

Three Lemmas in Geometry

Solutions

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1 Diameter of incircle

1. (IMO 1992) In the plane let \mathcal{C} be a circle, ℓ a line tangent to the circle \mathcal{C} , and M a point on ℓ . Find the locus of all points P with the following property: there exists two points Q, R on ℓ such that M is the midpoint of QR and \mathcal{C} is the inscribed circle of triangle PQR .

Solution: Let \mathcal{C} touch ℓ at D , and DE be a diameter of \mathcal{C} . For any such P, Q, R described in the problem, the line PE must intersect ℓ at a point F such that $MD = MF$, by the lemma. The point F depends only on M, ℓ , and \mathcal{C} . It follows that P must lie on the ray FE beyond E .

Conversely, given a point P lying on the ray FE beyond E , let the tangents from P to \mathcal{C} meet ℓ at Q and R . By the lemma we must have $QF = RF$, from which it follows that M is the midpoint of QR . Therefore, the locus is the ray FE beyond E .

2. (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E . Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G . Prove that the triangle AFG is isosceles.

Solution: Observe that F is the center of the excircle of ADC opposite to A (since the center satisfies the two defining properties of F). Let line AC touch this excircle at X . Then, using fact that $GACF$ is cyclic, we have

$$\angle GAF = \angle GCF = \angle XCF = \angle AGF,$$

and therefore AFG is isosceles.

3. (IMO Shortlist 2005) In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E , respectively. Let K and L be the symmetric points of D and E with respect to I . Prove that the quadrilateral $ACKL$ is cyclic.

Solution: Let $x = AD$, $y = BD = BE$, $z = CE$. Then $AB = x + y$, $BC = y + z$, $AC = x + z$, so the condition that $AB + BC = 3AC$ is equivalent to $x + y + y + z = 3x + 3z$, or equivalently $y = x + z$.

Let line CK meet AB at M and line AL meet BC at N . Then by the lemma, $BM = AD = x$, so $MD = BD - BM = y - x = z$. Similarly $NE = x$. By comparing the lengths of the legs, we see that the right triangles MDK and CEL are congruent, and so are ADK and NEL . Therefore $\angle MKD = \angle CLE$ and $\angle AKD = \angle NLE$. Adding gives $\angle MKA = \angle CLN$, so $\angle AKC = \angle ALC$, and hence $ACKL$ is cyclic.

4. (Nagel line) Let ABC be a triangle. Let the excircle of ABC opposite to A touch side BC at D . Similarly define E on AC and F on AB . Then AD, BE, CF concur (why?) at a point N known as the *Nagel point*.

Let G be the centroid of ABC and I the incenter of ABC . Show that I, G, N lie in that order on a line (known as the *Nagel line*, and $GN = 2IG$).

Solution: Let the incircle of ABC touch BC at X , and let XY be a diameter of the incircle. By the lemma, A, Y, D are collinear. Let M be the midpoint of BC . Then MI is a midline of triangle XYD , so IM and YD are parallel. The dilation centered at G with ratio -2 takes M to A , and thus it takes line IM to the line through A parallel to IM , namely the line AD . Hence the image of I under the dilation lies on the line AD . Analogously, it must also lie on BE and CF , and therefore the image of I is precise N . This proves that I, G, N are collinear in that order with $GN = 2IG$.

5. (USAMO 2001) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC , respectively. Denote by D_2 and E_2 the points on sides BC and AC , respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2 . Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q . Prove that $AQ = D_2P$.

Solution: From the lemma we know that D_1Q is a diameter of the incircle. Let the incenter of ABC be I , its centroid be G , and the midpoint of BC be M . Note that P is the Nagel point of ABC . From the previous problem, we know that the dilation centered at G with ratio -2 sends M to A and I to P , and hence sends segment IM to PA , thus $PA = 2IM$. On the other hand, a dilation centered at D_1 with ratio 2 sends IM to QD_2 , so $QD_2 = 2IM = PA$. Therefore, $AQ = PA - QP = QD_2 - QP = D_2P$.

6. (Tournament of Towns 2003 Fall) Triangle ABC has orthocenter H , incenter I and circumcenter O . Let K be the point where the incircle touches BC . If IO is parallel to BC , then prove that AO is parallel to HK .

Solution: Let KE be a diameter of the incircle, and let line AE meet BC at D . Let M be the midpoint of BC . By the lemma, M is also the midpoint of KD . Since IO is parallel to BC , $KMOI$ is a rectangle. Since I is the midpoint of KE and M is the midpoint of KD , we see that O must be the midpoint of ED . Thus lines AE and AO coincide.

Let G be the centroid of ABC . A dilation centered at G with ratio -2 takes M to A and O to H (by Euler line). So it takes segment MO to AH , and hence $AH = 2MO = EK$. Since AH and EK are both perpendicular to BC , it follows that $AHKE$ is a parallelogram, and hence HK is parallel to AE , which coincides with line AO .

7. (IMO 2008) Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C , which is also tangent to the lines AD and CD . Prove that the common external tangents of ω_1 and ω_2 intersect on ω .

(Hint: show that $AB + AD = CB + CD$. What does this say about the lengths along AC ?)

Solution: Chasing some lengths using equal tangents yields $AB + AD = CB + CD$ (details omitted). Let ω_1 and ω_2 touch AC at P and Q respectively. Then $AP = \frac{1}{2}(AB + AC - BC) = \frac{1}{2}(CD + AC - AD) = CQ$.

Let PP' be a diameter of ω_1 , and let QQ' be a diameter of ω_2 . By the lemma, B, P', Q are collinear, and P, Q', D are collinear.

Choose point T on ω so that the tangent to ω at T is parallel to AC and furthermore puts ω and B on different sides. Then the dilation centered at B that sends ω_1 to ω must send P' to T , so B, P', Q, T are collinear. Analogously, the dilation centered at D (with negative ratio) that sends ω_2 to ω must take Q' to T , so P, Q', D, T are collinear.

Now, PP' and QQ' are parallel diameters of ω_1 and ω_2 , and lines $P'Q$ and PQ' meet at T . It follows that there is a dilation with positive ratio centered at T that takes ω_1 to ω_2 , and hence T is the intersection of the common external tangents of ω_1 and ω_2 . Since T lies on ω , we are done.

2 Center of spiral similarity

1. (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle CBA = \angle DCA = \angle EDA.$$

Diagonals BD and CE meet at P . Prove that line AP bisects side CD .

Solution: Since A is the center of the spiral similarity sending BC to DE , by the lemma we know that $ABCP$ and $APDE$ are both cyclic. Furthermore, since $\angle ACD = \angle ABC$, the circumcircle of $ABCP$ is tangent to CD . Since $\angle ADC = \angle AEC$, the circumcircle of $APDE$ is also tangent to CD . Let AP meet CD at M . Then by power of a point, $MC^2 = MP \cdot MA = MD^2$, so $MC = MD$, as desired.

2. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC , respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T , respectively. Prove that the circumcircles of triangles SAE , SBF , TCF , and TDE pass through a common point.

Solution: Note that $ABCD$ cannot be a parallelogram, for otherwise FE and BA would be parallel. Let P be the center of spiral similarity that carries AD to BC . It must also carry E to F since $AE/ED = BF/FC$. Since the spiral similarity carries AE to BF , it follows by the lemma that $PAES$ and $PBFS$ are concyclic. Similarly, since the spiral similarity carries DE to CF , $PEDT$ and $PFCT$ are concyclic. Therefore the circumcircles of SAE , SBF , TCF , TDE all pass through P .

3. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O . Diagonals AC and BD meet at P . The circumcircles of triangles ABP and CDP meet at P and Q . Assume that points O , P , and Q are distinct. Prove that $\angle OQP = 90^\circ$.

Solution: Let M and N be the midpoints of AC and BD , respectively. Let \mathbf{T} be the spiral similarity that carries A to B and C to D . By the lemma (and the fact after it), Q is the center of the spiral similarity. Since \mathbf{T} carries AC to BD , it preserves midpoints, so \mathbf{T} brings M to N . Using the lemma again, we see that M, N, P, Q are concyclic. Since $\angle OMP = \angle ONP = 90^\circ$, points O, P, M, N are concyclic with diameter OP . Therefore, M, N, P, Q, O are concyclic with diameter OP , and therefore $\angle OQP = 90^\circ$.

4. Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P . Let O_1 and O_2 be the circumcenters of APD and BPC . Let M , N and O be the midpoints of AC , BD and O_1O_2 . Show that O is the circumcenter of MPN .

Solution: Let the circumcircles of APD and BPC meet at P and Q . Let \mathbf{T} denote the spiral similarity that sends AD to CB . Then \mathbf{T} is centered at Q by the lemma. Let \mathbf{id} denote the identity transformation, and consider the transformation $\mathbf{R} = \frac{1}{2}(\mathbf{id} + \mathbf{T})$. This is another spiral similarity centered at Q (if you're not convinced, think about it in terms of multiplication by complex numbers). Then $\mathbf{R}(A) = M$, $\mathbf{R}(D) = N$, $\mathbf{R}(O_1) = O$. Since O_1 is the circumcenter of QAD , the transformation yields that O is the circumcenter of QMN , whose circumcircle must again pass through P by the lemma. This proves the desired fact.

5. (Miquel point of a quadrilateral) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane, no two parallel. Let \mathcal{C}_{ijk} denote the circumcircle of the triangle formed by the lines ℓ_i, ℓ_j, ℓ_k (these circles are called *Miquel circles*). Then $\mathcal{C}_{123}, \mathcal{C}_{124}, \mathcal{C}_{134}, \mathcal{C}_{234}$ pass through a common point (called the *Miquel point*).

(It's not too hard to prove this result using angle chasing, but can you see why it's almost an immediate consequence of the lemma?)

Solution: Let P_{ij} denote the intersection of ℓ_i and ℓ_j . Let \mathcal{C}_{134} and \mathcal{C}_{234} meet at P . Then by the lemma, P is the center of the spiral similarity that sends P_{13} to P_{23} and P_{14} to P_{24} . It follows that P is also the center of the spiral similarity that sends P_{13} to P_{14} and P_{23} to P_{24} . Applying the lemma again, we find that \mathcal{C}_{123} and \mathcal{C}_{124} also pass through P , as desired.

6. (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P , the lines BD and EF meet at Q , the lines EF and AC meet at R . Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P .

Solution: Let S be the center of the spiral similarity that carries AD to CB , then it must also carry F to E . Using the lemma, we see that $SPAD$, $SRAF$, $SQFD$ are all cyclic. Note that they are the Miquel circles of the quadrilateral formed by the lines AD, AP, PD, QF , and thus S is the Miquel point of these circles. The remaining Miquel circle passes through P, Q, R, S , and hence S lies on the circumcircle of PQR . Note that S is the desired point, as it does not depend on the choice of E and F .

7. (IMO Shortlist 2006) Points A_1, B_1 and C_1 are chosen on sides BC, CA , and AB of a triangle ABC , respectively. The circumcircles of triangles AB_1C_1, BC_1A_1 , and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2 , and C_2 , respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3 , and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA , and AB , respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Solution: By the lemma, C_2 is the center of the spiral similarity that takes A_1 to B and B_1 to A . So triangles $C_2A_1B_1$ and C_2BA are similar. But C_2 is also the center of the spiral similarity that takes A_1 to B_1 and B to A . Then because $BA_1 = CA_3$ and $AB_1 = CB_3$,

$$\frac{C_2A_1}{C_2B_1} = \frac{BA_1}{AB_1} = \frac{CA_3}{CB_3}.$$

Since $\angle A_1C_2B_1 = \angle A_3CB_3$, triangles $CA_3B_3, C_2A_1B_1$, and C_2BA are similar. So $\angle CA_3B_3 = \angle C_2BA$. Similarly, $\angle BA_3C_3 = \angle B_2CA$. Then

$$\begin{aligned} \angle B_2A_2C_2 &= \angle B_2AC_2 = 180^\circ - \angle B_2C_2A - \angle C_2B_2A = 180^\circ - \angle B_2CA - \angle C_2BA \\ &= 180^\circ - \angle BA_3C_3 - \angle CA_3B_3 = \angle B_3A_3C_3. \end{aligned}$$

Similarly $\angle A_2B_2C_2 = \angle A_3B_3C_3$, hence triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

3 Symmedian

1. (Poland 2000) Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB , then show that $\angle APM + \angle BPC = 180^\circ$.

Solution: Since $\angle CAB = \angle CBA$, $\angle PAB = \angle PBC$ implies that $\angle PAC = \angle PBA$, and thus the circumcircle of ABP is tangent to CA and CB . It follows by the lemma that line CP is a symmedian of APB , and therefore $\angle APM = 180^\circ - \angle BPC$.

2. (IMO Shortlist 2003) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC . Denote by P the intersection of the tangents to Γ at A and C . Suppose Γ meets the segment PB at Q . Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ .

Solution: We follow the method of the first proof of the lemma. Let the bisector of $\angle AQC$ meet A at R , then $AR/RC = AQ/QC$, so it suffices to show that AQ/QC does not depend on Γ .

Applying Sine law repeatedly, we find that

$$\frac{AQ}{QC} = \frac{\sin \angle ACQ}{\sin \angle CAQ} = \frac{\sin \angle PAQ}{\sin \angle PCQ} = \frac{\frac{PQ}{AP} \sin \angle AQP}{\frac{PQ}{CP} \sin \angle CQP} = \frac{\sin \angle AQB}{\sin \angle CQB} = \frac{\frac{AB}{AQ} \sin \angle ABQ}{\frac{CB}{CQ} \sin \angle CBQ} = \frac{AB}{CB} \frac{QC}{AQ}.$$

Thus $AQ/QC = \sqrt{AB/CB}$, which is independent of Γ , as desired.

3. (Vietnam TST 2001) In the plane, two circles intersect at A and B , and a common tangent intersects the circles at P and Q . Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S , and let H be the reflection of B across the line PQ . Prove that the points A , S , and H are collinear.

Solution: We will only do the configuration where B is closer to line PQ than A . You should think about what happens in the other configuration, which is analogous.

Since AS coincides with the symmedian of APQ , it suffices to show that H lies on this symmedian. Note that AB coincides with a median of APQ . Indeed, let line AB meet PQ at M , then by Power of a Point, $MP^2 = MB \cdot MA = MQ^2$, so $MP = MQ$.

Since $\angle PHQ = \angle PBQ = 180^\circ - \angle BPQ - \angle BQP = 180^\circ - \angle BAP - \angle BAQ = 180^\circ - \angle PAQ$, we see that $APHQ$ is cyclic. Then $\angle HAQ = \angle HPQ = \angle BPQ = \angle BAP$. Since AB coincides with a median of APQ , it follows that AH coincides with a symmedian of APQ , and hence A, H, S are collinear.

4. (USA TST 2007) Triangle ABC is inscribed in circle ω . The tangent lines to ω at B and C meet at T . Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

Solution: Let M be the midpoint of BC . Since BT is tangent to ω , we have $\angle TBA = 180^\circ - \angle BCA$. By the lemma, we have $\angle BAT = \angle CAM$. Applying Sine law to triangles BAT and CAM , we get

$$\frac{BT}{AT} = \frac{\sin \angle BAT}{\sin \angle TBA} = \frac{\sin \angle CAM}{\sin \angle BCA} = \frac{MC}{AM}.$$

Since $TB = TC_1$, we have $TC_1/TA = MC/MA$. Note that $\angle TMS = \angle TAS = 90^\circ$, so $TMAS$ is cyclic, and hence $\angle AMS = \angle ATS$. Therefore, triangles AMC and ATC_1 are similar. Analogously, triangles AMB and ATB_1 are similar. Combine the two results, and we see that ABC and AB_1C_1 are similar.

5. Let ABC be a triangle. Let X be the center of spiral similarity that takes B to A and A to C . Show that AX coincides with a symmedian of ABC .

Solution: Let the tangents to the circumcircle of ABC at B and C meet at D . Let AD meet the circumcircle of BCD again at X' . Then

$$\angle ABX' = \angle BX'D - \angle BAX' = \angle BCD - \angle BAD = \angle BAC - \angle BAD = \angle X'AC,$$

and analogously we have $\angle ACX' = \angle X'AB$. Therefore $X = X'$, and it lies on the symmedian AD .

Second solution, not using the lemma: Let Y be a point on the circumcircle of ABC so that AY coincides with a symmedian of ABC . Let X' be the midpoint of AY . Let M be the midpoint of BC and N the midpoint of AC . Since AY is a symmedian, $\angle BAY = \angle MAC$. Additionally we have $\angle BYA = \angle MCA$, so triangles ABY and AMC are similar. Since X' is the midpoint of Y and N is the midpoint of AC , we see that ABX' and AMN are similar. Hence

$$\angle ABX' = \angle AMN = \angle MAB = \angle CAX'.$$

Analogously $\angle ACX' = \angle BAX'$. Therefore $X = X'$, and it lies on a symmedian.

6. (USA TST 2008) Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC . Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC , the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

Solution: Let X be the center of spiral similarity \mathbf{T} that carries B to A and A to C , as in the previous problem. Triangles BRP and PQC are similar as the corresponding sides are parallel. Since $AR/RB = QP/RB = QC/RP = CQ/QA$, we see that \mathbf{T} must carry R to Q . Thus triangles ARX and XQC are similar, so $\angle ARX = \angle XQC$, and hence $ARXQ$ is cyclic. Note that $\angle BAG = \angle CAX$ since X lies on the symmedian. Therefore X has the required properties.

7. (USA 2008) Let ABC be an acute, scalene triangle, and let M , N , and P be the midpoints of BC , CA , and AB , respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F , inside of triangle ABC . Prove that points A , N , F , and P all lie on one circle.

Solution: Let X be as in Problem 5. Then $\angle ABX = \angle XAC = \angle BAM = \angle ABF$, and analogously $\angle ACX = \angle ACF$, so $X = F$. Then this problem follows as a special case of the previous problem.

8. Let A be one of the intersection points of circles ω_1, ω_2 with centers O_1, O_2 . The line ℓ is tangent to ω_1, ω_2 at B, C respectively. Let O_3 be the circumcenter of triangle ABC . Let D be a point such that A is the midpoint of O_3D . Let M be the midpoint of O_1O_2 . Prove that $\angle O_1DM = \angle O_2DA$. (Hint: use Problem 5.)

Solution: We have $\angle AO_3C = 2\angle ABC = \angle AO_1B$, and triangles AO_3C and AO_1B are both isosceles, hence they are similar. So the spiral similarity at A that carries O_1 to B also carries O_3 to C , and it follows that AO_1O_3 and ABC are similar. Analogously, triangles ABC and AO_3O_2 are similar. So triangles AO_1O_3 and AO_3O_2 are similar.

Compare triangles AO_1D and ADO_2 . We have $\angle O_1AD = 180^\circ - \angle O_1AO_3 = 180^\circ - \angle O_3AO_2 = \angle DAO_2$. Also $AO_1/AD = AO_1/AO_3 = AO_3/AO_2 = AD/AO_2$. It follows that triangles AO_1D and ADO_2 are similar. It follows by Problem 5 that DA is a symmedian of DO_1O_2 , and thus $\angle O_1DM = \angle O_2DA$.