Circles

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1 Warm up problems

1. Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Prove that triangles OAB and OCD are similar.

2. (Miquel's theorem) Let ABC be a triangle. Points X, Y, and Z lie on sides BC, CA, and AB, respectively. Prove that the circumcircles of triangles AYZ, BXZ, CXY meet at a common point.

3. (Also Miquel's theorem) Let a, b, c, d be four lines in space, no two parallel, no three concurrent. Let ω_a denote the circumcircle of the triangle formed by lines b, c, d. Similarly define $\omega_b, \omega_c, \omega_d$.
   (a) Prove that $\omega_a, \omega_b, \omega_c, \omega_d$ pass through a common point. This is called the Miquel point.
   (b) Continuing the above notation, prove that the centers of $\omega_a, \omega_b, \omega_c, \omega_d$, and the Miquel point all lie on a common circle.

4. (Simson line) Let ABC be a triangle, and let P be another point on its circumcircle. Let X, Y, Z be the feet of perpendiculars from P to lines BC, CA, AB respectively. Prove that X, Y, Z are collinear.

5. Let $\angle AOB$ be a right angle, M and N points on rays OA and OB, respectively. Let $MNPQ$ be a square such that MN separates the points O and P. Find the locus of the center of the square when M and N vary.

6. Let $ABCD$ be a convex quadrilateral such that the diagonals AC and BD are perpendicular, and let P be their intersection. Prove that the reflections of P with respect to AB, BC, CD, DA are concyclic.

7. An interior point P is chosen in the rectangle $ABCD$ such that $\angle APD + \angle BPC = 180°$. Find $\angle DAP + \angle BCP$.

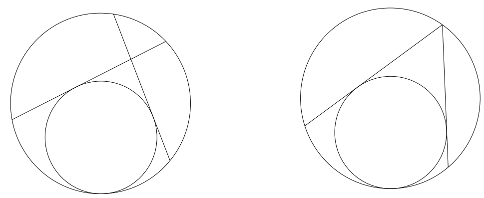
8. Let AB be a chord in a circle and P a point on the circle. Let Q be the projection of P on AB and R and S the projections of P onto the tangents to the circle at A and B. Prove that $PQ^2 = PR \cdot PS$.

9. Let ABC be an acute triangle. The points M and N are taken on the sides AB and AC respectively. The circles with diameters BN and CM intersect at points P and Q. Prove that P, Q, and the orthocenter H are collinear.

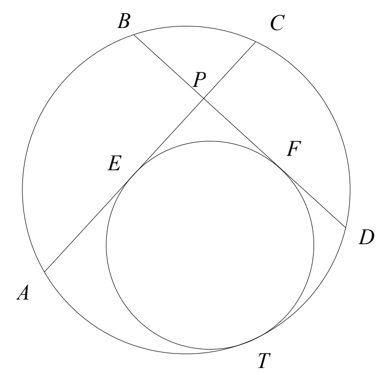
10. Among the points A, B, C, D no three are collinear. The lines AB and CD intersect at E, and BC and DA intersect at F. Prove that either the circles with diameters AC, BD, EF pass through a common point, or no two of them have any common point. (The line through the midpoints of AC, BD, EF is called the *Newton-Gauss* line.)

2 Tangent circles

In this section, we explore the following two configurations.[1]



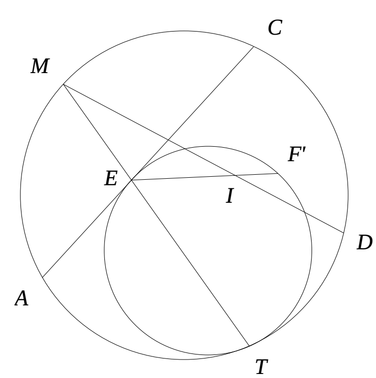
1. Let chords AC and BD of a circle ω intersect at P. A smaller circle ω_1 is tangent to ω at T and to segments AP and DP at E and F respectively.



   (a) Prove that ray TE bisects arc ABC of ω.

   (b) Let I be the incenter of triangle ACD and M be the midpoint of arc ABC of ω. Prove that $MA = MI = MC$.

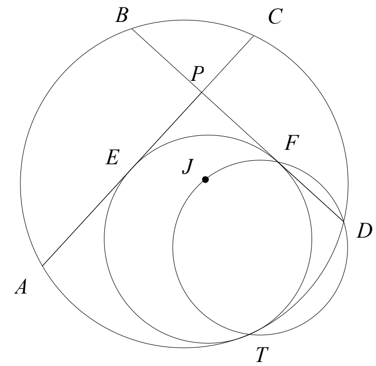
   (c) Let F' be the common point of ω_1 and line EI other than E. Prove that I, F', D, T are concyclic.



   (d) Prove that DF' is tangent to ω_1. This means that $F = F'$, so that E, F, I are collinear. (Remember this fact.)

   (e) Let J be the incenter of triangle APD. Prove that T, D, F, J are concyclic.

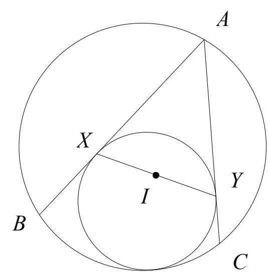
[1] Thanks to Oleg Golberg for providing some of these problems.



(f) Prove that ray TJ bisects angle ATD.

2. Now we consider a special case of the part (d) in the previous problem. Try to find a short proof of the following result. (Hint: use Pascal's theorem)

   Let ABC be a triangle and I its incenter. Let Γ be the circle tangent to sides AB, AC, as well as the circumcircle of ABC. Let Γ touch AB and AC at X and Y, respectively. Show that I is the midpoint of XY.



3. Let Ω be the circumcircle of ABC. A circle ω is tangent to sides AB, AC and circle Ω at points X, Y and Z, respectively. Let M be the midpoint of the arc BC of Ω which does not contain A. Prove that lines XY, BC and ZM have a common point.

   (Can you prove the result when Ω only passes through B and C and contains A in the interior?)

4. A circle ω is tangent to sides AB, AC of triangle ABC and to its circumcircle at points X, Y and Z. Segments AZ and XY meet at T. Prove that $\angle BTX = \angle CTY$.

5. (Sawayama-Thébault) Let ABC be a triangle with incenter I. Let D a point on side BC. Let P be the center of the circle that touches segments AD, DC, and the circumcircle of ABC, and let Q be the center of the circle that touches segments AD, BD, and the circumcircle of ABC. Show that P, Q, I are collinear.

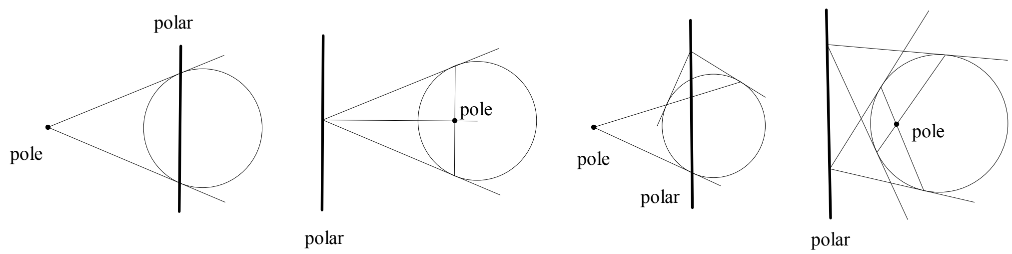
3 Projective quickies

Warning: The introductory text in this section is very rough. It is not meant for you to learn from. Rather, I assume that you already know the basics of projective geometry, and this should serve as a quick review of the important concepts.

You should be familiar with the notion of *pole* and *polar*. A quick definition is as follows:

Definition. Suppose that ω is a circle with center O, and P and P' are inverses respect to ω (i.e., P' lies on ray OP such that $OP \cdot OP' = r^2$, where r is the radius of ω). Let ℓ be the line through P' perpendicular to OP. Then we say that ℓ is the *polar* of P and P is the *pole* of ℓ.

Here are a few diagrams that may help you to remember the common setups of poles and polars.



Make sure that you understand the *duality* behind poles and polars. If you need to prove that something is true, it suffices to prove its polar dual. The polar map transforms points to lines, lines to points. It transforms the intersection of two lines to the line joining the two points, and vice-versa. For instance:

- To show that line ℓ passes through point P, it suffices to show that the pole of ℓ lies on the polar of P.
- To show that three points are collinear, it suffices to show that their poles are concurrent.

The cross ratio of four collinear points A, B, C, D is defined as

$$(A, B; C, D) = \frac{AC \cdot BD}{AD \cdot BC}$$

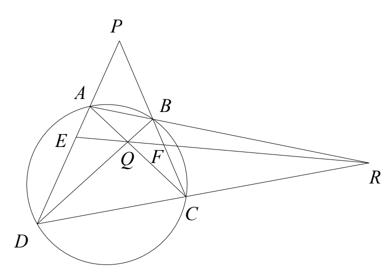
where the lengths are directed. *Cross ratios are preserved under projective transformations.* (Besides coincidence, this is pretty much the only thing that's projective-invariant.)

The most significant case is when $(A, B; C, D) = -1$, in which case we say that $(A, B; C, D)$ is *harmonic*. This notion of cross ratios and harmonic quadruples is not limited to collinear points, but also applies to pencils of lines, and concyclic points. Harmonic quadruples arise frequently, so learn to recognize them!

Possibly the most useful fact from polar geometry is the *self-polarity* of the diagonal triangle of a cyclic quadrangle. (Part of the reason why it's powerful if because it is not easy to prove simply using Eucliean geometry.)

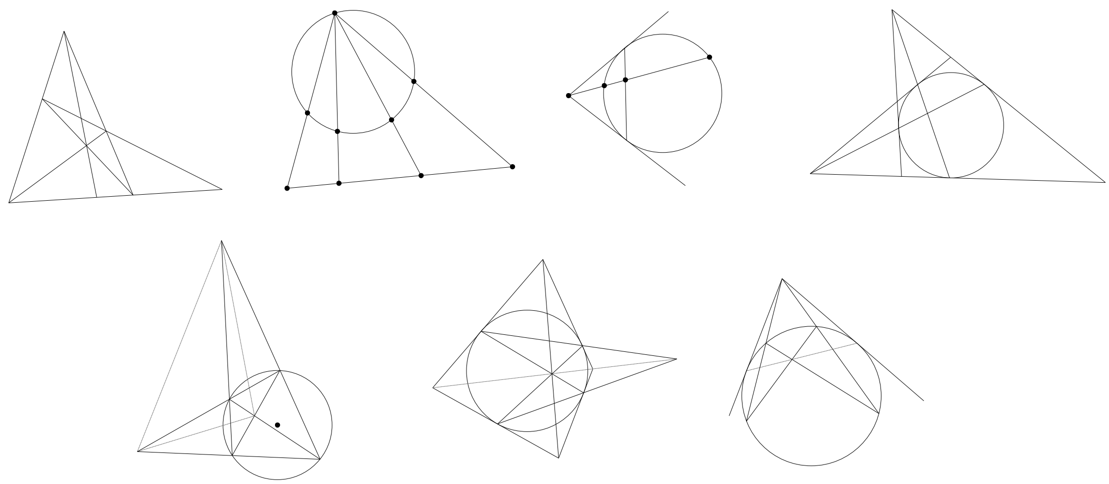
Theorem. Let $ABCD$ be a cyclic quadrilateral with circumcircle ω, and let $AD \cap BC = P$, $AC \cap BC = Q$ $AB \cap CD = R$. Then PQR is self-dual with respect to ω. That is, P is the pole of QR, Q is the pole of PR, and R is the pole of PQ.

In particular, we see that O, P, Q, R forms an orthocentric quadruple, where O is the center of ω. (This means that any of the four points is the orthocenter of the triangle formed by the other three).



Here is a quick sketch of the proof. Let line QR meet AD and BC at E and F respectively. Then from the configuration of lines, we see that $(A, D; P, E)$ and $(C, B; P, F)$ are both harmonic. It follows the polar of P is EF, which coincides with QR. □

You should develop the skill of recognizing when to use projective geometry. Here is a gallery of diagrams showing the patterns that tend to hint the use of projective geometry, in particular the use of poles and polars. Try to figure out the projective significance of each diagram. Ask Yufei if you need help.



1. Let ABC be a triangle and I be its incenter. Let the incenter of ABC touch sides BC, CA, AB at D, E, F respectively. Let S denote the intersection of lines EF and BC. Prove that $SI \perp AD$.

2. Let UV be a diameter of a semicircle, and let P, Q are two points on the semicircle. The tangents to the semicircle at P and Q meet at R, and lines UP and VQ meet at S. Prove that $RS \perp UV$.

3. A circle is inscribed in quadrilateral $ABCD$ so that it touches sides AB, BC, CD, DA at E, F, G, H respectively.
   (a) Show that lines AC, EF, GH are concurrent. In fact, they concur at the pole of BD.
   (b) Show that lines AC, BD, EG, FH are concurrent.

4. (China 1996) Let H be the orthocentre of triangle ABC. From A construct tangents AP and AQ to the circle with diameter BC, where P, Q are the points of tangency. Prove that P, H, Q are collinear.

5. (China 1997) Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

6. (Butterfly theorem) Let ω be a circle and let XY be a chord. Let M be the midpoint of XY, and let AB and CD be two chords of ω, both passing through M. Let XY meet chords AD and BC are P and Q respectively. So that $MP = MQ$.

7. Points C, M, D and A lie on line ℓ in that order with $CM = MD$. Circle ω is tangent to line ℓ at A. Let B be the point on ω that is diametrically opposite to A. Lines BC and BD meet ω at P and Q. Prove that the lines tangent to ω at P and Q and line BM are concurrent.
   (Hint: what is the significance of midpoints in terms of harmonic conjugates?)

8. **A very important fact.** Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Then AD coincides with a symmedian of $\triangle ABC$. (The *symmedian* is the reflection of the median across the angle bisector, all through the same vertex.)
(Hint: consider a reflection about the angle bisector of $\angle A$.)

9. Let $ABCD$ be a quadrilateral (not necessarily cyclic), and let X be the intersection of diagonals AC and BD. Suppose that BX is a symmedian of triangle ABC and DX is a symmedian of triangle ADC. Prove that AX is a symmedian of triangle ABD.

10. Let ABC be a triangle and tangent at point C to the circumcircle of ABC meets AB in M. The line perpendicular to OM in M intersects BC and AC in P and Q respectively. Prove that $MP = MQ$.

11. (Iberoamerican 1998) The incircle of triangle ABC is tangent to sides BC, CA, AB at D, E, F, respectively. Let segment AD meet the incircle again at Q. Show that the line EQ passes through the midpoint of segment AF if and only if $AC = BC$.

12. Let $ABCD$ be a convex quadrilateral (not necessarily cyclic). Let lines AB and CD meet at E, AD and BC meet at F, and AC and BD meet at P. Let M be the projection of P onto EF.
    (a) Show that $\angle AMP = \angle CMP$.
    (b) Show that $\angle BMC = \angle AMD$.

    (Hint: what do internal and external angle bisectors have to do with harmonic divisions?)

13. Let $ABCD$ be a circumscribed about a circle quadrilateral with the incenter I. Let M be the projection of I onto AC. Prove that $\angle AMB = \angle AMD$.

14. Let $ABCD$ be a convex quadrilateral (not necessarily cyclic). Let lines AB and CD meet at E, AD and BC meet at F. Suppose X is a point inside the quadrilateral such that $\angle AXE = \angle CXF$. Prove that $\angle AXB + \angle CXD = 180°$.
(Hint: apply a *polar transformation* centered at X and see what happens. And yes, you don't start with any circle. It's kind of weird I know ...)

15. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let lines AD and BC meet at E and lines AB and CD meet at F. Let M be the projection of O onto line EF. Prove that MO bisects $\angle BMD$.

16. (IMO 1985) A circle with center O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangle ABC and KBN intersect at exactly two distinct points B and M. Prove that angle $\angle OMB = 90°$.

17. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let lines AB and CD meet at E, AD and BC meet at F, and AC and BD meet at P. Furthermore, let EP and AD meet at K, and let M be the projection of O onto AD be M. Prove that $BCMK$ is cyclic.

18. Let ABC be a triangle with incenter I. Fix a line ℓ tangent to the incircle of ABC (not BC, CA or AB). Let A_0, B_0, C_0 be points on ℓ such that

$$\angle AIA_0 = \angle BIB_0 = \angle CIC_0 = 90°$$

Show that AA_0, BB_0, CC_0 are concurrent.

19. Let ABC be a triangle with incenter I. Fix a line ℓ tangent to the incircle of ABC (not BC, CA or AB). Let ℓ intersect the sides of the triangle at M, N, P. At I, erect perpendiculars to IM, IN and IP and let them intersect the corresponding sides of the triangle at M_0, N_0, and P_0 respectively. Show that M_0, N_0, P_0 lie on a line tangent to the incircle.

20. Let ABC be a triangle, and ω its incircle. Let ω touch the sides BC, CA, AB at D, E, F respectively. Let X be a point on AD and inside ω. Let segments BX and CX meet ω at Q and R respectively. Show that lines EF, QR, BC are concurrent.

Food for thought. Let ABC be a triangle with inscribed circle ω. There exists a projective transformation that preserves ω but sends ABC to any other desired triangle inscribed in ω! Can you prove this?

If a geometry problem is completely projective-invariant, then perhaps sending ABC to an equilateral triangle or an isosceles right triangle might be a good idea. Note that when the circumcircle is not present, we can simply apply an affine transformation to get what we want.