Lecture 10 : A Contest of Contests

Yufei Zhao

July 31, 2007

1. (IMC 2002) Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

2. (IMO 1998) In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that
$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

3. (China TST 1992) Sixteen students took part in a math competition where every problem was a multiple choice question with four choices. After the contest, it is found that any two students had at most one answer in common. Determine the maximum number of questions.

4. (IMO 2005) In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there are at least 2 contestants who each solved exactly 5 problems each.

Problem Solving Session

July 31, 2007

1. Show that if the edges of K_6, the complete graph with 6 vertices, are colored in 2 colors, then graph contains two monochromatic triangles. (Hint: count the number of monochromatic "angles")

2. (China 1993) Ten students ordered books. Each student ordered 3 different books. Each pair of students had ordered at least one same book. The book *Mathematics Olympiads* was the one which most (a tie being allowed) students ordered. What was the minimum number of students who ordered *Mathematics Olympiads*?

3. Twenty-five people form some committees with each committee has 5 members and each pair of committees have at most one common member. Determine, with justification, the maximum number of committees.

4. (USA TST 2005) Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$-element set T such that

   (a) each element of T is an m-element subset of S_m;

   (b) each pair of elements of T shares at most one common element; and

   (c) each element of S_m is contained in exactly two elements of T.

   Determine the maximum possible value of m in terms of n.

5. (APMO 2006) In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colors. Each clown is required to use at least five different colors. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colors and no more than 20 clowns may use any one particular color. Find the largest number n of clowns so as to make the ringmasters order possible.

6. (IMO Shortlist 2004) There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

   (a) Each pair of students is in exactly one club.

   (b) For each student and each society, the student is in exactly one club of the society.

   (c) Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.

   Find all possible values of k.

Lecture 11 : Generating Functions I

Yufei Zhao

August 1, 2007

Suppose that $(a_n)_{n=0}^{\infty}$ is a sequence of (complex) numbers. The *generating function*[1] of the sequence (a_n) is the following (formal) power series

$$F(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

1. **Vandermonde's identity.** By comparing the coefficient of x^n in $(x+1)^{a+b} = (x+1)^a(x+1)^b$, Show that
$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}.$$

2. Find the closed form expressions for the following generating functions:
   (a) $\displaystyle\sum_{n\geq 0} x^n$
   (b) $\displaystyle\sum_{n\geq 0} n x^n$
   (c) $\displaystyle\sum_{n\geq 0} \binom{m+n}{m} x^n$, where m is some positive integer.

3. **Fibonacci numbers.** Determine the generating function for the Fibonacci sequence, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

4. (a) Show that $\displaystyle\binom{2n}{n} = (-4)^n \binom{-\frac{1}{2}}{n}$.
   (b) Show that $\displaystyle(1-4x)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} x^n$.
   (c) Show that $\displaystyle\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$.

5. **Catalan numbers.** The Catalan numbers C_n, satisfies $C_0 = 1$ and
$$C_n = C_{n-1}C_0 + C_{n-2}C_1 + C_{n-3}C_2 + \cdots + C_0 C_{n-1}, \quad n \geq 1.$$
Find the generating function for C_n, and use it to obtain an explicit formula for C_n.

6. **Root of unity filter.** Let n be a positive integer, and let $\zeta = e^{2k\pi i/n}$ for some $0 < k < n$. For any polynomial $F(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ (with finitely many terms), show that the sum $f_0 + f_n + f_{2n} + \cdots$ is given by
$$f_0 + f_n + f_{2n} + \cdots = \frac{1}{n}\left(F(1) + F(\zeta) + F(\zeta^2) + \cdots + F(\zeta^{n-1})\right).$$

[1] In this course, we will only work with ordinary generating function, since this is the type that occurs in olympiads most often. Other useful generating functions include exponential generating functions, and Dirichlet series (used in number theory).

Problem Solving Session

August 1, 2007

1. Find the generating function for the sequence $(a_n)_{n=0}^{\infty}$ defined by $a_0 = 2, a_1 = 0, a_2 = -2$ and $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$, $n > 0$.

2. Express
$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2$$
in closed form.

3. Express in closed form: $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$.

4. (HMMT 2007) Let S denote the set of all triples (i, j, k) of positive integers where $i+j+k = 17$. Compute
$$\sum_{(i,j,k)\in S} ijk.$$

5. Let F_n be the n-th Fibonacci number. Evaluate the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{F_n}{4^n}$.

6. (Romania 2003) How many n-digit numbers, whose digits are in the set $\{2, 3, 7, 9\}$, are divisible by 3?

7. Adrian tosses 2003 fair coins, Andrea and Claudia each toss 2004 fair coins, and Zachary tosses 2005 fair coins. Show that the two events are equally likely, and determine the probability.

   (i) Claudia gets exactly one more head than Andrea does,

   (ii) Adrian and Zachary get exactly the same number of heads.

8. (Britain 1994) An increasing sequence of integers is said to be alternating if it starts with an odd term, the second term is even, the third term is odd, the fourth is even, and so on. The empty sequence is counted as alternating.

   Let A_n denote the number of alternating sequences involving only the integers $1, 2, \ldots, n$. Find $A(20)$.

Lecture 12 : Generating Functions II—Partitions

Yufei Zhao

August 2, 2007

A *partition* of an integer n is a nonincreasing sequence of positive integers $a_1 \geq a_2 \geq \cdots \geq a_k$ such that $n = a_1 + a_2 + \cdots + a_k$. We say that the a_i's are the *parts* of the partition.

For example, $p(5) = 7$, as the 7 partitions of $n = 5$ are $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$. By convention, $p(0)$ is defined to be 1.

1. Let $p(n)$ denote the number of partitions of n. Show that the generating function for $p(n)$ is

$$\sum_{n\geq 0} p(n)x^n = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots.$$

2. Let n be a positive integer. Let $f(n)$ denote the number of partitions of n with distinct parts, and let $g(n)$ denote the number of partitions of n with all parts being odd numbers. Prove that $f(n) = g(n)$.

3. (a) Show that every positive integer can be uniquely written as a sum of distinct powers of 2.

   (b) Show that every integer can be uniquely written as a sum of the form $\sum_{k=0}^{\infty} a_k 3^k$, where $a_k \in \{-1, 0, 1\}$.

4. (Putnam 1957) Let $\alpha(n)$ be the number of representations of a positive integer n as sum of 1's and 2's, taking order into account. For example, since

$$4 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2 = 1+1+1+1,$$

   we have $\alpha(4) = 5$. Let $\beta(n)$ be the number of representations of n that are sums of integers greater than 1, again taking order into account. For example, since

$$6 = 4+2 = 2+4 = 3+3 = 2+2+2,$$

   we have $\beta(6) = 5$. Show that $\alpha(n) = \beta(n+2)$.

5. (IMO Shortlist 1998) Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine a_{1998}.

6. **Euler's Pentagonal Numbers Theorem.** Show that

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^n) &= 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{k(3k-1)/2} + x^{k(3k+1)/2}\right)\\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots.\end{aligned}$$

Problem Solving Session

August 2, 2007

1. Let n be a positive integer. In how many ways can we fill a bag with n fruits subject to the following constraints?
   - The number of apples must be even.
   - The number of bananas must be a multiple of 5.
   - There can be at most four oranges.
   - There can be at most one pear.
2. Show that every integer has a unique base (-4) representation. That is, every integer can be written uniquely as $\sum_{k=0}^{\infty} a_k(-4)^k$, where $a_k \in \{0, 1, 2, 3\}$.
3. Let n be a positive integer. Show that the number of partitions of n into parts which have at most one of each distinct even part (e.g. $1+1+1+2+3+4$) equals the number of partitions of n in which each part can appear at most three times (e.g. $1+1+1+2+2+4+4+4$).
4. Does there exist a subset S of the positive integers satisfying: for each positive integer n, the number of partitions of n, where each part occurs at most twice, equals to the number of partitions of n into parts that are elements of S?
5. Let n be a positive integer. Show that the number of partitions of n, where each part appears at least twice, is equal to the number of partitions of n into parts all of which are divisible by 2 or 3.
6. How many polynomials P with coefficients $0, 1, 2$ or 3 have $P(2) = n$, where n is a given positive integer?

Lecture 13 : Generating Functions III

Yufei Zhao

August 3, 2007

1. Suppose the positive integers have been expressed as a disjoint union of arithmetic progressions $\{a_i + nd_i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, k$.

   (a) Show that $\dfrac{1}{d_1} + \dfrac{1}{d_2} + \cdots + \dfrac{1}{d_k} = 1$.

   (b) Show that $\dfrac{a_1}{d_1} + \dfrac{a_2}{d_2} + \cdots + \dfrac{a_k}{d_k} = \dfrac{k+1}{2}$.

   (c) Show that $d_i = d_j$ for some $i \neq j$.

2. (Putnam 2000) Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

3. (Putnam 2003) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S$, $s_1 \neq s_2$ and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

4. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two different unordered n-tuples of integers such that the sequences

   $$a_1 + a_2, a_1 + a_3, \ldots, a_{n-1} + a_n \quad \text{(all pairwise sums } a_i + a_j, 1 \leq i < j \leq n\text{)}$$

   and

   $$b_1 + b_2, b_1 + b_3, \ldots, b_{n-1} + b_n \quad \text{(all pairwise sums } b_i + b_j, 1 \leq i < j \leq n\text{)}$$

   coincide up to a permutation. Prove that n is a power of two.

5. (IMO 1995) Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are there such that the sums of its elements are divisible by p?

Problem Solving Session

August 3, 2007

1. (Putnam 2001) Adrian has the coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of showing heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

2. Each vertex of a regular polygon is colored with one of a finite number of colors so that the points of the same color are the vertices of some new regular polygon. Prove that at least two of the polygons obtained are congruent.

3. A standard die is labeled $1, 2, 3, 4, 5, 6$ (one integer per face). When you roll two standard dice, it is easy to compute the probability of the various sums. For example, the probability of rolling two dice and getting a sum of 2 is just $1/36$, while the probability of getting a 7 is $1/6$.

   Is it possible to construct a pair of *nonstandard* six-sided dice (possibly different from one another) with positive integer labels that nevertheless are indistinguishable from a pair of standard dice, if the sum of the dice is all that matters? For example, one of these nonstandard dice may have the label 8 on one of its faces, and two 3's. But the probability of rolling the two and getting a sum of 2 is still $1/36$, and the probability of getting a sum of 7 is still $1/6$.

4. Let p be a prime. Compute the number of subsets T of $\{1, 2, \ldots, p\}$ such that p divides the sum of the elements in T.

5. (Putnam 1997) Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all $k \geq 0$,
$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

6. A finite sequence $a_1, a_2, \ldots, a_n$ of real numbers is called k-balanced if
$$a_m + a_{m+k} + a_{m+2k} + \cdots$$
   is the same for any $k = 1, 2, \ldots, p$. Suppose that the sequence $a_0, a_1, \ldots, a_{49}$ is k-balanced for $k = 3, 5, 7, 11, 13$, and 17. Prove that $a_0 = a_1 = \cdots = a_{49} = 0$.

Lecture 14 : When Worlds Collide—Algebraic Combinatorics

Yufei Zhao

August 4, 2007

1. (St. Petersburg) Students in a school go for ice cream in groups of at least two. After $k > 1$ groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at most k.

2. **Oddtown and Eventown**. In a certain town with n citizens, a number of clubs are set up. No two clubs have exactly the set of members. Determine the maximum number of clubs that can be formed under each of the following constraints:

   (a) The size of every club is odd, and every pair of clubs share an even number of members.

   (b) The size of every club is even, and every pair of clubs share an even number of members.

3. (a) (China West 2002) Let $A_1, A_2, \ldots, A_{n+1}$ be non-empty subsets of $\{1, 2, \ldots, n\}$. Prove that there exists nonempty disjoint subsets $I, J \subset \{1, 2, \ldots, n+1\}$ such that

   $$\bigcup_{k \in I} A_k = \bigcup_{k \in J} A_k.$$

   (b) (Lindstrom) Let $A_1, A_2, \ldots, A_{n+2}$ be non-empty subsets of $\{1, 2, \ldots, n\}$. Prove that there exists nonempty disjoint subsets $I, J \subset \{1, 2, \ldots, n+2\}$ such that

   $$\bigcup_{k \in I} A_k = \bigcup_{k \in J} A_k, \qquad \text{and} \qquad \bigcap_{k \in I} A_k = \bigcap_{k \in J} A_k.$$

4. (Russia 2001) A contest with n question was taken by m contestants. Each question was worth a certain (positive) number of points, and no partial credits were given. After all the papers have been graded, it was noticed that by reassigning the scores of the questions, any desired ranking of the contestants could be achieved. What is the largest possible value of m?

5. (Crux 3037) There are 2007 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that for every senator in the senate, the number of enemies of that senator in the set K is an even number.

6. Let $a_1, a_2, \ldots, a_{2n+1}$ be real numbers, such that for any $1 \le i \le 2n+1$, we can remove a_i and separate the remaining $2n$ numbers into two groups with equal sums. Show that $a_1 = a_2 = \cdots = a_{2n+1}$.

7. **Odd vertex cover.** (Iran TST 1996, Germany TST 2004) Let G be a finite simple graph, and there is a light bulb at each vertex of G. Initially, all the lights are off. Each step we are allowed to chose a vertex and toggle the light at that vertex as well as all its neighbors'. Show that we can get all the lights to be on at the same time.

8. (Graham–Pollak) Show that the complete graph with n vertices, K_n, cannot be covered by fewer than $n - 1$ complete bipartite graphs so that each edge of K_n is covered exactly once.

Evaluation Test 3

August 4, 2007

Instructions: Read the problems carefully. Write your solutions neatly and concisely, but make sure to justify all your steps. Start each new solution on a new page and write your name and the problem number on every page.

1. Short answer questions: correct answers are worth full points in this problem.

   (a) [**5**] Let α, β be complex numbers. Determine the generating function for the sequence $(a_n)_{n=0}^{\infty}$ defined by $a_0 = \alpha$, $a_1 = \beta$, and $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$.

   (b) [**5**] Determine the generating function for the sequence $(b_n)_{n=0}^{\infty}$ defined by $b_0 = 0$, and $b_n = 2b_{n-1} + n$ for $n \geq 1$.

2. [**10**] Let n be a positive integer. Show that the number of partitions of n into odd parts greater than 1 is equal to the number of partitions of n into unequal parts none of which is a power of two.

3. [**10**] Seven students take a mathematical exam. Every problem was solved by at most 3 students. For every pair of students, there is at least one problem that they both solved. Determine, with proof, the minimum number of problems on this exam.

4. [**10**] Find the number of subsets of $\{1, 2, \ldots, 2007\}$, the sum of whose elements is divisible by 17.

Evaluation Test 3

Solutions

August 4, 2007

1. Short answer questions: correct answers are worth full points in this problem.

   (a) [**5**] Let α, β be complex numbers. Determine the generating function for the sequence $(a_n)_{n=0}^{\infty}$ defined by $a_0 = \alpha$, $a_1 = \beta$, and $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$.

   Solution: Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating function. Then

$$\begin{aligned} A(x) = \sum_{n\geq 0} a_n x^n &= \alpha + \beta x + \sum_{n\geq 2} a_n x^n \\ &= \alpha + \beta x + \sum_{n\geq 2} a_{n-1}x^n + \sum_{n\geq 2} a_{n-2}x^n \\ &= \alpha + \beta x + x(A(x) - \alpha) + x^2 A(x). \end{aligned}$$

   Solving for $A(x)$ gives $A(x) = \dfrac{\alpha + (\beta - \alpha)x}{1 - x - x^2}$.

   (b) [**5**] Determine the generating function for the sequence $(b_n)_{n=0}^{\infty}$ defined by $b_0 = 0$, and $b_n = 2b_{n-1} + n$ for $n \geq 1$.

   Solution: Let $B(x) = \sum_{n\geq 0} b_n x^n$ be the generating function. Then

$$B(x) = \sum_{n\geq 0} b_n x^n = \sum_{n\geq 1} b_n x^n = \sum_{n\geq 1} 2b_{n-1}x^n + \sum_{n\geq 1} nx^n = 2xB(x) + \frac{x}{(1-x)^2}.$$

   Solving for $B(x)$ gives $B(x) = \dfrac{x}{(1-x)^2(1-2x)}$.

2. [**10**] Let n be a positive integer. Show that the number of partitions of n into odd parts greater than 1 is equal to the number of partitions of n into unequal parts none of which is a power of two.

   Solution: Let a_n denote the number of partitions of n into odd parts greater than 1. The generating function for (a_n) is

$$A(x) = \frac{1}{(1-x^3)(1-x^5)(1-x^7)\cdots}.$$

   Also, let b_n denote the number of partitions of n into unequal parts none of which is a power of two. The generating function for (b_n) is

$$B(x) = \frac{(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots}{(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots}.$$

   Note that we have

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = \frac{(1-x^2)(1-x^4)(1-x^8)\cdots}{(1-x)(1-x^2)(1-x^4)\cdots} = \frac{1}{1-x},$$

and

$$\begin{aligned}(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots &= \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} \\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots}.\end{aligned}$$

Combining the two, we find that $A(x) = B(x)$. Therefore, $a_n = b_n$ for all n, as desired.

3. [**10**] Seven students take a mathematical exam. Every problem was solved by at most 3 students. For every pair of students, there is at least one problem that they both solved. Determine, with proof, the minimum number of problems on this exam.

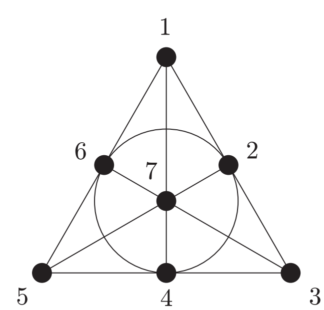
Solution: Assume that there are n problems. Let T denote the number of pairs (P, S), where P is a problem, and S is a pair of students who both solved P. On one hand, since every problem was solved at at most 3 students, we have $T \le \binom{3}{2}n = 3n$. On the other hand, since there is such a pair (P, S) for every pair of students, we have $T \ge \binom{7}{2} = 21$. Thus, $3n \ge 21$, so $n \ge 7$.

Let us show that $n = 7$ can be attained. Label the problems 1, 2, 3, 4, 5, 6, 7, and the let each of the following set denote the set of problems that an individual a student has solved:

$$\{1,2,3\}, \{1,4,7\}, \{1,5,6\}, \{2,5,7\}, \{2,4,6\}, \{3,4,5\}, \{3,6,7\}.$$

Therefore, the minimum number of problems on this exam is 7.

Remarks: Although the construction can be found by trial and error, it can be motivated by the Fano plane, which is depicted below. Each vertex represents a problem, and each line (including the center circle) represents a student who solved the problems that the line passes through.



4. [**10**] Find the number of subsets of $\{1, 2, \ldots, 2007\}$ (including the empty set) the sum of whose elements is divisible by 17.

Solution: Consider the polynomial

$$f(x) = (1+x)(1+x^2)(1+x^3)\cdots(1+x^{2007}).$$

Every subset $\{a_1, a_2, \ldots, a_k\} \subset \{1, 2, \ldots, 2007\}$ corresponds to a term in the expansion of $f(x)$, namely $x^{a_1+a_2+\cdots+a_k}$. Therefore, for each m, the coefficient of x^m is $f(x)$ equals to the number of subsets of $\{1, 2, \ldots, 2007\}$ the sum of whose elements is m. Therefore, we need the find the sum of the coefficients of $x^0, x^{17}, x^{34}, \ldots, x^{2006}$. This can be done using the roots of unity. Let $\zeta = e^{2\pi i/17}$. Then, the answer equals to

$$\frac{1}{17}\left(f(1) + f(\zeta) + f(\zeta^2) + \cdots + f(\zeta^{16})\right).$$

We must evaluate this expression.

First, notice that since the zeros of $x^{17} - 1$ are $1, \zeta, \zeta^2, \ldots, \zeta^{16}$, so

$$x^{17} - 1 = (x-1)(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{16}).$$

Setting $x = -1$ yields the identity

$$(1+1)(1+\zeta)(1+\zeta^2)\cdots(1+\zeta^{16}) = 2.$$

Note that for $1 \le k \le 16$, $\{0, k, 2k, 3k, \ldots, 16k\}$ is a permutation of $\{0, 1, 2, 3, \ldots, 16\}$ as residues in mod 17. It follows that

$$(1+1)(1+\zeta^k)(1+\zeta^{2k})\cdots(1+\zeta^{16k}) = 2.$$

Consequently, since $2007 = 17 \cdot 118 + 1$, we have

$$\begin{aligned} f(\zeta^k) &= (1+\zeta^k)(1+\zeta^{2k})\cdots(1+\zeta^{2007k}) \\ &= \left[(1+1)(1+\zeta^k)(1+\zeta^{2k})\cdots(1+\zeta^{16k})\right]^{118}(1+\zeta^k) \\ &= 2^{118}(1+\zeta^k). \end{aligned}$$

Also, $f(1) = 2^{2007}$. Therefore,

$$\begin{aligned} &\frac{1}{17}\left(f(1) + f(\zeta) + f(\zeta^2) + \cdots + f(\zeta^{16})\right) \\ =& \frac{1}{17}\left(2^{2007} + 2^{118}(16 + \zeta + \zeta^2 + \zeta^3 + \cdots + \zeta^{16})\right) \\ =& \frac{1}{17}2^{118}\left(2^{1889} + 15\right). \end{aligned}$$

Thus, the answer is $\frac{2^{118}\left(2^{1889}+15\right)}{17}$.