not due to the mapping method itself; rather, they were due to the fact that the problem is not well represented by a low-order disturbing function. Henrard *et al.* (1987) showed that a good representation of the motion is obtained only when terms of very high order are included. An analytic description is thus extremely difficult, and a map cannot be made for such a high-order disturbing function in the same way as the 3/1 map was derived.

The only alternative appears to be direct numerical integration. Numerical explorations have previously been performed by Froeschlé and Scholl (1976, 1981), but these studies suffer from the same limitation as their numerical explorations of the other gaps. Primarily, the integrations were too short; they typically spanned only 12,000 years. The integrations of Froeschlé and Scholl did find chaotic behavior, but the extent of the chaotic zone which the short integrations were able to detect was small. A new numerical exploration must be undertaken which spans a greater time interval.

The structure of the phase space near the principal commensurabilities is currently being explored with the Digital Orrery. The preliminary results of this survey are quite interesting. Figure 15 shows the chaotic zone near the 2/1 commensurability. In this figure a + indicates initial conditions (eccentricity e and semimajor axis a, referred to Jupiter's semimajor axis) under which a trajectory of the planar elliptic problem is chaotic. All of the trajectories have initial mean longitude equal to Jupiter's mean longitude and initial longitude of perihelion equal to Jupiter's longitude of perihelion. The integrations spanned 100,000 years each. The basic grid of initial conditions is clear from the figure. Though the exploration is not yet complete, the outline of the chaotic region is probably well represented. This figure indicates a fairly sizable chaotic zone near the 2/1 commensurability. At eccentricities below 0.1 the chaotic zone is in good agreement with the chaotic zone found with the second-order map. The corresponding plot for the region near the 3/2 commensurability is shown in Fig. 16. The nominal location of the 3/2 resonance is at a/a_J = 0.76. Note that this region is devoid of

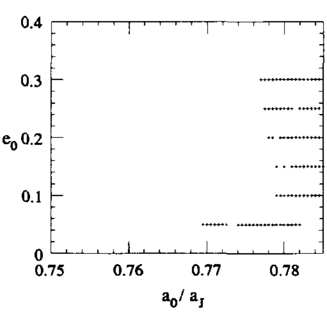

FIG. 16. Chaotic trajectories near the 3/2 commensurability in the planar elliptic approximation. A + marks the chaotic orbits. The resonance region near a/a_J = 0.76 is devoid of chaotic trajectories.

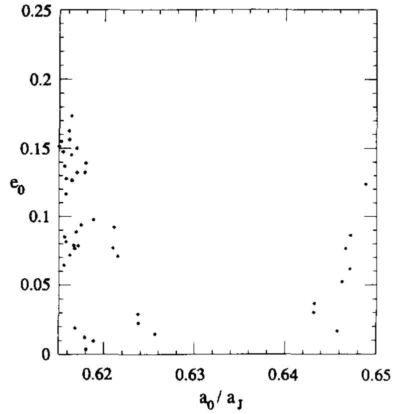

FIG. 17. Actual distribution of asteroids near the 2/1 commensurability, each evolved to the same longitudes used in the survey. There is a good qualitative agreement between the gap and the region of chaotic behavior shown in Fig. 15. The large semimajor axis boundaries are in excellent agreement, while there seems to be a discrepancy for the boundaries on the low semimajor axis side.

chaotic behavior. There is thus a *qualitative* difference in the structure of the phase space near the 2/1 and 3/2 commensurabilities which corresponds to the qualitative difference in the observed distribution of asteroids. There is a large chaotic zone at the 2/1 resonance and there is also a Kirkwood gap at that resonance. The Hildas are located near the 3/2 resonance and this region is devoid of chaotic behavior.

While the qualitative agreement just noted is remarkable, a quantitative comparison with the observed distribution does not show perfect agreement. Figure 17 shows the actual distribution of numbered asteroids near the 2/1 resonance, each evolved to a point where the longitudes nearly align with the longitudes of the initial conditions on the survey. (Without this correction a direct comparison to the observed distribution is not reasonable, nor are the boundaries of the observed distribution sharp. So far the correction has been done only with the second-order map, but a few test cases have shown that the correction is accurate.) A comparison with the chaotic zone in Fig. 15 shows perfect agreement on the high semimajor axis side of the gap. However, the agreement on the other side is not as good. There is a region at low eccentricity ($e \approx 0.05$) which contains no asteroids and also does not show chaotic behavior in the planar elliptic problem. Does this discrepancy indicate that a cosmogonic mechanism has operated? It is still too early to jump to this conclusion.

There are a number of possible explanations for the discrepancy. The survey was carried out in the context of the planar elliptic problem. A more limited survey in the three-dimensional elliptic problem does not change the conclusion. It is still possible that the variations of Jupiter's orbit resulting from the perturbations of the other planets are important. To determine whether this could be the source of the discrepancy, three integrations were carried out with the Digital Orrery of test particles placed in the discrepant region moving in the field of the four major planets. The initial semimajor axes and eccentricities were (0.624, 0.075), (0.626, 0.075), and (0.628, 0.05). The first two of these appeared to have regular trajectories, but the third was clearly chaotic. The fact that one of the test trajectories was chaotic indicates that the discrepancy may indeed merely reflect the inadequacy of the planar elliptic approximation used in the survey. Further investigation is warranted.

To complete the picture the uncorrected distribution of Hilda asteroids is shown in Fig. 18. The phase space under the Hildas is quasiperiodic, as might have been expected. One might note though that the Hildas are somewhat more bunched than seems to be required by the extent of the quasiperiodic region. Perhaps this again reflects the inadequacy of the planar elliptic approximation. This also warrants further investigation.

The association of chaotic behavior with a gap in the distribution of asteroids does not by itself explain the formation of the gap. For the 3/1 resonance an essential

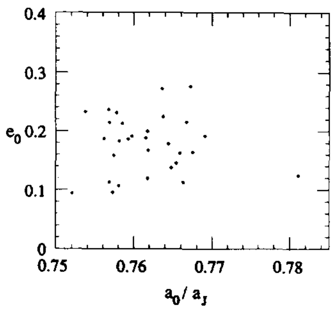

FIG. 18. The raw distribution of Hilda asteroids. The region in which they are found is devoid of chaotic behavior in the planar elliptic problem (cf. Fig. 16).

ingredient was the fact that chaotic trajectories, as well as resonant quasiperiodic trajectories, crossed the orbits of Mars and Earth. The sweeping action of these planets can clear the 3/1 gap. It seems a priori unlikely that a similar mechanism can account for the formation of those gaps which are significantly more distant from Mars and Earth. There is a large chaotic zone near the 2/1 resonance which at least qualitatively corresponds to the 2/1 Kirkwood gap. What is it about chaotic trajectories that leads to the demise of those unfortunate asteroids which follow them? As is always the case, the first step is simply to determine the nature of the motion. Froeschlé and Scholl (1976, 1981) attempted to answer this question. They performed 100,000-year integrations, both averaged and unaveraged, of Giffen's chaotic trajectory near the 2/1 commensurability, in the planar elliptic approximation. They found that the chaotic zone in which Giffen's trajectory moved was constrained to eccentricities below about 0.15. This is certainly not a catastrophic eccentricity, and in fact it is typical of the asteroids as a whole! The clear association of a chaotic zone with the gap requires a reevaluation of this problem. Evidently, the planar elliptic problem is not an accurate enough representation of the dynamics. The Digital Orrery is being brought to bear on this problem. Five test particles with initial conditions in the 2/1 chaotic zone were numerically integrated with the Digital Orrery for 5 my each. The test particles were allowed full three-dimensional motion and perturbations of the four major planets were taken into account. The initial eccentricities were in each case 0.1 or 0.05. The maximum eccentricities reached by each of the test particles were 0.38, 0.185, 0.23, 0.48, and 0.53. Interestingly enough the second trajectory remained stuck in the discrepant region for 4 my. The last two trajectories showed the remarkable behavior shown in Fig. 19. This rather messy plot shows the eccentricity of the fifth test particle plotted against its inclination. In the beginning while the inclination is low the eccentricity seems to be limited to values below about 0.25, just as Froeschlé and Scholl found in the planar elliptic problem. Over the span of the integration though the trajectory seems to trace out a pathway to high eccentricity which temporarily takes it through inclinations as

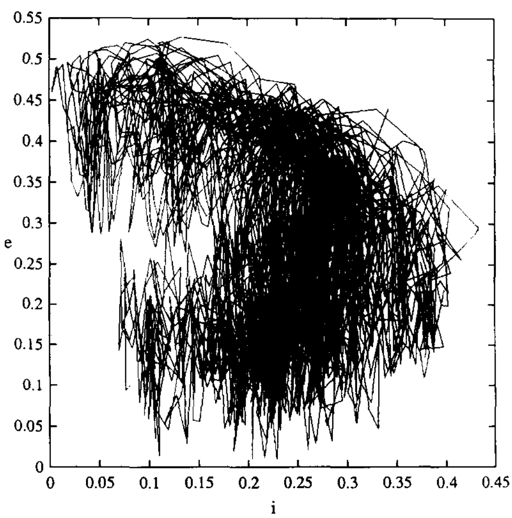

FIG. 19. Chaotic trajectories of test particles perturbed by the Jovian planets show this remarkable correlation between eccentricity and inclination. There is a path in the phase space which takes the trajectory from low eccentricity to high eccentricity which requires that it temporarily take moderate inclination. The three-dimensional nature of the motion is crucial.

high as 0.44 radians. Thus the three-dimensional nature of the motion is crucial. (Froeschlé and Scholl (1983) also found large eccentricities in the three-dimensional problem, but the trajectories in their study were begun with unphysically large inclinations.) At the peak in eccentricity the fifth trajectory is marginally Mars crossing. This does provide a mechanism for the removal of the asteroids on chaotic trajectories, but I am not convinced this is the final answer.

The numerical integrations on the Orrery have so far all used the 12th-order Stormer predictor as the numerical integration algorithm. This integrator has excellent stability properties at low eccentricity. It is the integrator of choice for the integration of planetary orbits. At high eccentricities though the Stormer method has markedly poorer stability. As the eccentricity of the orbit is increased, the step length above which the integrator is unstable becomes shorter and shorter. The integration of the test particles described above was carried out with a step size of 1 day, which should allow the stable integration of orbits with eccentricities below 0.6. The trajectory in Fig. 19 is thus on the borderline of believability. More work is required to verify the phenomenon.

OUTER PLANETS AND PLUTO

The determination of the stability of the solar system is one of the oldest problems in dynamical astronomy. While Arnold's proof of the stability of a large measure of solar systems with sufficiently small planetary masses, eccentricities, and inclinations marks tremendous progress toward a rigorous answer to this question, the stability of the actual solar system remains unknown. Certainly, the great age of the solar system demands a high level of stability, but weak instabilities may still be present. Experience with the motion of asteroids has demonstrated that weak instabilities may sometimes even lead to sudden, dramatic changes in orbits. The stability of the solar system should thus not be taken for granted. As with most other dynamical systems, the stability of the solar system is still best examined through numerical experiment.

The first application of the Digital Orrery was to the long-term evolution of the outer planets. For many years the million-year integration of Cohen *et al.* (1973) held the title of the longest integration of the solar system. With the Orrery, the interval of integration has been extended to 210 my (Applegate *et al.* 1986). Only the principal results of these integrations will be mentioned.

The integrations showed that the best analytic approximations of the motion of the outer planets were in serious need of improvement. Bretagnon (1974) listed over 200 corrections to the Lagrange solutions. It turns out that there are other contributions to the motion of the Jovian planets which are larger than all but 7 of those corrections! Higher-order terms are more important than the terms taken into account by Bretagnon. The construction of approximate analytic theories is quite a difficult undertaking even with the aid of computer algebra. More recent work of Duriez (1979) and Bretagnon (1982) has improved the analytic theory. The new semianalytic theory of Laskar (1986, and recent unpublished work) is in particularly good agreement and provides independent confirmation of the numerical results. The motions of the Jovian planets are surprisingly complicated. A sample power spectrum of Jupiter is shown in Fig. 20. Here at last are the true harmonies of this world. For a discussion of the numerical errors see Applegate *et al.* (1986).

The motion of Pluto is extraordinarily complicated. Pluto's orbit is unique among the planets. It is both highly eccentric ($e \approx 0.25$) and highly inclined ($i \approx 16°$). The orbits of Pluto and Neptune cross one another, a condition which is only permitted by the libration of a resonant argument associated with the 3/2 mean-motion commensurability. This resonance assures that

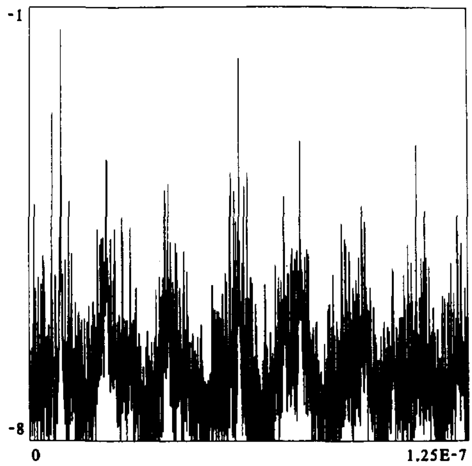

FIG. 20. The power spectrum of Jupiter. The logarithm of the amplitude in the numerically determined spectrum of $e \cos \tilde{\omega}$ is plotted against frequency (in cycles per day).

Pluto is at aphelion when Pluto and Neptune are in conjunction and thus prevents close encounters. The dynamics of such a resonance is well understood, however, being again just an application of Poincaré's problem! The real complexity of Pluto's orbit is appreciated only when it is studied over time intervals much greater than the period of libration of the resonant argument, which is about 19,857 years. The next level of complexity, that the argument of perihelion of Pluto librates about $\pi/2$ with a period near 3.8 my, was discovered by Williams and Benson (1971), who studied the evolution of Pluto with a numerical averaging scheme similar to that used by Schubart. The Orrery integrations confirmed this libration but found that the picture was not yet complete. The amplitude of libration of the argument of perihelion is modulated with a period of 34 my. The appearance of this new period, however, is not surprising. After averaging over the resonance timescale the long-period evolution of Pluto is governed by a Hamiltonian with two degrees of freedom. Quasiperiodic trajectories of the long-period problem are thus expected to have two independent frequencies. Without the modulation there would be only a single frequency. The really surprising result is that much longer periods are present in the orbital elements of Pluto. In fact the frequency of the second-largest contribution to the eccentricity of Pluto corresponds to a period of 137 my. This small frequency results from a near commensurability between the frequency of circulation of the longitude of the ascending node of Pluto and one of the fundamental frequencies in the motion of the Jovian planets. This is surprising enough but the motion of Pluto presents even more surprises. Figure 21 shows the inclination of Pluto over 214 my. There seems to be an even longer period present, or perhaps even a secular drift! In dynamical systems the most prominent zones of chaotic behavior are usually associated with resonances, though of course a resonance need not lead to irregular behavior. The motion of Pluto is suspiciously rich in resonances; a new resonance has been discovered each time the interval of integration has been extended. It should be recalled that only three resonances are re-

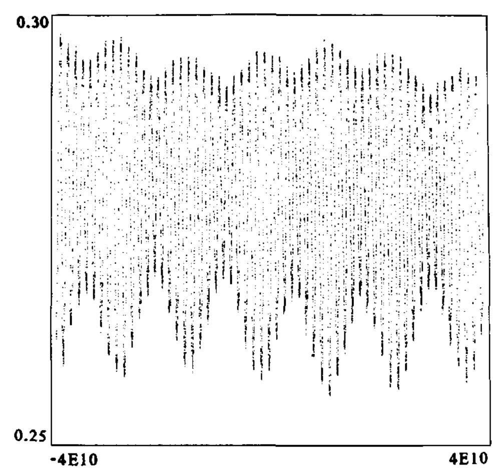

FIG. 21. The inclination (in radians) of Pluto for 214 my. Time (on the abscissa) is given in days. Besides the 34-my modulation of the 3.80-my oscillation, there is evidence of much longer period variations or perhaps even a secular drift.

quired to give rise to the multidimensional instability called Arnold diffusion (see Chirikov 1979). However, the computation of the Lyapunov characteristic exponent for Pluto does not yet show any objective evidence for chaotic behavior. Much longer integrations seem to be required to determine the true nature of Pluto's motion.

SUMMARY AND CONCLUSION

It is no longer possible to investigate the motion of the planets and their satellites without being fully conscious of the possibilities introduced by the recent discoveries of nonlinear dynamics. Foremost among these is the remarkable similarity in the behavior of diverse Hamiltonian systems. The phase space of most Hamiltonian systems is divided; for some initial conditions the trajectories are chaotic; while for other initial conditions the trajectories are quasiperiodic. Chaotic behavior is a ubiquitous feature in Hamiltonian systems. The solar system must be recognized for what it is, just another dynamical system, and as such, the discovery that chaotic behavior plays a role in numerous situations in the solar system should come as no surprise.

Several physical examples of chaotic behavior in the solar system have been presented. Hyperion most likely tumbles irregularly as a consequence of its out-of-round shape, large orbital eccentricity, and tidally evolved rotation. Hyperion is currently the only example of this chaotic tumbling in the solar system. However, *all* of the tidally evolved, irregularly shaped satellites in the solar system tumbled chaotically in the past. The synchronous spin–orbit state, the state in which all tidally evolved satellites in the solar system are found today, is surrounded by a chaotic zone. This chaotic zone is sizable if the body is out-of-round, even if the orbital eccentricity is relatively small. This chaotic separatrix is attitude unstable. Thus as a satellite enters the chaotic zone through the action of the tides, the spin axis falls away from the orbit normal and the satellite tumbles chaotically. The time spent in the chaotic tumbling state is unknown but is probably comparable to the timescale for tidal evolution to synchronous rotation. While the tumbling episode occurred for a number of satellites, there are no firm physical consequences for the satellites, though some possibilities were mentioned. The enhanced dissipation in a tumbling satellite, however, may have a significant effect on the orbital evolution of the satellite. The anomalously low eccentricity of Deimos may be a result of a period of chaotic tumbling which lasted on the order of 100 my.

Physical examples of chaotic orbital behavior have also been presented. The distribution of asteroids seems to be, in several instances, a reflection of the character of the trajectories in the underlying phase space. This is clearly the case for the 3/1 Kirkwood gap. There is a sizable chaotic zone and the phase space boundary of the distribution of asteroids corresponds quite well with the outer boundary of the chaotic zone. In this case, the fact that both chaotic and quasiperiodic trajectories cross planetary orbits explains the removal of any asteroids originally in the gap. Taking into account the perturbations of the outer planets and allowing three-dimensional motion, trajectories in the 3/1 chaotic zone reach Earth-crossing eccentricities. These trajectories thus provide the long-sought dynamical route for the transport of meteoritic material from the asteroid belt to Earth. Studies by Wetherill have shown this source to be consistent with the ordinary chondrite data.

The 2/1 gap and the Hilda group have long presented a paradox to classical dynamical astronomy. Why is there a paucity of asteroids at one resonance and a surplus at another resonance? The phase space near the 2/1 resonance is largely chaotic, while the phase space near the Hildas is quasiperiodic. The qualitative difference in the distributions is reflected in the qualitative difference in the underlying dynamics.

Where the phase space is chaotic there are no asteroids, and where it is quasiperiodic asteroids are found. An analytic explanation for this qualitative difference in the dynamics is not yet known. In detail the distribution of asteroids does not agree precisely with the boundaries of the chaotic zones. This is probably a result of the approximations involved in the systematic survey. Chaotic behavior resulted from a more realistic integration of a test particle started in one of the regions of discrepancy. The fate of asteroids on chaotic trajectories is not yet known. In two integrations of chaotic test particles perturbed by the Jovian planets the eccentricity and inclination showed an interesting coupling. When the inclination was low the eccentricity was limited to values below 0.25, but there exists a path to high eccentricity and low inclination which temporarily takes the trajectory through relatively high inclination. Thus the three-dimensional aspect of the trajectories is essential. The high eccentricities obtained are marginally Mars crossing. This provides a possible mechanism for the removal of material in the 2/1 chaotic zone, but this may not be the final word on the origin of this gap.

The stability of the solar system itself has been examined through a 210-my integration of the outer planets. The motion of the Jovian planets themselves seems to be regular, though perhaps a bit more complicated than might have been expected. On the other hand, the motion of Pluto is extraordinarily complicated. Besides the well-understood mean-motion resonance which prevents the close approach of Pluto and Neptune even though their orbits cross, Pluto participates in at least two other resonances. First, it has been known for some time that the argument of perihelion librates about $\pi/2$. Then, the frequency of the circulation of Pluto's ascending node is nearly commensurate with one of the fundamental frequencies in the motion of the Jovian planets. This near commensurability gives rise to strong variations in the eccentricity with a 137-my period. There is also evidence of much longer periods in the inclination, which appears to be secularly declining over the 210-my integration. While the abundance of resonances raises suspicions about the stability of Pluto, there is not yet any objective evidence that the motion of Pluto is chaotic.

In the early part of this century, classical mechanics was eclipsed by quantum mechanics, and rightly so. Quantum mechanics provides a better description of the world. However, the relevance of classical mechanics is once again beginning to be appreciated. For the most part, the world of our everyday lives is classical, and classical mechanics is not at all simple. Newton could not have dreamt of the beauty and complexity of the mechanics that he brought forth. The final state of Hyperion is completely unpredictable. Apparently, even in a classical world God does, after all, play dice.

ACKNOWLEDGMENT

This research was supported in part by the NASA Planetary Geology and Geophysics Program under Grant NAGW-706.

REFERENCES

Andersson, L. E. 1974. *A Photometric Study of Pluto and Satellites of the Outer Planets*. Ph.D. thesis, University of Indiana, Bloomington.

Applegate, J. F., M. R. Douglas, Y. Gursel, P. Hunter, C. Seitz, and G. J. Sussman 1985. A digital orrery. *IEEE Trans. Comput.*, Sept.

Applegate, J. F., M. R. Douglas, Y. Gursel, G. J. Sussman, and J. Wisdom 1986. The outer solar system for 200 million years. *Astron. J.* **92,** 176–194.

Arnold, V. I. 1961. Small denominators and the problem of stability in classical and celestial mechanics. In *Report to the IVth All-Union Mathematical Congress, Leningrad,* pp. 85–191.

Arnold, V. I. 1974. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York.

Arnold, V. I., and A. Avez 1968. *Ergodic Problems of Classical Mechanics*. Benjamin, New York.

Benettin, G., M. Casartelli, L. Galgani, A. Giorgilli, and J-M. Strelcyn 1978. On the reliability of numerical studies of stochasticity. *Nuovo Cimento* **44,** 183–195.

Binzel, R., J. Green, and C. Opal 1986. Chaotic rotation of Hyperion? *Nature* **320,** 511.

BRETAGNON, P. 1974. Termes a longues périodes dans le système solaire. *Astron. Astrophys.* **30,** 141–154.

BRETAGNON, P. 1982. Théorie du mouvement de l'ensemble des planètes. Solution VSOP82. *Astron. Astrophys.* **114,** 278–288.

BURNS, J. A., AND V. S. SAFRONOV 1973. Asteroid mutation angles. *Mon. Not. Roy. Astron. Soc.* **165,** 403–411.

CHANNON, S. R., AND J. I. LEBOWITZ 1980. Numerical experiments in stochasticity and homoclinic oscillation. In *Nonlinear Dynamics* (R. Helleman, Ed.), pp. 108–118. New York Academy of Sciences, New York.

CHIRIKOV, B. V. 1979. A universal instability of many dimensional oscillator systems. *Phys. Rep.* **52,** 263–379.

COHEN, C. J., E. C. HUBBARD, AND C. OESTERWINTER 1973. Elements of the outer planets for one million years. In *Astronomical Papers of the American Ephemeris,* Vol. XXII, Part I.

CONNER, S. R. 1984. *Photometry of Hyperion.* Masters thesis, Massachusetts Institute of Technology, Cambridge, MA.

DARWIN, G. 1879. On the bodily tides of viscous and semi-elastic spheroids. *Philos. Trans.,* Part I.

DURIEZ, L. 1979. *Approche d'une théorie générale planétaire en variables elliptiques héliocentric.* Thesis, Lille, France.

DUXBURY, T. C., AND J. D. CALLAHAN 1982. Phobos and Deimos cartography. *Lunar Planet. Abstr.* **13,** 190.

FERMI, E. 1923. Reprinted in *Collected Works of Enrico Fermi.* Univ. of Chicago Press, Chicago, 1965.

FROESCHLÉ, C., AND H. SCHOLL 1976. On the dynamical topology of the Kirkwood gaps. *Astron. Astrophys.* **48,** 389–393.

FROESCHLÉ, C., AND H. SCHOLL 1979. New numerical experiments to deplete the outer asteroid belt. *Astron. Astrophys.* **72,** 246.

FROESCHLÉ, C., AND H. SCHOLL 1981. The stochasticity of peculiar orbits in the 2/1 Kirkwood gap. *Astron. Astrophys.* **93,** 62–66.

FROESCHLÉ, C., AND H. SCHOLL 1983. A systematic exploration of three-dimensional asteroid motion at the 2/1 resonance. In *Asteroids, Comets, Meteors* (C. I. Lagerkvist and H. Rickmann, Eds.), Uppsala University.

GIFFEN, R. 1973. A study of commensurable motion in the solar system. *Astron. Astrophys.* **23,** 387–403.

GOGUEN, J. 1983. Paper presented at IAU Colloquium 77, Natural Satellites, Ithaca, NY, July 5–9.

GOLDREICH, P. 1963. On the eccentricity of satellite orbits in the solar system. *Mon. Not. Roy. Astron. Soc.* **126,** 257–268.

GOLDREICH, P., AND S. J. PEALE 1966. Spin–orbit coupling in the solar system. *Astron. J.* **71,** 425–438.

HÉNON, M. 1966a. Exploration numérique du problème restreint. III. Masses égales, orbites non périodiques *Bull. Astron. Paris* **1,** fasc. 1, 57.

HÉNON, M. 1966b. Exploration numérique du problème restreint. IV. Masses égales, orbites non périodiques *Bull. Astron. Paris* **1,** fasc. 2, 49.

HÉNON, M. 1970. Numerical exploration of the restricted problem. VI. Hill's case: Nonperiodic orbits. *Astron. Astrophys.* **9,** 24–36.

HÉNON, M., AND C. HEILES 1964. The applicability of the third integral of motion: Some numerical experiments. *Astron. J.* **69,** 73–79.

HENRARD, J., A. LEMAITRE, A. MILANI, AND C. D. MURRAY 1987. The reducing transformation and apocentric librators. Submitted for publication.

JEFFERYS, W. H. 1966. Some dynamical systems of two degrees of freedom in celestial mechanics. *Astron. J.* **71,** 306–313.

LASKAR, J. 1986. Secular terms of classical planetary theories using the results of general theory. *Astron. Astrophys.* **157,** 59–70.

LECAR, M., AND F. FRANKLIN 1973. On the original distribution of the asteroids, I. *Icarus* **73,** 422–436.

LORENZ, E. N. 1963. Deterministic non-periodic flow. *J. Atmos. Sci.* **20,** 130–141.

MCFADDEN, L. A., AND F. VILAS 1987. The 3:1 Kirkwood gap as sources of ordinary chondrites: Perspectives from spectral reflectance. Paper presented at the Lunar and Planetary Science Conference, March 1987.

MURRAY, C. D. 1986. Structure of the 2:1 and 3:2 Jovian resonances. *Icarus* **65,** 70–82.

MURRAY, C. D., AND K. FOX 1984. Structure of the 3:1 Jovian resonance: A comparison of numerical methods. *Icarus* **59,** 221–233.

PEALE, S. J. 1977. Rotation histories of the natural satellites. In *Planetary Satellites* (J. Burns, Ed.), pp. 87–112. Univ. of Arizona Press, Tucson.

PETTENGILL, G. H., AND R. B. DYCE 1965. A radar determination of the rotation of the planet Mercury. *Nature* **206,** 1240.

POINCARÉ, H. 1892. *Les méthodes nouvelles de la méchanique céleste.* Gauthier–Villars, Paris.

POINCARÉ, H. 1902. Sur les planètes du type d'Hecube. *Bull Astron.* **19,** 289.

SCHOLL, H., AND C. FROESCHLÉ 1974. Asteroidal motion at the 3/1 commensurability. *Astron. Astrophys.* **33,** 455–458.

SCHUBART, J. 1964. Long period effects in nearly commensurable cases of the restricted three-body problem. *Smithsonian Astrophys. Spec. Rep. No. 149.*

SESSIN, W., AND S. FERRAZ-MELLO 1984. Motion of two planets with periods commensurable in the ratio 2:1. Solution of the Hori auxiliary problem. *Celest. Mech.* **32,** 307–332.

SMITH, B., *et al.* 1982. A new look at the Saturn system: The Voyager 2 images. *Science* **215,** 504–537.

THOMAS, P., *et al.* 1984. Hyperion: 13-Day rotation from Voyager data. *Nature* **307,** 716–717.

THOMAS, P., AND J. VEVERKA 1985. Hyperion: Analysis of Voyager observations. *Icarus* **64,** 414–424.

UREY, H. C., W. M. ELSASSER, AND M. G. ROCHESTER 1959. Note on the internal structure of the Moon. *Astrophys. J.* **129,** 842–848.

WETHERILL, G. W. 1968. Stone meteorites: Time of fall and origin. *Science* **159,** 79–82.

WETHERILL, G. W. 1985. Asteroidal source of ordinary chondrites. *Meteoritics* **18,** 1–22.

WILLIAMS, J. G., AND G. S. BENSON 1971. Resonances in the Neptune–Pluto system. *Astron. J.* **76,** 167–177.

WISDOM, J. 1980. The resonance overlap criterion and the onset of stochastic behavior in the restricted three-body problem. *Astron. J.* **85,** 1122–1133.

WISDOM, J. 1982. The origin of the Kirkwood gaps: A mapping for asteroidal motion near the 3/1 commensurability. *Astron. J.* **87,** 577–593.

WISDOM, J. 1983. Chaotic behavior and the origin of the 3/1 Kirkwood gap. *Icarus* **56,** 51–74.

WISDOM, J. 1985a. Meteorites may follow a chaotic route to Earth. *Nature* **315,** 731–733.

WISDOM, J. 1985b. A perturbative treatment of motion near the 3/1 commensurability. *Icarus* **63,** 272–289.

WISDOM, J. 1986. Canonical solution of the two critical argument problem. *Celest. Mech.* **38,** 175–180.

WISDOM, J. 1987. Rotational dynamics of irregularly shaped satellites. Submitted for publication.

WISDOM, J., AND S. J. PEALE 1984. Do current observations support the hypothesis of the chaotic rotation for Hyperion? Paper presented at the Division for Planetary Sciences, 1984, Kona, Hawaii, Oct. 8–13.

WISDOM, J., S. J. PEALE, AND F. MIGNARD 1984. The chaotic rotation of Hyperion. *Icarus* **58,** 137–152.

YODER, C. F. 1982. Tidal rigidity of Phobos. *Icarus* **49,** 327–346.