The Number of Independent Sets in a Regular Graph

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**Introduction**

Let \( G = (V, E) \) be a graph. An **independent set** is a subset of the vertices with no two adjacent. Let \( i(G) \) denote the number of independent sets of \( G \).

**Figure 1:** The independent sets of a 4-cycle: \( i(C_4) = 7 \).

The following question is motivated by applications in combinatorial group theory [1] and statistical mechanics [4].

**Question.** In the family of \( N \)-vertex, \( d \)-regular graphs, when is the number of independent sets maximized?

Alon [1] in 1991 and Kahn [4] in 2001 conjectured that, when \( N/2d \in \mathbb{Z} \), \( i(G) \) should be maximized when \( G \) is a disjoint union of \( N/2d \) copies of \( K_{d,d} \), which has \( i(K_{d,d})^{N/2d} \) independent sets since \( i(G_1 \cup G_2) = i(G_1) i(G_2) \) for any graphs \( G_1 \) and \( G_2 \). More precisely, it was conjectured that:

**Conjecture (Alon and Kahn).** For any \( N \)-vertex, \( d \)-regular graph \( G \):

\[
i(G) \leq i(K_{d,d})^{N/2d} = (2^{d-1} - 1)^{N/2d}.
\]

Note equality holds if \( G \) is a disjoint union of \( K_{d,d} \)'s.

Our result confirms and generalizes this conjecture.

**Example:** Two 6-vertex 3-regular graphs:

- 13 independent sets
- 15 independent sets

**Previous results**

Alon [1] \( i(G) \leq (2^{d-1} + 1)^{N/2} \)

Kahn [4] Proved conjecture for bipartite \( G \)

Sapozhenko [6] \( i(G) \leq (2^{d-1} + 1)^{dN/2} \)

Kahn [5] \( i(G) \leq 2^{(1/2 + \epsilon)N} \)

Galvin [2] \( i(G) \leq 2^{(1/2 + \epsilon)N} \)

**Main Result**

For any \( N \)-vertex, \( d \)-regular graph \( G \), and any \( \lambda \geq 0 \),

\[
P(\lambda, G) \leq P(\lambda, K_{d,d})^{N/2d} = (2^{1+\lambda d} - 1)^{N/2d},
\]

with equality if \( G \) is a disjoint union of \( K_{d,d} \)’s. Here

\[
P(\lambda, G) = \sum_{k \geq 2} i(G^k)^{\lambda^k / k} = \sum_{k \geq 2} (\# \text{ ind. sets of size } k)^{\lambda^k / k}.
\]

Setting \( \lambda = 1 \) yields the Alon-Kahn conjecture.

**Proof**

We prove our main result by reducing it to the bipartite case, which was proven by Galvin and Tetali [3] (and by Kahn [4] for the non-weighted case).

From \( G \) we build \( G \cup G \) and \( G \times K_2 \):

**Key Lemma**

For any graph \( G \), there exists a size-preserving injection from \( I(G \cup G) \) to \( I(G \times K_2) \), where \( I(\cdot) \) denotes the collection of independent sets of a graph.

**Construction of the injection:**

- Start with an independent set \( A \cup B \) of \( G \cup G \):
  - *Merge* the two layers. Obtain \( A \cup B \subset V(G) \).
  - The induced subgraph \( G[A \cup B] \) is a bipartite graph since it is induced by the union of two independent sets. Choose the lexicographically first \( S \subset V(G) \) so that all edges of \( G[A \cup B] \) lie between \( S \) and \( V(G) \setminus S \).
  - Back to \( G \cup G \). Swap each pair of vertices in \( S \), and we obtain an independent set of \( G \times K_2 \).

**Claim.** This is an injection whose image consists of all independent sets \( C \cup D \) of \( G \times K_2 \) such that \( C \cap D \) is bipartite. Here \( C \subset V \) corresponds to the two “layers” of \( G \times K_2 \).

**Proof.** The construction always produces an independent set of \( G \times K_2 \) since swapping the vertices of \( S \) eliminates all possible adjacencies in \( G \times K_2 \).

We obtain the inverse map by basically the same procedure. See [7] for details.

**Further Questions**

**Non-regular graphs.** Kahn [4] also conjectured that, for any graph \( G \) without isolated vertices

\[
i(G) \leq \prod_{v \in V(G)} (2^{\deg(v)} - 1)^{1/(\deg(v))} \cdot \frac{1}{d}.
\]

**Non-entropy proof of bipartite case?** So far the only known proofs of the bipartite case of these results use entropy methods [3, 4]. It would be nice to have an elementary and completely combinatorial proof.

**Counting graph homomorphisms.** Galvin and Tetali [3] generalized Kahn’s result and showed that for any \( d \)-regular, \( N \)-vertex bipartite graph \( G \), and any graph \( H \) (possibly with self-loops),

\[
|\text{Hom}(G, H)| \leq |\text{Hom}(K_{d,d}, H)|^{N/2d}.
\]

Graph homomorphisms generalize the notion of independent sets as well as colorings. It is suspected that the inequality holds also for non-bipartite \( G \) as long as \( H \) is “nice,” but we do not have a proof.

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**References**