ON THE BRUHAT ORDER OF THE SYMMETRIC GROUP AND ITS SHELLABILITY

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Abstract. In this paper we discuss the Bruhat order of the symmetric group. We give two criteria for comparing elements in this poset and show that the poset is Eulerian. We also discuss the notion of shellability and EL-shellability, and use this concept to show that the order complex associated to the Bruhat order triangulates a sphere.

1. Introduction.

We start on a rather light note. The following problem was shortlisted for the 2006 International Mathematical Olympiad in Slovenia:

> A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.
>
> Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of *switches*, defined as follows:
>
> A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

It is not hard to see that the above description gives a partial ordering on the set of arrangements, which correspond to the set $\mathfrak{S}_n$ of permutations of $\{1, 2, 3, \ldots, n\}$. This is actually a well-known ordering called the *Bruhat order*. In this paper, we explore this partial order. We derive a number of results related to the Bruhat order of the symmetric group, including criteria for comparison, a result of Verma [10] that the the poset is Eulerian, and some results by Björner [1] and Edelman [6] related to the shellability of the poset that imply that the order complex of the poset triangulates a sphere.

Many of the results in the paper hold in the more general setting of *Coxeter groups*, which include symmetric groups as a special case. However, we will only discuss the symmetric group here. For discussion Coxeter groups, see Björner and Brenti [2]. Some of the proofs presented here differ from the ones found in the references as we make extensive use of the comparison criterion in Proposition 3.1, which is simply a restatement of the problem given at the beginning of this introduction. We believe that this approach offers a more visual and intuitive perspective on the Bruhat order of symmetric groups. However, due to space constraints, we leave out some of the technical details in some of the proofs and only sketch the ideas.

Our presentation of the results is inspired by Stanley [9]. In Section 2 we review some of the terminology related to partially ordered sets and introduce the Bruhat order of the symmetric group. In Section 3 we present two criteria for comparing elements in the Bruhat order. In Section 4 we prove the result of Verma that $\mathfrak{S}_n$ is Eulerian. In Section 5 we introduce the concept of

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EL-labeling, and show that $\mathfrak{S}_n$ admits an EL-labeling. In Section 6 we show that $\mathfrak{S}_n$ is shellable and thus $\Delta(x, y)$ triangulates a sphere for any $x < y$ in $\mathfrak{S}_n$.

2. Preliminaries.

2.1. **Partially ordered sets.** We follow [8, Ch. 3] for poset notation and terminology. We use $x \lessdot y$ to denote that y *covers* x, that is, $x < y$ and there is no z such that $x < z < y$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the covering relations, and such that if $x < y$ then y is drawn "above" x. A poset is said to be *bounded* if it has a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$ respectively. If $x, y \in P$ with $x \leq y$, we let $[x, y] = \{z \in P : x \leq z \leq y\}$, and we call it an *interval* of P. If $x, y \in P$, with $x < y$, a *chain* from x to y of length k is a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ such that $x = x_0 < x_1 < \cdots < x_k = y$, denoted simply by "$x_0 < x_1 < \cdots < x_k$". A chain is said to be *saturated* if all the relations in it are covering relations, and in this case we denote it by "$x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$". e A finite poset is said to be *graded* of *rank* n if all maximal chains of P have the same length n. In this case there is a unique *rank function* $\rho : P \to \{0, 1, \ldots, n\}$ such that $\rho(x) = 0$ is x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if y covers x in P.

For a finite poset P, we define the *Möbius function* $\mu : P \times P \to \mathbb{Z}$ recursively by

$$\mu(x, y) = \begin{cases} 0, & \text{unless } x \leq y \\ 1, & x = y \\ -\sum_{x \leq z < y} \mu(x, z), & x < y \end{cases}$$

We say that a poset is *Eulerian* if $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ for all $x \leq y$.

An *(abstract) simplicial complex* Δ on a vertex set V is a collection of subsets of V satisfying (i) if $x \in V$ then $\{x\} \in \Delta$ and (ii) if $S \in \Delta$ and $T \subset S$ then $T \in \Delta$. An element $S \in \Delta$ is called a *face* of Δ, the the *dimension* of S is defined to be $|S| - 1$. Also, define the *dimension* of Δ to be the supremum of $\dim F$ over all faces $F \in \Delta$. If P is a poset, then we can define the *order complex* of P to be the following simplicial complex: the vertices of $\Delta(P)$ are the elements of P, and the faces of $\Delta(P)$ are the chains in P.

2.2. **Bruhat order of the symmetric group.** Let $\mathfrak{S}_n$ denote the set of all permutations $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. If $w \in \mathfrak{S}_n$, then we can represent w as a *word* by $w = w_1 w_2 \cdots w_n$ with $w(i) = w_i$. We can also represent a permutation in a product of cycles. For example, if $w = 364152$, then we can also write $w = (1, 3, 4)(2, 6)$. Given $\sigma, \tau \in \mathfrak{S}_n$, we let $\sigma\tau = \sigma \circ \tau$ (composition of functions).

For $w \in \mathfrak{S}_n$, and $1 \leq k \leq n-1$, we say that k is a *descent* of w if $w(k) > w(k+1)$. Let $D(w) = \{k \mid w(k) > w(k+1)\}$ denote the set of descents of w. Define $\ell(w) = \#\{(i, j) \mid i < j, w(i) > w(j)\}$ to be the number of *inversions* of w.

We can define a partial order $\leq$ on $\mathfrak{S}_n$, call the (strong) *Bruhat order*, to be the transitive and reflexive closure of

$$u < (i, j)u, \quad \text{if } \ell((i, j)u) = 1 + \ell(u).$$

For instance, we have 6**2**718453 < 6**4**718**2**53 because all the numbers appearing between 2 and 4 (i.e., 7, 1, 8) are all greater or less than both 2 and 4. If $w \leq v$ in $\mathfrak{S}_n$, we will use $\ell(w, v) = \ell(v) - \ell(w)$ to denote the length of the interval $[w, v]$.

Let us say a word about the motivation behind the Bruhat order. Consider the set $\mathcal{F}(\mathbb{C}^n)$ of all (complete) flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

of subspaces of $\mathbb{C}^n$ (so $\dim V_i = i$). Then, for every such flag, after some row elimination, we can associate to it a unique sequence of vectors $v_1, v_2, \ldots, v_n \in \mathbb{C}^n$ such that (i) $\{v_1, \ldots, v_i\}$ is a basis for V_i and (ii) the $n \times n$ matrix with rows $v_1, \ldots, v_n$ has the form as shown by the example below,

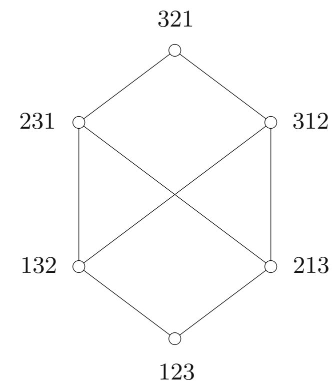

FIGURE 1. Bruhat Order of $\mathfrak{S}_3$ (taken from [2, p. 30])

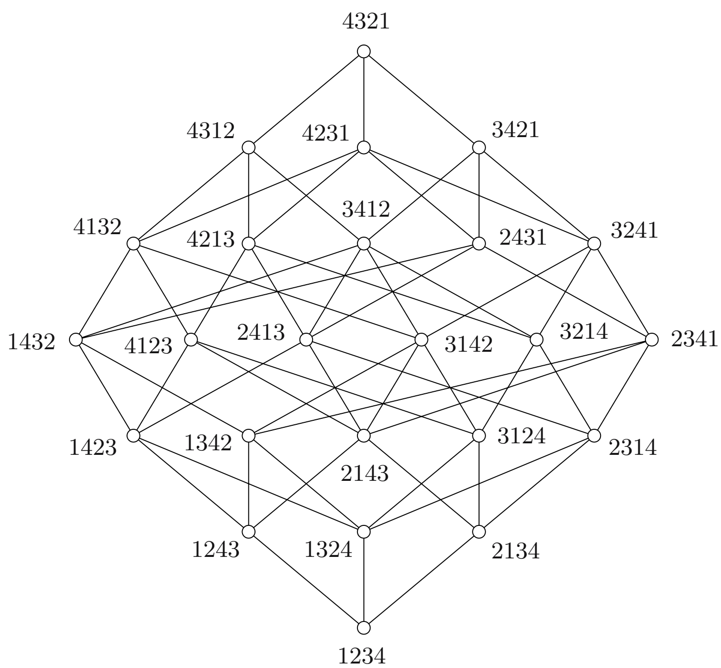

FIGURE 2. Bruhat Order of $\mathfrak{S}_4$ (taken from [2, p. 31])

where each $*$ represents an arbitrary element of $\mathbb{C}$ and the $*$'s are found at position where there is a 1 directly below and directly to the right:

$$\begin{matrix} * & * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & 1 \\ 0 & * & 0 & 1 & 0 & 0 \\ 0 & * & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix}$$

Then the positions of the 1's define a permutation associated to the flag. For instance, the above matrix gives the permutation 316452. For any $w \in \mathfrak{S}_n$, we can define the *Bruhat cell* (or *Schubert* *cell*) Ω_w to be the set of all flags associated to w. Then

$$\mathcal{F}(\mathbb{C}^n) = \bigsqcup_{w \in \mathfrak{S}_n} \Omega_w$$

and this is actually a cell decomposition in the sense of topology. Let $\overline{\Omega}_w$ be the topological closure of a Bruhat cell. Then how are these cells arranged? It turns out that, as shown by Ehresmann [7], $\overline{\Omega}_v \subseteq \overline{\Omega}_w$ if and only if $v \le w$ in the Bruhat order.

3. Two criteria for comparison.

Given two elements of $\mathfrak{S}_n$, how can we decide whether how they compare in the Bruhat order? In this section we address this question by providing two classic criteria for comparing elements in the Bruhat order.

Given a permutation $\sigma \in \mathfrak{S}_n$, define the *diagram* of σ to be a square of $n \times n$ cells, with the cell (i, j) (i.e., the cell on row i column j) filled with a dot if and only if $\sigma(i) = j$. Also, let $\sigma[i, j]$ denote the number of dots in contained in the upper-left $i \times j$ rectangle of the diagram of x (note that our notation here differs from the ones that appear in [2]).

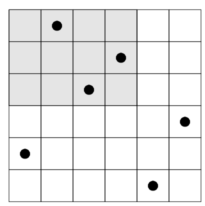

Figure 3. The diagram of $\sigma = 243615 \in \mathfrak{S}_6$. The shaded box illustrates $\sigma[3, 4] = 3$.

Our first criterion provides a solution to the problem given at the beginning. An equivalent, although slightly different phrased, criterion can be found in [2, Thm. 2.1.5].

Proposition 3.1. *Let $x, y \in \mathfrak{S}_n$. Then, $x \le y$ if and only if $x[i, j] \ge y[i, j]$ for all $1 \le i, j \le n$.*

Proof. To prove the "only if" part of the equivalence, we only need to check that whenever $x \lessdot x'$, we have $x[i, j] \ge x'[i, j]$ for all $1 \le i, j \le n$. Note that $x \lessdot x'$ means that x can be obtained from x' by switching two rows in the diagram, and so the inequality is quite easy to check.

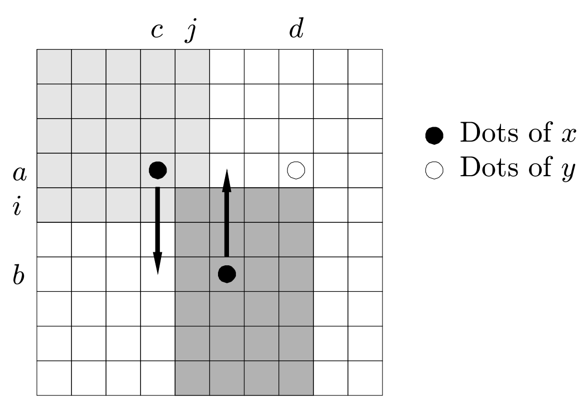

Figure 4. Illustrating the proof of the "if" part of Proposition 3.1.

To prove "if" part of the equivalence, we give a (greedy) algorithm to go from x up to y one inversion at a time. Suppose that $x[i, j] \ge y[i, j]$ for all $1 \le i, j \le n$. Let the a-th row be the first row where the two diagrams differ. On this row, suppose that x has a dot on the c-th column, and y has a dot on the d-th column (so that we must have $c < d$ since $x[a, c] \geq y[a, c]$). Consider the region $[a+1, n] \times [c+1, d]$ and let the uppermost dot of x in this region appear on the b-th row (this exists because the dot of x on the column d lies below row a). Then, no dots of x line in $[a+1, b-1] \times [c, d]$. Now, let $x' = x(a, b)$, so that the diagram of x' is obtained from the diagram of x by switching the rows a and b. We claim that $x'[i, j] \geq y[i, j]$ for all $1 \leq i, j \leq n$. Notice that $x'[i, j] = x[i, j]$ unless $a \leq i < b$ and $c \leq j < d$, in which case $x'[i, j] = x[i, j] - 1 = x[i, d] - 1 \geq y[i, d] - 1 \geq y[i, j]$. So that we can repeat this algorithm until we climb all the way up to y. ☐

We give another classic criterion, although we will not use it in the rest of this paper.

Proposition 3.2 (Tableau Criterion)**.** *Let $x, y \in \mathfrak{S}_n$. Then $x \leq y$ if and only if $x_{i,j} \leq y_{i,j}$ for all $1 \leq i \leq j \leq n$, where $x_{i,j}$ is the i-th entry in the increasing rearrangement of $x_1, x_2, \ldots, x_j$, and similarly for $y_{i,j}$.*

For instance, suppose that we want to compare $x = 35124$ and $y = 45123$. Construct the following the tableaux for x and y, where the j-th row of each tableau is constructed by removing the last $j - 1$ terms of the corresponding permutation and writing everything else in increasing order. Then we see that $x \leq y$ as the left tableau is componentwise less than or equal to the right tableau.

$$x : \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & \\ 1 & 3 & 5 & & \\ 3 & 5 & & & \\ 3 & & & & \end{array} \quad \leq \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & \\ 1 & 4 & 5 & & \\ 4 & 5 & & & \\ 4 & & & & \end{array} \quad : y$$

Proof. It is not hard to see that the numbers $x[i, j]$ and $x_{i,j}$ are related by the formulas

$$x[i, j] = \max\{k \mid x_{k,i} \leq j\}$$
$$x_{i,j} = \min\{k \mid x[j, k] = i\}$$

From this we see that $x[i, j] \geq y[i, j]$ for all i, j if and only if $x_{i,j} \leq y_{i,j}$ for all $i \leq j$. So Proposition 3.1 and 3.2 are equivalent. ☐

An improved version of the tableau criterion by Björner and Brenti may be found in [3] or [2, Theorem 2.6.3].

4. S_n is Eulerian.

In this section, we prove a result of Verma [10] that the Bruhat order is Eulerian. In fact, Verma showed this result for Coxeter groups in general, although here we will only prove it for the symmetric group. The only difference lies in the proof of the lifting property, whose proof for the Coxeter group in general can be found in [2, Prop. 2.2.7].

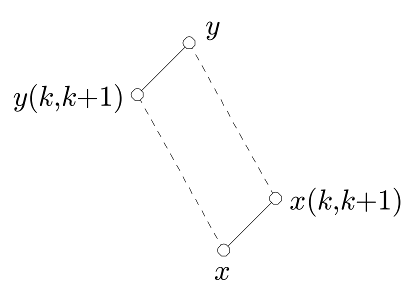

Figure 5. The lifting property.

Lemma 4.1 (Lifting Property)**.** *Suppose that $x < y$ in $\mathfrak{S}_n$ and $k \in D(y) \setminus D(x)$. Then $x \le y(k,k+1)$ and $x(k,k+1) \le y$.*

Proof. We give a proof of this lemma using the criterion in Proposition 3.1. Let $x' = x(k,k+1)$. (Notice that the diagram of x' is the diagram of x with the k-th row and the $(k+1)$-th row switched. So $x'[i,j] = x[i,j]$ for all i,j unless $i = k$ and $x_i \le j < x_{i+1}$, in which case $x'[i,j] = x[i,j] - 1$. So it suffices to show that for $x_k \le j < x_{k+1}$, we have $x'[k,j] \ge y[k,j]$, or equivalently, for $x[k,j] > y[k,j]$.

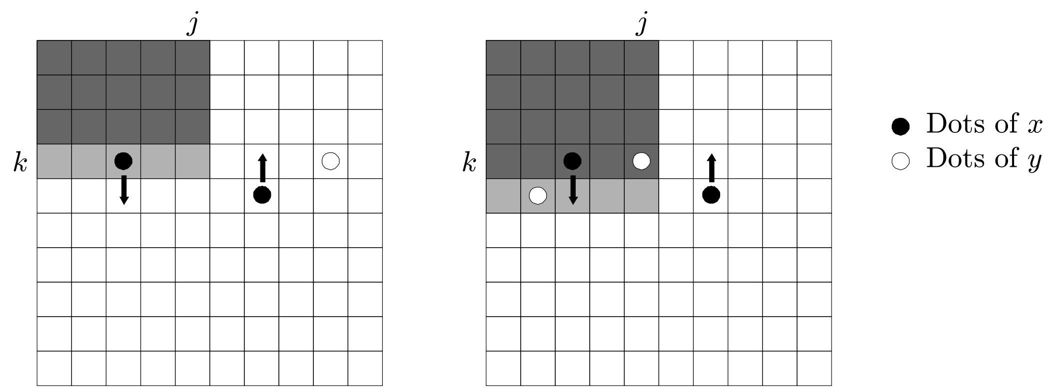

Figure 6. The two cases in the proof of the lifting property.

We consider two cases. If $y_k > j$, then we have $x[k,j] = x[k-1,j] + 1 \ge y[k-1,j] + 1 = y[k,j] + 1 > y[k,j]$. On the other hand, if $y_k \le j$, then $y[k+1,j] = y[k,j] + 1$ since $y_{k+1} < y_k \le j$, and thus $x[k,j] = x[k+1,j] \ge y[k+1,j] = y[k,j] + 1 > y[k,j]$.

So we have proved that $x(k,k+1) \le y$. Since k is a descent for both $x(k,k+1)$ and y, we can compose both permutations with $(k,k+1)$ on the right to get $x \le y(k,k+1)$. ☐

Theorem 4.2 (Verma [10])**.** *If $x \ge y$ in $\mathfrak{S}_n$, then*

$$\sum_{x \le z \le y} (-1)^{\ell(z)} = 1.$$

Proof. We use induction on $\ell(x) + \ell(y)$. The base case occurs when $\ell(x) = 0$ and $\ell(y) = 1$, then $x \lessdot y$ and the claim is clearly true.

For, let us fix $x < y$, and let $k \in D(y)$. We shall consider elements z between x and y. Let us denote $x' = x(k,k+1)$,$y' = y(k,k+1)$, and $z' = z(k,k+1)$. We consider two cases depending on whether $k \in D(x)$.

Case 1: $k \notin D(x)$. Then $x' > x$ and $y' < y$, so we can use Lemma 4.1 to deduce that $x \le z \le y$ if and only if $x \le z' \le y$ for all z. Indeed, suppose that $x \le z < y$ and $z < z'$, then we have $x \le z < z'$ and applying the lemma to the interval $[z,y]$ shows that $z' \le y$, and thus $x \le z' < y$. The other parts are analogous. Then, by pairing up z with z', we see that there are as many even permutations between x and y as there are odd permutations, and thus claim the true (we did not need the inductive hypothesis in this case).

Case 2: $k \in D(x)$. Then $x' < x$ and $y' < y$. We have

$$\{z \mid x \le z \le y\} = \{z \mid x' \le z \le y\} - \{z \mid x' \le z \le y, x \not\le z\}.$$

Since we can apply the induction hypothesis on the first set, it suffices to show that the second set has as many even permutations as odd permutations. Let us show that

$$\{z \mid x' \le z \le y, x \not\le z\} = \{z \mid x' \le z \le y', x \not\le z\}.$$

Indeed, suppose that $x' \leq z \leq y$ and $x \not\leq z$, we need to show that $z \leq y'$. We must have $z' > z$ since otherwise applying Lemma 4.1 to $[x', z]$ would imply that $x \leq z$, which is false. Thus, $z' > z$, and applying Lemma 4.1 again to $[z, y]$ show that $z \leq y'$.

Then,

$$\{z \mid x' \leq z \leq y', x \not\leq z\} = \{z \mid x' \leq z \leq y'\} - \{z \mid x \leq z \leq y'\}$$

and we can apply the induction hypothesis to the RHS to obtain the result. □

Using the recursive definition of the Mobius function, we obtain the following consequence as a corollary.

Corollary 4.3. *$\mathfrak{S}_n$ is Eulerian, that is, $\mu(x, y) = (-1)^{\ell(x,y)}$ for all $x \leq y$ in $\mathfrak{S}_n$.*

It is known that posets that arise from triangulations of spheres are always Eulerian[8, Prop. 3.8.9]. Then, seeing that $\mathfrak{S}_n$ is Eulerian, we may wonder whether $\Delta(\mathfrak{S}_n - \{\hat{0}, \hat{1}\})$ triangulates a sphere. It turns out that this is indeed the case. The rest of the paper will be focused on developing this result.

5. Lexicographic shellability

The goal of this section and the following section is to show that for all $x < y$ in $\mathfrak{S}_n$, the order complex $\Delta(x, y)$ triangulates a sphere.

The proof uses the concept of shellability of simplicial complexes. It is known that a shellable pseudomanifold triangulates a sphere. The fact that $\Delta(x, y)$ is a pseudomanifold is an easy consequence of the result that $\mathfrak{S}_n$ is Eulerian. So the main difficulty lies in showing $\mathfrak{S}_n$ is shellable. Björner [1] introduced the concept of edge-lexicographic(EL)-shellability, implies shellability. Then, Edelman [6] proved that $\mathfrak{S}_n$ is EL-shellable, thereby proving that $\Delta(x, y)$ triangulates a sphere.

For a finite poset P, let $\mathcal{E}_P = \{(x, y) \in P \times P \mid x \lessdot y\}$ the covering relation, or equivalently the set of edges in the Hasse diagram of P. An *edge-labeling* of P is a map $\lambda : \mathcal{E}_P \to \{1, 2, \dots\}$.

Definition 5.1. An edge labeling λ of P is called an *EL-labeling* if (i) for all $x < y$, there exists a unique saturated increasing chain $\mathbf{c} : x = x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_r = y$, that is, $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \lambda(x_{r-1}, x_r)$, and (ii) the label sequence of $\mathbf{c}$ lexicographically precedes that of all other saturated chains from x to y.

Definition 5.2. A poset is *EL-shellable* if it is bounded and graded and admits an EL-labeling.

Theorem 5.3 (Edelman [6])**.** *$\mathfrak{S}_n$ is EL-shellable.*

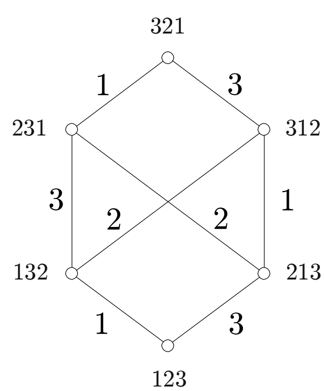

Figure 7. An EL-labeling of S_3.

Proof. (Sketch) We give an explicit labeling. Let $\tau_1, \tau_2, \ldots, \tau_{\binom{n}{2}}$ be the transpositions in $\mathfrak{S}_n$ in lexicographic order. For instance, for $n = 4$, we have $\tau_1 = (1,2), \tau_2 = (1,3), \tau_3 = (1,4), \tau_4 = (2,3), \tau_5 = (2,4), \tau_6 = (3,4)$. Now, let $\lambda(x,y) = j$ if $x \lessdot y$ and $\tau_j x = y$. We claim that λ is an EL-labeling of $\mathfrak{S}_n$.

We omit the proof that λ is an EL-labeling (consult [6] for details), but we will give an idea of the proof. Recall the proof of Proposition 3.1. It is not hard to see that given greedy algorithm in fact produces the lexicographically first chain, and this chain is increasing. We can also use the same setup to show that this is also the unique increasing chain. □

6. Shellability and sphere triangulation

Let Δ be a finite (abstract) simplicial complex. We say that Δ is *shellable* if Δ is pure and the facets of Δ can be given a linear order $F_1, F_2, \ldots, F_t$ in such a way that if $1 \le i < k \le t$ then there is a j, $1 \le j < k$, and an $x \in F_k$ such that $F_i \cap F_k \subset F_j \cap F_k = F_k - \{x\}$. Equivalently, the facet F_k is required to intersect the complex $\bigcup_{i=1}^{k-1} F_i$ in a non-empty union of maximal proper faces of F_k for each k. A linear order of the facets which satisfies this requirement is called a *shelling*. We say that a finite poset P is shellable if its order complex $\Delta(P)$ is shellable.

Theorem 6.1 (Björner [1])**.** *Let P be a EL-shellable poset. Then P is shellable.*

Proof. Let $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_t$ be the set of all maximal chains of P arranged according to the lexicographic ordering of their label sequence. We will show that this is in fact a shelling. To prove this, we need to show that whenever $1 \le i < k \le t$, there is a j, $1 \le j < k$, and an $x \in \mathbf{m}_k$ such that $\mathbf{m}_i \cap \mathbf{m}_k \subset \mathbf{m}_j \cap \mathbf{m}_k = \mathbf{m}_k - \{x\}$.

Suppose that $1 \le i < k \le t$. Denote the elements of the two maximal chains by $\mathbf{m}_i : \hat{0} = x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n = \hat{1}$ and $\mathbf{m}_k : \hat{0} = y_0 \lessdot y_1 \lessdot y_2 \lessdot \cdots \lessdot y_n = \hat{1}$. Let d be the greatest integer such that $x_i = y_i$ for $i = 0, 1, \ldots, d$, and let g be the least integer greater than d such that $x_g = y_g$. Then $g - d \ge 2$ and $x_i \ne y_i$ whenever $d < i < g$. Since the label sequence of m_i lexicographically precedes that m_k, the chain $x_d \lessdot x_{d+1} \lessdot \cdots \lessdot x_g$ cannot be the unique rising chain the interval $[x_d, x_g]$. Therefore, there must be some e with $d < e < g$ and $\lambda(x_{e-1}, x_e) > \lambda(x_e, x_{e+1})$. Let $x_{e-1} \lessdot x'_e \lessdot x_{e+1}$ be the unique increasing sequence from x_{e-1} to x_{e+1}. Then by replacing x_e by x'_e in $\mathbf{m}_k$, we get another maximal chain $\mathbf{m}_j$ satisfying $1 \le j < k$ and $\mathbf{m}_i \cap \mathbf{m}_k \subset \mathbf{m}_j \cap \mathbf{m}_k = \mathbf{m}_k - \{x_e\}$. This completes the proof. □

Now let us finish the last piece of the puzzle and conclude that $\Delta(x,y)$ triangulates a sphere for all $x < y$ in $\mathfrak{S}_n$. Recall that a *pseudomanifold* of dimension d is a d-dimensional simplicial complex such that every $(d-1)$-face is contained in exactly two facets.

Proposition 6.2. *A shellable (simplicial) pseudomanifold triangulates a sphere.*

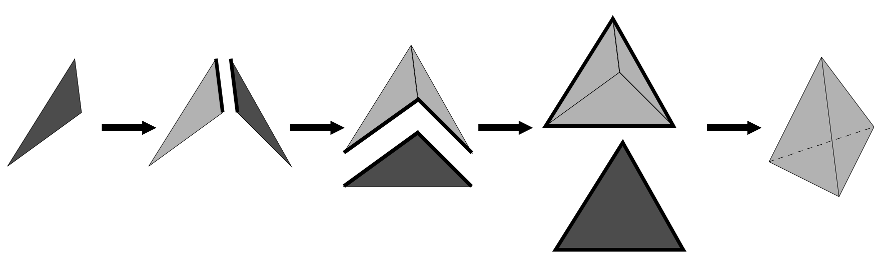

Figure 8. A shellable pseudomanifold triangulates a sphere.

Proof. (Sketch) Since this result uses concepts from PL (piecewise linear) topology, we shall only sketch the idea here. See [5, Prop. 1.2] or [4, Sec. 4.7] for details. Let $F_1, F_2, \ldots, F_t$ be the facets of the pseudomanifold arranged in a shelling order. Let Δ_i be the subcomplex generated by $\{F_1, F_2, \ldots, F_i\}$. At each step we construct Δ_i from Δ_{i-1} by gluing F_i to Δ_{i-1} via some union of the $(d-1)$-faces of F_i. Indeed, the condition for shelling implies that $\partial F_i \cap \Delta_{i-1}$ is a pure $(d-1)$-dimensional complex, and so in particular it can only be a $(d-1)$-ball or a $(d-1)$-sphere, with the latter occurring if $\partial F_i \subset \Delta_{i-1}$. Initially, we have $\Delta_1 = F_1$, which is a d-ball. At each step, if we glue F_i to Δ_{i-1} along a $(d-1)$-ball, we get another d-ball Δ_i. However, if we glue F_i to Δ_{i-1} along the $(d-1)$-sphere ∂F_i then we are gluing together the boundaries of two d-balls and so we must get a d-sphere, and we must stop here. Since the complex is a pseudomanifold, at the last step we must glue together the boundaries of F_t and Δ_{t-1}, thereby producing a d-sphere. □

Corollary 6.3. *If $x \leq y$ in $\mathfrak{S}_n$, then $\Delta(x, y)$ triangulates a sphere.*

Proof. Theorem 5.3 and Theorem 6.1 together imply that $\mathfrak{S}_n$ is shellable, so any interval $[x, y]$ must be shellable. Furthermore, from Theorem 4.2 we see that whenever $x < y$ with $\ell(x, y) = 2$, there are exactly two elements between x and y (in order words, every interval of rank 2 is isomorphic to B_2, the boolean algebra of rank 2). It follows that $\Delta(x, y)$ is a pseudomanifold. Then Proposition 6.2 implies that $\Delta(x, y)$ triangulates a sphere. □

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