UMA Putnam Talk Lecture Notes

Determinants: Evaluation and Manipulation

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Appetizer problem

(This problem doesn't actually use determinants.)

Problem 1. Do there exist square matrices A and B such that $AB - BA = I$?

Solution. No. Take the trace of both sides and using $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we get that $\operatorname{tr}(AB-BA) = 0$ while $\operatorname{tr}(I) \neq 0$. □

1. Introduction

In this talk I'll discuss some techniques on dealing with determinants that may be useful for the Putnam exam. We will focus on the evaluation and manipulation of determinants. I won't talk about applications of determinants to, say, combinatorics (maybe another time).

We will assume familiarity with basic properties of determinants. Just a reminder, if $A = (a_{ij})_{1\le i,j\le n}$ is an $n \times n$ matrix, then

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where the sum is taken over all permutations of $\{1, 2, \ldots, n\}$.

Here's an outline of techniques used to deal with determinants.

Evaluation
- Row and column operations
- Expansion by minors
- Setting variables / Vandermonde
- Eigenvalues / circulant matrices

Manipulation
- Assume invertibility
- Block decomposition
- Conjugation and positivity

2. Evaluation of determinants

I'll talk about how to evaluate determinants when the entries are given.

The most basic (and often extremely useful) method is **row/column operations** and **minor expansions**. Though I won't discuss them here, since I want to talk more exciting techniques.

The first example is everyone's favorite **Vandermonde determinant**.

Problem 2 (Vandermond determinant)**.** Let

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

Show that

$$\det V = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Solution. Let

$$p(x_1, x_2, \ldots, x_n) = \det V,$$

viewed as a polynomial in n variables. Now, suppose we view p as a single-variable polynomial in x_1 with coefficients in $\mathbb{Q}(x_2, \ldots, x_n)$. If we set x_1 to x_i, for any $i \neq 1$, then two rows of the matrix are equal and hence the determinant vanishes, and therefore $(x_1 - x_i)$ must be a factor of p.

Similarly, every $(x_i - x_j)$ for $i \neq j$ is a factor of $p(x_1, \ldots, x_n)$. But the degree of p is $\frac{1}{2}n(n-1)$ (from looking at the matrix), and we just showed that $\prod_{1 \le i < j \le n}(x_j - x_i)$ (which has degree $\frac{1}{2}n(n-1)$) divides p. Therefore,

$$p(x_1, x_2, \ldots, x_n) = k \prod_{1 \le i < j \le n} (x_j - x_i),$$

for some constant k. Comparing the coefficient of the term $x_2 x_3^2 x_4^3 \cdots x_n^{n-1}$ shows that $k = 1$. □

Our next example is the **circulant matrix**.

Problem 3 (Circulant matrix)**.** Let

$$C = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$

Then

$$\det C = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \zeta^{jk} a_k \right)$$

where $\zeta = e^{2\pi i/n}$.

Solution. We know that the determinant equals to the product of the eigenvalues. The eigenvectors of C are

$$v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ \zeta^2 \\ \zeta^4 \\ \vdots \\ \zeta^{2(n-2)} \end{bmatrix} \quad \cdots \quad v_{n-1} = \begin{bmatrix} 1 \\ \zeta^{n-1} \\ \zeta^{2(n-1)} \\ \vdots \\ \zeta^{(n-1)^2} \end{bmatrix}.$$

They are independent because of the Vandermonde determinant, so they form a complete set of eigenvalues. The corresponding eigenvectors are

$$\begin{aligned} \lambda_0 &= a_0 + a_1 + a_2 + \cdots + a_{n-1} \\ \lambda_1 &= a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{n-1}\zeta^n \\ &\cdots \\ \lambda_{n-1} &= a_0 + a_1\zeta^{n-1} + a_2\zeta^{2(n-1)} + \cdots + a_{n-1}\zeta^{(n-1)^2} \end{aligned}$$

Thus $\det C = \lambda_0 \lambda_1 \cdots \lambda_{n-1}$. □

Now you have the tools to solves the following problem, which appeared as Putnam 1999/B5. The highest score on his problem was 2 points by one contestant! By this measure, it is one of the most difficult Putnam problems in history; but knowing the above technique is becomes not so bad.

Problem 4 (Putnam 1999/B5)**.** Let $n \ge 3$. Let A be the $n \times n$ matrix with $A_{jk} = \cos(2\pi(j+k)/n)$. Find $\det(I + A)$.

3. Manipulation of matrices

Now I'll discuss some techniques on dealing with determinants of matrices without knowing their entires. We will make repeated uses of the fact that $\det AB = \det A \det B$ for square matricies.

Problem 5. Let A and B be $n \times n$ matrices. Show that $\det(I + AB) = \det(I + BA)$.

Solution. First, assume that A is invertible. Then

$$\det(I + AB) = \det(A(I + BA)A^{-1}) = \det A \det(I + BA) \det(A^{-1}) = \det(I + BA).$$

Now we give two ways of working around the assumption that A is invertible.
Method 1. For any $t \in \mathbb{R}$, let $A_t = A - tI$. Then A_t is non-invertible precisely when t is an eigenvalue of A. Thus, if t is not an eigenvalue, then $\det(I + A_t B) = \det(I + BA_t)$. Now, $\det(I + A_t B) - \det(I + BA_t)$ is a polynomial in t which vanishes everywhere except for the finitely many eigenvalues; hence $\det(I + A_t B) - \det(I + BA_t) = 0$ for all t. Setting $t = 0$ gives the result.

Method 2. View the entries of A and B as indeterminants, so that what we are proving is a polynomial identity in $\{a_{ij}\} \cup \{b_{ij}\}$. Work over the field $\mathbb{Q}(a_{11}, \ldots, b_{11}, \ldots)$. Then in this field, A is invertible, and the proof works. □

Remark. The set of invertible matrices form a Zariski (dense) open subset, and hence to verify a polynomial identity, it suffices to verify it on this dense subset.

Remark. The statement is also true when A and B are not square matrices. Specifically, suppose that A is an $n \times m$ matrix, and B an $m \times n$ matrix, then $\det(I_n + AB) = \det(I_m + BA)$. To prove this fact, extend A and B to square matrices by filling in zeros.

The technique of assuming invertibility is very powerful. Let us give another example.

Problem 6. Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Solution. First assume that A is invertible. Then

$$\begin{pmatrix} I & O \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ O & D - CA^{-1}B \end{pmatrix},$$

so that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} A & B \\ O & D - CA^{-1}B \end{pmatrix} = \det A \det(D - CA^{-1}B)$$
$$= \det(AD - ACA^{-1}B) = \det(AD - CB).$$

(We used the fact that A and C commute.)
Now we need to get rid of the invertibility assumption. Let $A_t = A - tI$. Since $AC = CA$, we get $A_t C = CA_t$ for all t. It follows that

$$\det \begin{pmatrix} A_t & B \\ C & D \end{pmatrix} = \det(A_t D - CB).$$

whenever t is not an eigenvalue of A. But this is a polynomial equation in t, which holds for all but finitely many t's, and hence it must hold for all t. In particular, setting $t = 0$ gives the desired result. □

Finally, let us look at a few problems involving inequalities.

Problem 7. Let A be a square matrix with real entries. Show that $\det(A^2 + I) \geq 0$.

One way to solve this problem is to look at the eigenvalues of A. If the eigenvalues of A are $\{\lambda_i\}$ (as a multiset, i.e., counting multiplicities), then the eigenvalues of A^2+I are $\{\lambda_i^2+1\}$, and hence $\det(A^2+1) = \prod_i(\lambda_i^2+1)$. Finally use the fact that all non-real eigenvalues λ_i come in conjugate pairs.

Here is a much slicker solution.

Proof. We have $A^2+I = (A+iI)(A-iI)$. So

$$\det(A^2+I) = \det(A+iI)\det(A-iI) = \det(A+iI)\overline{\det(A+iI)} = |\det(A+iI)|^2 \geq 0. \qquad \square$$

Problem 8. Let A, B, C be $n \times n$ real matrices that pairwise commute and $ABC = O$. Show that

$$\det(A^3+B^3+C^3)\det(A+B+C) \geq 0.$$

Solution. Recall the identity

$$A^3+B^3+C^3-3ABC = (A+B+C)(A+\omega B+\omega^2 C)(A+\omega^2 B+\omega C)$$

where $\omega = e^{2\pi/3}$ is a third root of unity. We used the assumption that A, B, C pairwise commute.

Hence,

$$\begin{aligned}\det(A^3+B^3+C^3)\det(A+B+C) &= \det(A^3+B^3+C^3-3ABC)\det(A+B+C)\\ &= (\det(A+B+C))^2\det(A+\omega B+\omega^2 C)\det(A+\omega^2 B+\omega C)\\ &= (\det(A+B+C))^2\det(A+\omega B+\omega^2 C)\overline{\det(A+\omega B+\omega^2 C)}\\ &\geq 0. \qquad \square\end{aligned}$$

Problem 9. Let A and B be two $n \times n$ real matrices that commute. Suppose that $\det(A+B) \geq 0$. Prove that $\det(A^k+B^k) \geq 0$ for all $k \geq 1$

Problem 10. Let A be real skew-symmetric square matrix (i.e., $A^t = -A$). Prove that $\det(I + tA^2) \geq 0$ for all real t.