Vector Calculus - Sample Final Exam

This would typically be a two-hour exam.

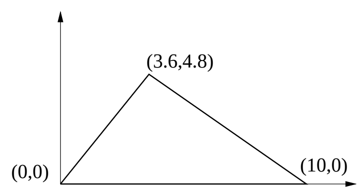
1. (a) Describe the graph of the function $f(x, y) = \sqrt[4]{x^2 + y^2}$. This means sketch it if you can, and you should probably compute some level sets and cross sections.

   (b) Write down the equation for the tangent plane to this graph at the point $(3, 4, \sqrt{5})$.

   (c) Consider the three-dimensional region which is bounded below by this graph and above by the disk $x^2 + y^2 \leq 5$, $z = 4$. Write down a formula for its volume—you *don't* have to compute the integral.

2. (a) State the formula for Green's Theorem.

   (b) Evaluate the integral $\int_C y\,dx - x\,dy$, where C is the boundary of the triangle shown below. (*Hint*: Use (a), together with some common sense.)



   (c) Let R be the region in the plane given by $0 \leq x \leq 1$, $x^2 \leq y \leq x$. Sketch a graph of the region, and evaluate the integral $\int_C x^2\,dx - xy\,dy$, where C is the boundary of R.

3. A metallic wire is shaped in the form of the path $\sigma(t) = \left(t, t^2, \frac{4}{3}t^{\frac{3}{2}}\right)$, $0 \leq t \leq 1$.

   (a) Find the length of the wire.

   (b) If the density of the wire at the point (x, y, z) is given by $\rho(x, y, z) = xy + z^2$, compute the mass of the wire.

   (c) Compute the x-coordinate for the *center of mass* of the wire.

4. Consider a force field in space given by $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + 2xy\,\mathbf{j} + 2z\,\mathbf{k}$.
   (a) Compute the work required to move a particle along the parabola $z = y^2$,$x = 0$ from the point $(0, -1, 1)$ to $(0, 2, 4)$.
   (b) Compute both the divergence and curl of $\mathbf{F}$ at the point (x, y, z).
   (c) A particle sits at the point $(1, 0, 0)$. Compute the total work required to move it in a circular path of radius one about the origin, back to its starting point. Stokes' Theorem might be helpful.

5. A beetle flies around in a circle of radius 3 meters, moving clockwise and making one revolution every 4 seconds (this is one really rapid beetle).
   (a) Write down a formula for the function $\sigma : \mathbf{R} \to \mathbf{R}^2$ which will describe the beetle's position in the x-y plane as a function of time. Take the origin of coordinates to be the center of the circle, and take the beetle's coordinates to be $(3, 0)$ at $t = 0$.
   (b) If the temperature at the point (x, y) is given by $T(x, y) = e^{xy} + x \cos y$, figure out how quickly the beetle feels the temperature changing at time $t = 1\,s$. (*Hint*: Use the chain rule.)
   (c) If the beetle leaves the circle at time 2.5 s and flies off in the tangent direction without changing speed, where will he be four seconds later?