# YOUNG TABLEAUX AND THE REPRESENTATIONS OF THE SYMMETRIC GROUP

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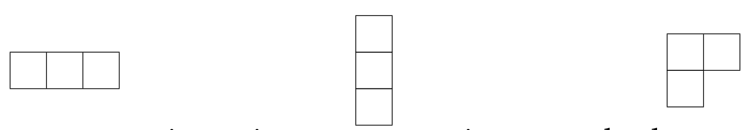
ABSTRACT. We explore an intimate connection between Young tableaux and representations of the symmetric group. We describe the construction of Specht modules which are irreducible representations of $S_n$, and also highlight some interesting results such as the branching rule and Young's rule.

## 1. INTRODUCTION

In this paper, we explore an intimate connection between two seemingly unrelated objects: on one hand we have representations of $S_n$; on the other hand we have combinatorial objects called Young tableaux, which are fillings of a certain configuration of boxes with entries from $\{1, 2, \ldots, n\}$, an example of which is shown below. We give a more precise definition of Young tableaux in Section 2.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

So how are representations of $S_n$ related to *Young tableau*? It turns out that there is a very elegant description of irreducible representations of $S_n$ through Young tableaux. Let us have a glimpse of the results. Recall that there are three irreducible representations of $S_3$. It turns out that they can be described using the set of *Young diagrams* with three boxes. The correspondence is illustrated below.



trivial representation sign representation standard representation

It is true in general that the irreducible representations of $S_n$ can be described using Young diagrams of $n$ boxes! Furthermore, we can describe a basis of each irreducible representation using *standard Young tableaux*, which are numberings of the boxes of a Young diagram with $1, 2, \ldots, n$ such that the rows and columns are all increasing. For instance, the bases of the standard representation of $S_3$ correspond to the following two standard Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

The dimension of the irreducible representations can be easily computed from its Young diagram through a result known as the hook-length formula, as we explain in Section 4.

*Date*: May 2008.

There are many other surprising connections between Young tableaux and representations of $S_n$, one of which is the following. Suppose we have an irreducible representation in $S_n$ and we want to find its induced representation in $S_{n+1}$. It turns out that the induced representation is simply the direct sum of all the representations corresponding to the Young diagrams obtained by adding a new square to the original Young diagram! For instance, the induced representation of the standard representation from $S_3$ to $S_4$ is simply

$$\mathrm{Ind}_{S_3}^{S_4} \ydiagram{2,1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}.$$

Similarly, the restricted representation can be found by removing a square from the Young diagram:

$$\mathrm{Res}_{S_2} \ydiagram{2,1} = \ydiagram{2} \oplus \ydiagram{1,1}.$$

In this paper, we describe the connection between Young tableaux and representations of $S_n$. The goal is to attract readers to the subject by showing a selection of very elegant and surprising results. Most proofs are omitted, but those who are interested may find them in [1], [2], or [3]. We assume familiarity with the basics of group representations, including irreducible representations and characters. Induced representations is used in Section 5. For references on group representations, see [2], [3] or [4].

In Section 2 we introduce Young diagrams and Young tableaux. In Section 3, we introduce tabloids and use them to construct a representation of $S_n$ known as the permutation module $M^\lambda$. However, permutation modules are generally reducible. In Section 4, we construct irreducible representations of $S_n$ known as Specht modules $S^\lambda$. Specht modules $S^\lambda$ correspond bijectively to Young diagrams $\lambda$, and they form a complete list of irreducible representations. In Section 5, we discuss the Young lattice and the branching rule, which are used to determine the induced and restricted representations of $S^\lambda$. Finally, in Section 6, we introduce Kostka numbers and state a result concerning the decomposition of permutation modules into the irreducible Specht modules.

## 2. Young Tableaux

First we need to settle some definitions and notations regarding partitions and Young diagrams.

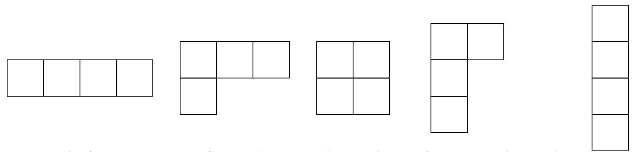
**Definition 2.1.** A *partition* of a positive integer $n$ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We write $\lambda \vdash n$ to denote that $\lambda$ is a partition of $n$.

For instance, the number 4 has five partitions: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$. We can also represent partitions pictorially using Young diagrams as follows.

**Definition 2.2.** A *Young diagram* is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing[1]. The Young diagram associated to the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is the one that has $l$ rows, and $\lambda_i$ boxes on the $i$th row.

[1]The notation used here is known as the *English notation*. Most Francophones, however, use the *French notation*, which is the upside-down form of the English notation. E.g. $(3,1)$ as $\ydiagram{1,3}$ .

For instance, the Young diagrams corresponding to the partitions of 4 are



(4) (3, 1) (2, 2) (2, 1, 1) (1, 1, 1, 1)

Since there is a clear one-to-one correspondence between partitions and Young diagrams, we use the two terms interchangeably, and we will use Greek letters $\lambda$ and $\mu$ to denote them.

A Young tableau is obtained by filling the boxes of a Young diagram with numbers.

**Definition 2.3.** Suppose $\lambda \vdash n$. A *(Young) tableau* $t$, of *shape* $\lambda$, is obtained by filling in the boxes of a Young diagram of $\lambda$ with $1, 2, \ldots, n$, with each number occurring exactly once. In this case, we say that $t$ is a $\lambda$-tableau.

For instance, here are all the tableaux corresponding to the partition $(2, 1)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

**Definition 2.4.** A *standard (Young) tableau* is a Young tableaux whose the entries are increasing across each row and each column.

The only standard tableaux for $(2, 1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Here is another example of a standard tableau:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}\,.$$

The definitions that we use here are taken from Sagan [3], however, other authors may have different conventions. For instance, Fulton [1], a Young tableau is a filling which is weakly increasing across each row and strictly increasing down each column, but may have repeated entries. We call such tableaux *semistandard* and we use them in Section 6.

Before we move on, let us recall some basic facts about permutations. Every permutation $\pi \in S_n$ has a decomposition into disjoint cycles. For instance $(123)(45)$ denotes the permutation that sends $1 \to 2 \to 3 \to 1$ and swaps 4 and 5 (if $n > 5$, then by convention the other elements are fixed by $\pi$). The *cycle type* of $\pi$ is the partition whose parts are the lengths of the cycles in the decomposition. So $(123)(45) \in S_5$ has cycle type $(3, 2)$. It is a basic result that two elements of $S_n$ are conjugates if and only if they have the same cycle type. The easiest way to see this is to consider conjugation as simply a relabeling of the elements when the permutation is written in cycle notation. Indeed, if

$$\pi = (a_1 a_2 \ldots a_k)(b_1 b_2 \ldots b_l) \cdots ,$$

and $\sigma$ sends $x$ to $x'$, then

$$\sigma \pi \sigma^{-1} = (a'_1 a'_2 \ldots a'_k)(b'_1 b'_2 \ldots b'_l) \cdots .$$

This means that the conjugacy classes of $S_n$ are characterized by the cycle types, and thus they correspond to partitions of $n$, which are equivalent to Young diagrams of size $n$. Recall from representation theory that the number of irreducible representations of a finite group is equal to the number of its conjugacy classes. So our goal for the next two sections is to construct an irreducible representation of $S_n$ corresponding to each Young diagram.

## 3. Tabloids and the Permutation Module $M^\lambda$

We would like to consider certain permutation representations of $S_n$. There is the obvious one: the permutation action of $S_n$ on the elements $\{1, 2, \ldots, n\}$, which extends to the *defining representation*. In this section, we construct other representations of $S_n$ using equivalence classes of tableaux, known as tabloids.

**Definition 3.1.** Two $\lambda$-tableaux $t_1$ and $t_2$ are *row-equivalent*, denoted $t_1 \sim t_2$, if the corresponding rows of the two tableaux contain the same elements. A *tabloid* of shape $\lambda$, or $\lambda$-tabloid is such an equivalence class, denoted by $\{t\} = \{t_1 \mid t_1 \sim t\}$ where $t$ is a $\lambda$-tabloid. The tabloid $\{t\}$ is drawn as the tableaux $t$ without vertical bars separating the entries within each row.

For instance, if

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

then $\{t\}$ is the tabloid drawn as

$$\begin{array}{|cc|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}$$

which represents the equivalence class containing the following two tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The notation is suggestive as it emphasizes that the order of the entries within each row is irrelevant, so that each row may be shuffled arbitrarily. For instance

$$\begin{array}{|ccc|}\hline 1 & 4 & 7 \\ \hline \multicolumn{2}{|c|}{3 \;\; 6} \\ \cline{1-2} \multicolumn{2}{|c|}{2 \;\; 5} \\ \cline{1-2}\end{array} = \begin{array}{|ccc|}\hline 4 & 7 & 1 \\ \hline \multicolumn{2}{|c|}{6 \;\; 3} \\ \cline{1-2} \multicolumn{2}{|c|}{2 \;\; 5} \\ \cline{1-2}\end{array} \neq \begin{array}{|ccc|}\hline 4 & 7 & 1 \\ \hline \multicolumn{2}{|c|}{6 \;\; 5} \\ \cline{1-2} \multicolumn{2}{|c|}{2 \;\; 3} \\ \cline{1-2}\end{array} \neq \begin{array}{|ccc|}\hline 4 & 7 & 1 \\ \hline \multicolumn{2}{|c|}{2 \;\; 3} \\ \cline{1-2} \multicolumn{2}{|c|}{6 \;\; 5} \\ \cline{1-2}\end{array}$$

We want to define a representation of $S_n$ on a vector space whose basis is exactly the set of tabloids of a given shape. We need to find a way for elements of $S_n$ to act on the tabloids. We can do this in the most obvious manner, that is, by letting the permutations permutate the entries of the tabloid. For instance, the cycle $(1\,2\,3) \in S_3$ acts on a tabloid by changing replacing its "1" by a "2", its "2" by a "3", and its "3" by a "1", as shown below:

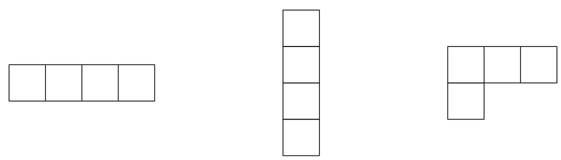
$$(1\,2\,3)\begin{array}{|cc|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array} = \begin{array}{|cc|}\hline 2 & 3 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1}\end{array}$$

We should check that this action is well defined, that is, if $t_1$ and $t_2$ are row-equivalent, so that $\{t_1\} = \{t_2\}$, then the result of permutation should be the same, that is, $\pi\{t_1\} = \pi\{t_2\}$. This is clear, as $\pi$ simply gives the instruction of moving some number from one row to another.

Now that we have defined a way for $S_n$ to act on tabloids, we are ready to define a representation of $S_n$. Recall that a representation of a group $G$ on a complex vector space $V$ is equivalent to extending $V$ to a $\mathbb{C}[G]$-module, so we often use the term *module* to describe representations.

**Definition 3.2.** Suppose $\lambda \vdash n$. Let $M^\lambda$ denote the vector space whose basis is the set of $\lambda$-tabloids. Then $M^\lambda$ is a representation of $S_n$ known as the *permutation module corresponding to $\lambda$*.

Let us show a few example of permutation modules. We see that the $M^\lambda$ corresponding to the following Young diagrams are in fact familiar representations.



**Example 3.3.** Consider $\lambda = (n)$. We see that $M^\lambda$ is the vector space generated by the single tabloid

$$\boxed{1 \quad 2 \quad \cdots \quad n}\,.$$

Since this tabloid is fixed by $S_n$, we see that $M^{(n)}$ is the one-dimensional trivial representation.

**Example 3.4.** Consider $\lambda = (1^n) = (1, 1, \ldots, 1)$. Then a $\lambda$-tabloid is simply a permutation of $\{1, 2, \ldots, n\}$ into $n$ rows and $S_n$ acts on the tabloids by acting on the corresponding permutation. It follows that $M^{(1^n)}$ is isomorphic to the regular representation $\mathbb{C}[S_n]$.

**Example 3.5.** Consider $\lambda = (n-1, 1)$. Let $\{t_i\}$ be the $\lambda$-tabloid with $i$ on the second row. Then $M^\lambda$ has basis $\{t_1\}, \{t_2\}, \ldots, \{t_n\}$. Also, note that the action of $\pi \in S_n$ sends $t_i$ to $t_{\pi(i)}$. And so $M^{(n-1,1)}$ is isomorphic to the defining representation $\mathbb{C}\{1, 2, \ldots, n\}$. For example, in the $n = 4$ case, the representation $M^{(3,1)}$ has the following basis:

$$t_1 = \begin{array}{|lll|}\hline 2 & 3 & 4 \\ \hline 1 & & \\ \hline\end{array}\,, \quad t_2 = \begin{array}{|lll|}\hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline\end{array}\,, \quad t_3 = \begin{array}{|lll|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline\end{array}\,, \quad t_4 = \begin{array}{|lll|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline\end{array}\,.$$

Now we consider the dimension and characters of the representation $M^\lambda$. First, we shall give a formula for the number of tabloids of each shape.

**Proposition 3.6.** *If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$,*

$$\dim M^\lambda = \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_l!}.$$

*Proof.* Since the basis for $M^\lambda$ is the set of $\lambda$-tabloids, the dimension of $M^\lambda$ is equal to the number of distinct $\lambda$-tabloids. So let us count the number of $\lambda$-tabloids.

Since there are $\lambda_i!$ ways to permute the $i$th row, the number of tableaux in each row-equivalence class is $\lambda_1! \lambda_2! \cdots \lambda_l!$. Since there are $n!$ tableaux in total, the number of equivalence classes is given by $\frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_l!}$. $\square$

Now we give a formula for the characters of $M^\lambda$.

**Proposition 3.7.** *Suppose $\lambda = (\lambda_1, \ldots, \lambda_l), \mu = (\mu_1, \ldots, \mu_m)$ are partitions of $n$. The character of $M^\lambda$ evaluated at an element of $S_n$ with cycle type $\mu$ is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ in*

$$\prod_{i=1}^{m} (x_1^{\mu_i} + x_2^{\mu_i} + \cdots + x_l^{\mu_i}).$$

*Proof.* Since $M^\lambda$ can be realized as a permutation representation on the $\lambda$-tabloids, its character at an element $\pi \in S_n$ is equal to the number of tabloids fixed by $\pi$. Now, let us consider the properties of a tabloid fixed by $\pi$. Consider the cycle decomposition of $\pi$. Then, the elements of a cycle must all belong to the same row of the tabloid. In other words, to construct a tabloid fixed by $\pi$, we need to select, for each cycle of $\pi$, a row of the tabloid that this cycle should belong to.

In fact, this is exactly what the polynomial enumerates. As we expand the polynomial, the term that we select in each factor corresponds to the choice of which row in the tabloid we would like to put the cycle in. Specifically, choosing the term $x_j^{\mu_i}$ corresponds to placing a cycle of length $\mu_i$ into the $j$-th row of the tabloid. Then, for any term in the expansion, the exponent of $x_j$ corresponds to the total number of elements placed in the $j$-th row. So the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ is precisely the number of tabloids of shape $\lambda$ that comes from this process, i.e., the number of $\lambda$-tabloids fixed by a permutation of cycle type $\mu$. $\square$

Note that Proposition 3.6 also follows as a corollary to the above result. Indeed, the dimension of a representation is simply the value of the character at the identity element, which has cycle type $\mu = (1^n)$. So Proposition 3.7 tells us that the dimension of $M^\lambda$ is the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ in $(x_1 + \cdots + x_n)^n$, which is equal to $\dim M^\lambda = \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_l!}$ by the multinomial expansion formula.

**Example 3.8.** Let us compute the full list of the characters of the permutation modules for $S_4$. The character at the identity element is equal to the dimension, and it can found through Proposition 3.6. For instance, the character of $M^{(2,1,1)}$ at $e \in s_4$ is $4!/2! = 12$.

Say we want to compute the character of $M^{(2,2)}$ at the permutation $(12)$, which has cycle type $(2,1,1)$. Using Proposition 3.7, we see that the character is equal to the coefficient of $x_1^2 x_2^2$ in $(x_1^2 + x_2^2)(x_1 + x_2)^2$, which is 2. Other characters can be similarly computed, and the result is shown in the following table.

| permutation<br>cycle type | $e$<br>$(1,1,1,1)$ | $(12)$<br>$(2,1,1)$ | $(12)(34)$<br>$(2,2)$ | $(123)$<br>$(3,1)$ | $(1234)$<br>$(4)$ |
|---|---|---|---|---|---|
| $M^{(4)}$ | 1 | 1 | 1 | 1 | 1 |
| $M^{(3,1)}$ | 4 | 2 | 0 | 1 | 0 |
| $M^{(2,2)}$ | 6 | 2 | 2 | 0 | 0 |
| $M^{(2,1,1)}$ | 12 | 2 | 0 | 0 | 0 |
| $M^{(1,1,1,1)}$ | 24 | 0 | 0 | 0 | 0 |

Note that in the above example, we did *not* construct the character table for $S_4$, as all the $M^\lambda$ are in fact reducible with the exception of $M^{(4)}$. In the next section, we take a step further and construct the irreducible representations of $S_n$.

## 4. Specht Modules

In the previous section, we constructed representations $M^\lambda$ of $S_n$ known as permutation modules. In this section, we consider an irreducible subrepresentation of $M^\lambda$ that corresponds uniquely to $\lambda$.

The group $S_n$ acts on the set of Young tableaux in the obvious manner: for a tableaux $t$ of size $n$ and a permutation $\sigma \in S_n$, the tableaux $\sigma t$ is the tableaux that puts the number $\pi(i)$ to the box where $t$ puts $i$. For instance,

$$(1\,2\,3)(4\,5)\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3&6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 2&3&5&4\\ \hline 1&6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}.$$

Observe that a tabloid is fixed by the permutations which only permute the entries of the rows among themselves. These permutations form a subgroup of $S_n$, which we call the row group. We can similarly define the column group.

**Definition 4.1.** For a tableau $t$ of size $n$, the *row group* of $t$, denoted $R_t$, is the subgroup of $S_n$ consisting of permutations which only permutes the elements within each row of $t$. Similarly, the *column group* $C_t$ is the the subgroup of $S_n$ consisting of permutations which only permutes the elements within each column of $t$.

For instance, if

$$t = \begin{array}{|c|c|c|}\hline 4&1&2\\ \hline 3&5 \\ \cline{1-2}\end{array}$$

then

$$R_t = S_{\{1,2,4\}} \times S_{\{3,5\}}, \quad \text{and} \quad C_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}}.$$

Let us select certain elements from the space $M^\lambda$ that we use to to span a subspace.

**Definition 4.2.** If $t$ is a tableau, then the associated *polytabloid* is

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi\{t\}.$$

So we can find $e_t$ by summing all the tabloids that come from column-permutations of $t$, taking into account the sign of the column-permutation used. For instance, if

$$t = \begin{array}{|c|c|c|}\hline 4&1&2\\ \hline 3&5 \\ \cline{1-2}\end{array},$$

then

$$e_t = \begin{array}{ccc}\hline 4&1&2\\ \hline 3&5 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 3&1&2\\ \hline 4&5 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 4&5&2\\ \hline 3&1 \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 3&5&2\\ \hline 4&1 \\ \cline{1-2}\end{array}.$$

Now, through the following technical lemma, we see that $S_n$ permutes the set of polytabloids.

**Lemma 4.3.** *Let $t$ be a tableau and $\pi$ be a permutation. Then $e_{\pi t} = \pi e_t$.*

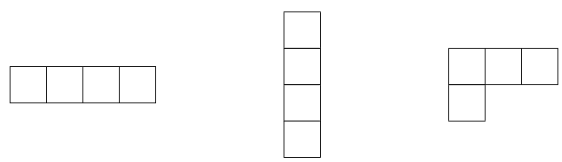
*Proof.* First observe that $C_{\pi t} = \pi C_t \pi^{-1}$, which can be viewed as a "relabeling" similar to the discussion at the end of Section 2. Then, we have

$$\begin{aligned} e_{\pi t} &= \sum_{\sigma \in C_{\pi t}} \operatorname{sgn}(\sigma)\sigma\{\pi t\} = \sum_{\sigma \in \pi C_t \pi^{-1}} \operatorname{sgn}(\sigma)\sigma\{\pi t\} \\ &= \sum_{\sigma' \in C_t} \operatorname{sgn}(\pi\sigma'\pi^{-1})\pi\sigma'\pi^{-1}\{\pi t\} = \pi \sum_{\sigma' \in C_t} \operatorname{sgn}(\sigma')\sigma'\{t\} = \pi e_t. \end{aligned}$$ □

Now we are ready to extract an irreducible subrepresentation from $M^\lambda$.

**Definition 4.4.** For any partition $\lambda$, the corresponding *Specht module*, denoted $S^\lambda$, is the submodule of $M^\lambda$ spanned by the polytabloids $e_t$, where $t$ is taken over all tableaux of shape $\lambda$.

Again, let us look at a few examples. We see that the Specht modules corresponding to the following Young diagrams are familiar irreducible representations.



**Example 4.5.** Consider $\lambda = (n)$. Then there is only one polytabloid, namely

$$\boxed{1 \quad 2 \quad \cdots \quad n}$$

Since this polytabloid is fixed by $S_n$, we see that $S^{(n)}$ is the one-dimensional trivial representation.

**Example 4.6.** Consider $\lambda = (1^n) = (1, 1, \ldots, 1)$. Let

$$t = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$$

Observe that $e_t$ is a sum of all the $\lambda$-tabloids multiplied by the sign of permutation it took to get there. For any other $\lambda$-tableau $t'$, we have either $e_t = e_{t'}$ if $t'$ is obtained from $t$ through an even permutation, or $e_t = -e_{t'}$ if $t'$ is obtained from $t$ through an odd permutation. So $S^\lambda$ is a one-dimensional representation. From Lemma 4.3 we have $\pi e_t = e_{\pi t} = \operatorname{sgn}(\pi)e_t$. From this we see that $S^{(1^n)}$ is the sign representation.

**Example 4.7.** Consider $\lambda = (n-1, 1)$. Continuing the notation from Example 3.5 where we use $\{t_i\}$ to denote the $\lambda$-tabloid with $i$ on the second row, we see that the polytabloids have the form $\{t_i\} - \{t_j\}$. Indeed, the polytabloid constructed from the tableau

$$\begin{array}{|c|c|c|c|} \hline i & a & b & \cdots \\ \hline j \\ \cline{1-1} \end{array}$$

is equal to $\{t_i\} - \{t_j\}$. Let us temporarily use $\mathbf{e}_i$ to denote the tabloid $\{t_i\}$. Then $S^\lambda$ is spanned by elements of the form $\mathbf{e}_i - \mathbf{e}_j$, and it follows that

$$S^{(n-1,1)} = \{c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots c_n\mathbf{e}_n \mid c_1 + c_2 + \cdots + c_n = 0\}.$$

This is an irreducible representation known as the standard representation. The direct sum of the standard representation and the trivial representation gives the defining representation, that is, $S^{(n-1,1)} \oplus S^{(n)} = M^{(n-1,1)}$.

We know that the $S_3$ has three irreducible representations: trivial, sign, and standard. These are exactly the ones described above. Furthermore, there are exactly three partitions of 3: $(3)$, $(1,1,1)$, $(2,1)$. So in this case, the irreducible representations are exactly the Specht modules. Amazingly, this is true in general.

**Theorem 4.8.** *The Specht modules $S^\lambda$ for $\lambda \vdash n$ form a complete list of irreducible representations of $S_n$ over $\mathbb{C}$.*

The proof may be found in Sagan [3]. We do not describe the proof here, but we remark that the fact that the number of irreducible representations of $S_n$ equals the number of Young diagrams with $n$ boxes has already been discussed at the end of Section 2.

Note that the polytabloids are generally not independent. For instance, as we saw in Example 4.6, any pair of polytabloids in $S^{(1^n)}$ are in fact linearly dependent. Since we know that $S^\lambda$ is spanned by the polytabloids, we may ask how to select a basis for vector space from the set of polytabloids. There is an elegant answer to this question: the set of polytabloids constructed from standard tableaux form a basis for $S^\lambda$. Recall that a standard tableau is a tableau with increasing rows and increasing columns.

**Theorem 4.9.** *Let $\lambda$ be any partition. The set*

$$\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$$

*forms a basis for $S^\lambda$ as a vector space.*

The proof may be found in Sagan [3]. We only sketch an outline here. First, an ordering is imposed on tabloids. If some linear combination of $e_t$ is zero, summed over some standard tableaux $t$, then by looking at a maximal tabloid in the sum, one can deduce that its coefficient must be zero and conclude that $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ is independent. Next, to prove that the set spans $S^\lambda$, a procedure known as the *straightening algorithm* is used to write an arbitrary polytabloid as a linear combination of standard polytabloids.

Now we look at some consequences of the result. Let $f^\lambda$ denote the number of standard $\lambda$-tableaux. Then the following result follows immediately from Theorem 4.9.

**Corollary 4.10.** *Suppose $\lambda \vdash n$, then* $\dim S^\lambda = f^\lambda$.

Let us end this section with a few results concerning $f^\lambda$.

**Theorem 4.11.** *If $n$ is a positive integer, then*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n,!$$

*where the sum is taken over all partitions of $n$.*

*Proof.* Recall from representation theory that the sum of the squares of the irreducible representation is equal to the order of the group. This theorem follows from that fact and Corollary 4.10. ☐

Theorem 4.11 also has an elegant combinatorial proof using the celebrated RSK correspondence. See [1] or [3] for details.

Given the partition $\lambda$, the number $\dim S^\lambda = f^\lambda$ can be computed easily using the *hook-length formula* of Frame, Robinson, and Thrall, which we state now.

**Definition 4.12.** Let $\lambda$ be a Young diagram. For a square $u$ in the diagram (denoted by $u \in \lambda$), we define the *hook* of $u$ (or at $u$) to be the set of all squares directly to the right of $u$ or directly below $u$, including $u$ itself. The number of squares in the hook is called the *hook-length* of $u$ (or at $u$), and is denoted by $h_\lambda(u)$.

For example, consider the partition $\lambda = (5,5,4,2,1)$. The figure on the left shows a typical hook, and the figure on the right shows all the hook-lengths.

| | | | | |
|---|---|---|---|---|
| | | | | |
| | $u$ | • | • | • |
| | • | | | |
| | • | | | |
| | | | | |

| | | | | |
|---|---|---|---|---|
| 9 | 7 | 5 | 4 | 1 |
| 8 | 6 | 4 | 3 | 1 |
| 6 | 4 | 2 | 1 | |
| 3 | 1 | | | |
| 1 | | | | |

**Theorem 4.13** (Hook-length formula)**.** *Let $\lambda \vdash n$ be a Young diagram. Then*

$$\dim S^\lambda = f^\lambda = \frac{n!}{\prod_{u\in\lambda} h_\lambda(u)}.$$

For instance, from the above example, we get

$$\dim S^{(5,5,4,2,1)} = f^{(5,5,4,2,1)} = \frac{17!}{9\cdot 7\cdot 5\cdot 4\cdot 1\cdot 8\cdot 6\cdot 4\cdot 3\cdot 1\cdot 6\cdot 4\cdot 2\cdot 1\cdot 3\cdot 1\cdot 1} = 3403400.$$

For proof of the hook-length formula, see Sagan [3].

Finally, we state a formula for the characters of the representation $S^\lambda$.

**Theorem 4.14** (Frobenius formula)**.** *Suppose $\lambda = (\lambda_1, \dots, \lambda_l), \mu = (\mu_1, \dots, \mu_m)$ are partitions of $n$. The character of $S^\lambda$ evaluated at an element of $S_n$ with cycle type $\mu$ is equal to the coefficient of $x_1^{\lambda_1+l-1} x_2^{\lambda_2+l-2} \cdots x_l^{\lambda_l}$ in*

$$\prod_{1\le i<j\le l} (x_i - x_j) \prod_{i=1}^m (x_1^{\mu_i} + x_2^{\mu_i} + \cdots + x_l^{\mu_i}).$$

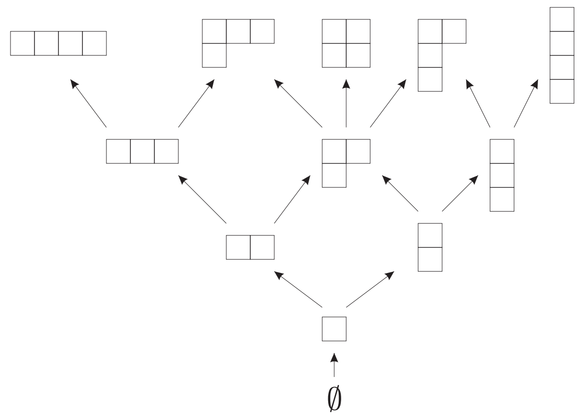
See Fulton and Harris [2] for proof. Observe the similarity between the statements of Proposition 3.6 and hook-length formula, and also between Proposition 3.7 and the Frobenius formula. The hook-length formula can also be derived from the Frobenius formula by evaluating the character at the identity element. Again, see [2] for details.

## 5. Young Lattice and Branching Rule

Now let us consider the relationships between the irreducible representations of $S_n$ and those of $S_{n+1}$.

Consider the set of all Young diagrams. These diagrams can be partially ordered by inclusion. The resulting partially ordered set is known as *Young's lattice*.

We can represent Young's lattice graphically as follows. Let $\lambda \nearrow \mu$ denote that $\mu$ can be obtained by adding a single square to $\lambda$. At the $n$th level, all the Young diagrams with $n$ boxes are drawn. In addition, $\lambda$ to connected to $\mu$ if $\lambda \nearrow \mu$. Here is a figure showing the bottom portion of Young's lattice (of course, it extends infinitely upwards).

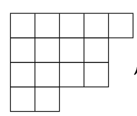


Now we consider the following question: given $S^\lambda$ a representation of $S_n$, how can we determine its restricted representation in $S_{n-1}$ and its induced representation in $S_{n+1}$? There is a beautiful answer to this question, given by Young's branching rule.

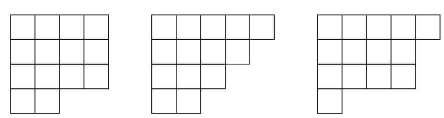
**Theorem 5.1** (Branching Rule). *Suppose $\lambda \vdash n$, then*

$$\operatorname{Res}_{S_{n-1}} S^\lambda \cong \bigoplus_{\mu:\mu \nearrow \lambda} S^\mu \quad \textit{and} \quad \operatorname{Ind}_{S_n}^{S_{n+1}} S^\lambda \cong \bigoplus_{\mu:\lambda \nearrow \mu} S^\mu.$$

For instance, if $\lambda = (5,4,4,2)$, so that

$\lambda =$  ,

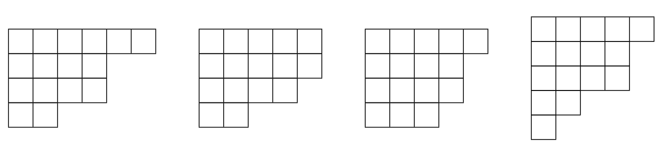
then the diagrams that can be obtained by removing a square are



So

$$\operatorname{Res}_{S_{14}} S^{(5,4,4,2)} = S^{(4,4,4,2)} \oplus S^{(5,4,3,2)} \oplus S^{(5,4,4,1)}.$$

Similarly, the diagrams that can be obtained by adding a square are



So

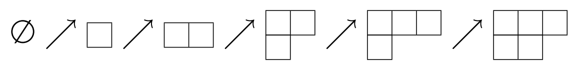
$$\operatorname{Ind}_{S_{15}}^{S_{16}} S^{(5,4,4,2)} = S^{(6,4,4,2)} \oplus S^{(5,5,4,2)} \oplus S^{(5,4,4,3)} \oplus S^{(5,4,4,2,1)}.$$

The proof of Theorem 5.1 may be found in Sagan [3]. We shall only mention that the two parts of the branching rules are equivalent through the Frobenius reciprocity theorem.

There is an interesting way to view this result. If we consider $S^\lambda$ only as a vector space, then the branching rule implies that

$$S^\lambda \cong \bigoplus_{\mu:\mu \nearrow \lambda} S^\mu \cong \bigoplus_{\nu:\nu \nearrow \mu \nearrow \lambda} S^\nu \cong \cdots \cong \bigoplus_{\emptyset=\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)}=\lambda} S^\emptyset$$

The final sum is indexed over all upward paths from $\emptyset$ to $\lambda$ in Young's lattice. Since $S^{\emptyset}$ is simply an one-dimensional vector space, it follows that we can construct a basis for $S^\lambda$ where each basis vector corresponds to a upward path in the Young lattice from $\emptyset$ to $\lambda$. However, observe that upward paths in the Young lattice from $\emptyset$ to $\lambda$ correspond to standard $\lambda$-tableaux! Indeed, for each standard $\lambda$-tableaux, we can associate to it a path in the Young lattice constructed by adding the boxes in order as labeled in the standard tableaux. The reverse construction is similar. As an example, the following path in the Young lattice



corresponds to the following standard tableau

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\,.$$

So we have recovered a basis for $S^\lambda$ which turned out to be the same as the one found in Theorem 4.9.

Now, one may object that this argument contains some circular reasoning, namely because the proof of the branching rule (as given in Sagan [3]) uses Theorem 4.9, that a basis of $S^\lambda$ can be found through standard tableaux. This is indeed the case. However, there is an alternative view on the subject, given recently by Okounkov and Vershik [5], in which we start in an abstract algebraic setting with some generalized form of the Young lattice. Then, we can form a basis known as the Gelfand-Tsetlin basis by taking upward paths as we did above. We then specialize to the symmetric group and "discover" the standard tableaux. This means that the standard tableaux in some sense form a "natural" basis for $S^\lambda$.

## 6. Decomposition of $M^\mu$ and Young's Rule

First, we constructed the permutation modules $M^\lambda$, and from it we extracted irreducible subrepresentations $S^\lambda$, such that $S^\lambda$ forms a complete list of irreducible representations of $S_n$ as $\lambda$ varies over all partitions of $n$.

Let us revisit $M^\mu$ and ask, how does $M^\mu$ decompose into irreducible representations. It turns out that $M^\mu$ only contains the irreducible $S^\lambda$ if $\lambda$ is, in some sense, "greater" than $\mu$! To make this notation more precise, let us define a partial order on partitions of $n$. (Note that this is not the same as the one used to define Young's lattice!)

**Definition 6.1.** Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ are partitions of $n$. Then $\lambda$ *dominates* $\mu$, written $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \geq 1$. If $i > l$ (respectively, $i > m$), then we take $\lambda_i$ (respectively, $\mu_i$) to be zero.

In other words, $\lambda \trianglerighteq \mu$ if, for every $k$, the first $k$ rows of the Young diagram of $\lambda$ contains more squares than that of $\mu$. Intuitively, this means that diagram for $\lambda$ is short and fat and the diagram for $\mu$ is long and skinny.

For example, when $n = 6$, we have $(3,3) \trianglerighteq (2,2,1,1)$. However, $(3,3)$ and $(4,1,1)$ are incomparable, as neither dominates the other. The dominance relations for partitions of 6 is depicted using the following figure. Such diagrams are known as *Hasse diagrams* which are used for representing partially ordered sets.

$$
\begin{tikzcd}[column sep=small]
 & (6) & \\
 & (5) \arrow[u] & \\
 & (4,2) \arrow[u] & \\
(3,3) \arrow[ur] & & (4,1,1) \arrow[ul] \\
 & (3,2,1) \arrow[ul] \arrow[ur] & \\
(3,1,1,1) \arrow[ur] & & (2,2,2) \arrow[ul] \\
 & (2,2,1,1) \arrow[ul] \arrow[ur] & \\
 & (2,1,1,1,1) \arrow[u] & \\
 & (1,1,1,1,1,1) \arrow[u] &
\end{tikzcd}
$$

Now we can precisely state what we wanted to say at the beginning of the section.

**Proposition 6.2.** *$M^\mu$ contains $S^\lambda$ as a subrepresentation if and only if $\lambda \trianglerighteq \mu$. Also, $M^\mu$ contains exactly one copy of $S^\mu$.*

We may ask how many copies of $S^\lambda$ is contained in $M^\mu$. It turns out that this answer has a nice combinatorial interpretation. In order to describe it, we need a few more definitions.

**Definition 6.3.** A *semistandard tableau of shape $\lambda$* is an array $T$ obtained by filling in the boxes of $\lambda$ with positive integers, repetitions allowed, and such that the rows weakly increase and the columns strictly increase. The *content* of $T$ is the composition $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$, where $\mu_i$ equals the number of $i$'s in $T$.

For instance, the following semistandard tableau has shape $(4,2,1)$ and content $(2,2,1,0,1,1)$:

$$
\begin{ytableau} 1 & 1 & 2 & 5 \\ 2 & 3 \\ 6 \end{ytableau} \; .
$$

The number of semistandard tableau of a given type and content is known as the Kostka number.

**Definition 6.4.** Suppose $\lambda, \mu \vdash n$, the *Kostka number $K_{\lambda\mu}$* is the number of semistandard tableaux of shape $\lambda$ and content $\mu$.

For instance, if $\lambda = (3,2)$ and $\mu = (2,2,1)$, then $K_{\lambda\mu} = 2$ since there are exactly two semistandard tableaux of shape $\lambda$ and content $\mu$:

$$
\begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \end{ytableau} \qquad \text{and} \qquad \begin{ytableau} 1 & 1 & 3 \\ 2 & 2 \end{ytableau} \; .
$$

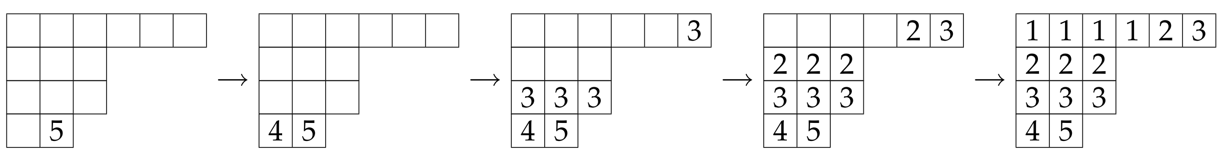
We are almost ready to state the result, but let us first make the following observation.

**Proposition 6.5.** *Suppose that $\lambda, \mu \vdash n$. Then $K_{\lambda\mu} \neq 0$ if and only if $\lambda \trianglerighteq \mu$. Also, $K_{\lambda\lambda} = 1$.*

*Proof.* Since the columns of a semistandard tableau are strictly increasing, an entry $k$ can only appear in one of the first $k$ rows. So, if $K_{\lambda\mu} \neq 0$, so that there exists some semistandard tableau $T$ of shape $\lambda$ and content $\mu$, then the first $k$ rows of $T$ must contain all the entries from the set $\{1, 2, \dots, k\}$. It follows that $\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k$. Therefore, $\lambda \trianglerighteq \mu$.

Conversely, suppose $\lambda \trianglerighteq \mu$. We give a procedure for constructing a $\lambda$-semistandard tableau. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m)$. First, fill in $\mu_m$ copies of the number $m$ as follows: starting from the last (bottommost) row, start filling in $m$ from the right. If we run out of space, then move to the previous row and fill the number in all the legal positions starting from the right (so that no two copies of $m$ lie on the same column), and repeat for the previous rows as many times if necessary. After we finish with $m$, repeat the same procedure for $m - 1$ on the remaining empty boxes, and so on.

For instance, if $\lambda = (6, 3, 3, 2)$ and $\mu = (4, 4, 4, 1, 1)$ then the construction yields the following:



To show that this works, observe that after each step, the set of empty squares form another Young diagram. So by induction it suffices to prove that (1) the first step is possible and (2) if $\lambda'$ is the diagram formed by the remaining empty squares of $\lambda$, and $\mu'$ is the partition $\mu$ without the last part, then $\lambda' \trianglerighteq \mu'$. For (1), observe that $\mu_m \leq \mu_1 \leq \lambda_1$, and so there are at least as many columns in $\lambda$ as $\mu_k$, and so there is enough space for all the copies of $k$ to be filled in. For (2), we need to prove that $\lambda'_1 + \cdots + \lambda'_k \geq \mu'_1 + \cdots + \mu'_k$ for all $k$. If none of the copies of $m$ are filled in the first $k$ rows of $\lambda$, then $\lambda_i = \lambda'_i$ for $1 \leq i \leq k$, and the inequality follows from the hypothesis $\lambda \trianglerighteq \mu$. Otherwise, note that it suffices to prove that $\lambda'_{k+1} + \lambda'_{k+2} + \cdots + \lambda'_l \leq \mu'_{k+1} + \mu'_{k+2} + \cdots + \mu'_l$ (note that $l \leq m$ follows from $\lambda \trianglerighteq \mu$). But this is true since the algorithm guarantees that $\lambda_i \leq \mu_m \leq \mu_i$ for $k < i \leq m$, and so $\lambda'_i \leq \mu_i = \mu'_i$ for $k < i < l \leq m$. It follows that $\lambda' \trianglerighteq \mu'$.

Finally, if $\lambda = \mu$, then since the numbers 1 to $k$ must only appear in the first $k$ rows, the number $k$ can only appear in the $k$th row, which gives a unique semistandard tableau. Thus, $K_{\lambda\lambda} = 1$. ∎

This means that if we arrange all the partitions of $n$ in some linear extension of the reverse dominance order (for instance, the reverse lexicographic order), then the matrix for $K_{\lambda\mu}$ is an upper-triangular matrix with 1's on the diagonal. For instance, the following table lists the values of $K_{\lambda\mu}$ for $n = 5$.

| $K_{\lambda\mu}$ | $\mu =$ | $(5)$ | $(4,1)$ | $(3,2)$ | $(3,1,1)$ | $(2,2,1)$ | $(2,1,1,1)$ | $(1,1,1,1,1)$ |
|---|---|---|---|---|---|---|---|---|
| $\lambda =$ | $(5)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | $(4,1)$ | 0 | 1 | 1 | 2 | 2 | 3 | 4 |
| | $(3,2)$ | 0 | 0 | 1 | 1 | 2 | 3 | 5 |
| | $(3,1,1)$ | 0 | 0 | 0 | 1 | 1 | 3 | 6 |
| | $(2,2,1)$ | 0 | 0 | 0 | 0 | 1 | 2 | 5 |
| | $(2,1,1,1)$ | 0 | 0 | 0 | 0 | 0 | 1 | 4 |
| | $(1,1,1,1,1)$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

We are now ready to state the result about the decomposition of $M^\lambda$ into irreducible representations.

**Theorem 6.6** (Young's Rule). $M^\mu \cong \bigoplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^\lambda.$

For instance, from the table above, we see that

$$M^{(2,2,1)} \cong S^{(2,2,1)} \oplus S^{(3,1,1)} \oplus 2S^{(3,2)} \oplus 2S^{(4,1)} \oplus S^{(5)}.$$

Note that Proposition 6.5 is a consequence of Young's rule. We shall end with a couple of examples illustrating Young's rule.

**Example 6.7.** Note that $K_{(n)\mu} = 1$ as there is only one $(n)$-semistandard tableau of content $\mu$, formed by filling in all the required entries in order. Then Young's Rule implies that every $M^\mu$ contains exactly one copy of the trivial representation $S^{(n)}$ (see Example 4.5).

**Example 6.8.** Since a semistandard tableau with content $(1^n)$ is just a standard tableau, we have $K_{\lambda(1^n)} = f^\lambda$ (the number of standard $\lambda$-tableaux). So Young's rule says that $M^{(1^n)} \cong \bigoplus_\lambda f^\lambda S^\lambda$. But from Example 3.4 we saw that $M^{(1^n)}$ is simply the regular representation. By taking the magnitude of the characters of both sides, we get another proof of the identity $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$, which we saw in Theorem 4.11.

## References

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