I.M.O. Winter Training Camp 2008: Invariants and Monovariants

On math contests, you will often find yourself trying to analyze a *process* of some sort. For example, consider the following two problems.

Sample Problem 1. *Several stones are placed on an infinite (in both directions) strip of squares. As long as there are at least two stones on a single square, you may pick up two such stones, then move one to the preceding square and one to the following square. Is it possible to return to the starting configuration after a finite sequence of such moves?*

Sample Problem 2. (Kvant) *In the sequence* $1, 0, 1, 0, 1, 0, 3, 5, \ldots$, *each term starting with the seventh is equal to the last digit of the sum of the preceding six terms. Prove that this sequence does not contain six consecutive terms equal to* $0, 1, 0, 1, 0, 1$, *respectively.*

In the first problem, the process consists of moving stones, and in the second problem, it consists of choosing numbers according to a recurrence. To understand these processes better, it is helpful to consider *invariants* and *monovariants*. These are simple properties that do not change, or change in very predictable ways as a process continues. If you pick the right property, it can explain a great deal about what the process is doing, and very difficult problems can become almost trivial! Let's see how this can be done here.

Solution to Sample Problem 1. Label the strip with consecutive integers, and let n_i denote the label of the square containing stone $\#i$.

Let $X = \sum_i n_i^2$, and consider what happens to X each time we do a move. First X decreases by $2t^2$ as we remove two stones from some square t. Next, X increases by $(t-1)^2 + (t+1)^2 = 2t^2 + 2$ as we replace the stones in squares $t-1$ and $t+1$. Therefore, every move causes X to increase by *exactly* 2.

In particular, after any sequence of moves, X will always be higher than where it began, so we could not possibly be in the same position we began in. □

Solution to Sample Problem 2. Let n_i denote the i^{th} term in the given sequence, and define:

$$X_i = n_i + 2n_{i+1} + 3n_{i+2} + 4n_{i+3} + 5n_{i+4} + 6n_{i+5}.$$

Note that $X_1 = 1 \cdot 1 + 3 \cdot 1 + 5 \cdot 1 = 9$. Also for any $i \geq 2$, we have:

$$\begin{aligned}
X_i &= n_i + 2n_{i+1} + 3n_{i+2} + 4n_{i+3} + 5n_{i+4} + 6n_{i+5} \\
&\equiv n_i + 2n_{i+1} + 3n_{i+2} + 4n_{i+3} + 5n_{i+4} + 6(n_{i-1} + n_i + n_{i+1} + n_{i+2} + n_{i+3} + n_{i+4}) \pmod 5 \\
&\equiv n_{i-1} + 2n_i + 3n_{i+1} + 4n_{i+2} + 5n_{i+3} + 6n_{i+4} \pmod 5 \\
&\equiv X_{i-1} \pmod 5.
\end{aligned}$$

It follows that $X_i \equiv X_1 \equiv 4 \pmod 5$ for all i. If we had a subsequence $\{0, 1, 0, 1, 0, 1\}$, however, then the corresponding X_i would be $1 \cdot 2 + 1 \cdot 4 + 1 \cdot 6 = 12 \not\equiv 4 \pmod 5$, which is impossible. $\square$

Usually an invariant problem is pretty easy once you find the right invariant (or monovariant), but finding it can be pretty tough! After all, why would you think to try exactly X_i in Sample Problem 2? In truth, finding the right invariant is an art, and that is what makes these problems hard. Nonetheless, there are a few things that you should always be thinking about:

- *Colorings:* Color all the squares in a grid with two or more colors. Usually the chessboard pattern is a good choice, but other patterns are also sometimes useful. Consider squares of each color separately.

- *Algebraic expressions:* Given a set of values, look at their differences, their sum, the sum of their squares, or occasionally their product. If you are working with integers, try looking at these values modulo n. (Usually n should be a small prime power.)

- *Corners and edges:* For grid-based problems, consider any shapes formed. How many boundary edges do they have? How many corners?

- *Inversions:* If you are permuting a sequence of numbers, consider the number of inversions – that is, the number of pairs (i, j) such that i and j are listed in the wrong order. Both the absolute number of inversions and its parity are useful.

- *Integers and rationals:* Can you find a positive integer that keeps decreasing? Or does the denominator of a rational number keep decreasing?

- *Symmetries:* Can you ensure that after each step, a figure is symmetrical in some way? Perhaps you can logically pair up objects, and two paired objects are always in the same state? Perhaps the problem can be divided into two essentially identical subproblems? This is especially useful for game-theory type problems.

You can apply invariants in a number of interesting ways, but often, you will know what to do when you get there. Instead of listing all the possibilities, I will just say practice makes perfect, and give you lots of problems to try! There are hints to some of the problems, but try to solve them on your own first.

Starter problems

1. An 8×8 chessboard has two opposite corners removed. Is it possible to tile the remaining 62 squares with 31 dominoes?

2. *(Bernoulli Trials, 1998)* Arya and Bran are playing a game. They begin with 2008 coins arranged in a circle, and alternate turns, starting with Arya. On his or her turn, a player may remove any one coin, or if two adjacent coins remain, he or she may instead remove both. The player who removes the last coin wins. Show that Bran has a winning strategy, no matter how Arya plays.

3. The numbers $1, 2, 3, \ldots, n$ are written in a row. It is permitted to swap any two numbers. If 2007 such operations are performed, is it possible that the final arrangement of numbers coincides with the original?

4. *(AM-GM inequality)*

   (a) Suppose we are given positive numbers a, b, x with $a < x < b$. Show that $x(a+b-x) > ab$.

   (b) Given positive real numbers $a_1, a_2, \ldots, a_n$, prove that:

   $$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

   Remark: This is a general technique called "smoothing". Suppose you want to show some function f satisfies $f(a_1, a_2, \ldots, a_n) \geq 0$ for all choices of $\{a_i\}$. Start with an arbitrary choice. Then alter it slightly so as to decrease f. Continue doing this until you reach $\{a'_i\}$ for which you already know $f(a'_1, a'_2, \ldots, a'_n) = 0$. Then, $f(a_1, a_2, \ldots, a_n) \geq f(a'_1, a'_2, \ldots, a'_n) = 0$, as required. You need to show that $\{a_i\}$ actually reaches $\{a'_i\}$ after a finite number of steps, and that's where invariant theory comes in!

   As an aside, if you happened to know that there is some choice of $\{a_i\}$ for which f takes on its minimum value, you could side-step some of this work. However, this is not always the case (for example, imagine if f ranged over all positive real numbers).

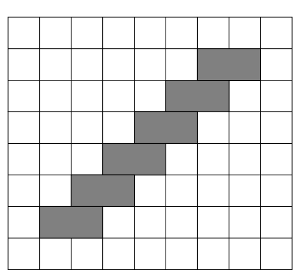
Olympiad-level problems

5. *(USAMO 1997, #1)* Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

   $$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \frac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

   where $\{x\}$ denotes the factional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where x is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

6. *(CMO 2007, #1)* What is the maximum number of non-overlapping 2×1 dominoes that can be placed on an 8×9 chessboard if six of them are placed as shown? Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board.



7. *(Stanford Putnam training 2007)* On an $n \times n$ board, there are n^2 squares, $n-1$ of which are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.

8. *(Adapted from IOI 2002)* A computer screen shows an $n \times n$ grid, colored black and white in some way. One can select with a mouse any rectangle with sides on the lines of the grid and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Let X denote the minimum number of mouse clicks required to make the grid all white. Also, let Y denote the number of grid vertices adjacent to an odd number of black squares. Show that $\frac{Y}{4} \leq X \leq \frac{Y}{2}$.
Remark: USAMO 1998 #4 is essentially a special case of this result.

9. You have a stack of $2n+1$ cards, which you can shuffle using the two following operations:

   1. Cut: Remove any number of cards from the top of the pile, and put them on the bottom.
   2. Perfect riffle shuffle: Remove the top n cards from the deck and place them in order in the spaces between the other $n+1$ cards.

   Prove that, no matter how many operations you perform, you can reorder the cards in at most $2n(2n+1)$ different ways.

10. *(USAMO 2004, #4)* Alice and Bob play a game on a 6×6 grid. On his or her turn, a player chooses a rational number not appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a path from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she cannot. (If two squares share a vertex, Alice can draw a path from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the two players.

11. *(IMO training camp, 1999)* The vertices of a regular n-sided polygon have integer coordinates. Show that $n = 4$. It might help to first show that n is even.

12. *(AIME 1998, #15)* Define a *domino* to be an ordered pair of *distinct* positive integers. A *proper sequence* of dominoes is a list of distinct dominoes in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not *both* appear for any i and j. Let D_{40} be the set of all dominoes whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominoes that can be formed using the dominoes of D_{40}.

13. The numbers from 1 through 2008 are written on a blackboard. Every second, Dr. Math erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

14. *(APMO 2007, #5)* A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially, all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

15. *(IMO 2000, #3)* Let $n \geq 2$ be a positive integer and λ be a positive real number. Initially, there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas, at some points A and B, with A to the left of B, and letting the flea from A jump to the point C to the right of B such that $\overline{BC}/\overline{AB} = \lambda$.

    Determine all values of λ such that, for any point M on the line and any initial position of the n fleas, there exists a sequence of moves that will take them all to positions right of M.

16. *(IMO training camp, 1999)* Recall the scenario of Sample Problem 1. Prove that any sequence of moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves.

17. *(MOP 1998 homework)* Several stones are placed on an infinite (in both directions) strip of squares, indexed by the integers. We perform a sequence of moves, each move being one of the following two types:

    (a) Remove one stone from each of the squares $n-1$ and n and place one stone on square $n+1$.

    (b) Remove two stones from square n and place one stone on each of the squares $n-2$ and $n+1$.

    Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves.

18. Anya and Borya are playing a game. They begin with a "pyramid" consisting of 10 rows, where row i has exactly i coins arranged in a line. On his or her turn, a player chooses any number of contiguous coins within a single row, and then removes them. (Note that if a coin is removed, the coins remaining to its left and to its right are not considered contiguous with each other.) The player who removes the last coin wins. Anya goes first and then the players alternate. Find, with proof, a winning strategy for one of the two players.

Unusual problems

19. *(IMO Correspondence Program 1998 and CM+MM 1997)* How many distinct acute angles α are there for which $\cos(\alpha) \cdot \cos(2\alpha) \cdot \cos(4\alpha) = 1/8$?

20. *(MOP 1998, Po-Shen Loh)* Let ω_1 and ω_2 be equal-radius circles meeting at points B and C. Let X denote the midpoint of $\overline{BC}$ and let A be a point on ω_1 that is not contained inside ω_2. Extend $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to hit ω_2 at A_1 and A_2 respectively. Then extend $\overrightarrow{A_1X}$ and $\overrightarrow{A_2X}$ to hit ω_1 at P_1 and P_2 respectively. Show that $\overline{AP_1} = \overline{AP_2}$.

21. Several checkers are placed on a board. Each turn, a checker may jump diagonally over an adjacent piece if the opposite square is empty. If a checker is jumped over in this way, it is removed from the board. Is it possible to make a sequence of such jumps to remove all but one checker from the board shown below?

O		O		O		O	
	O		O		O		O
O						O	
	O						O
O						O	
	O						O
O		O		O		O	
	O		O		O		O

Selected hints

2. As with many 2-player games, symmetry is the key.

10. It's always symmetry with games! Try pairing up squares.

11. Invariants can show n is even. This further implies n is a power of 2, but why?

12. Think of dominoes as edges on a graph. Could you use all the dominoes if several vertices had odd degree (i.e., an odd number of incident edges)? What if all the vertices had even degree?

14. A careful coloring can tell you where the light *can't* be. Unfortunately that's the easy part!

15. If $\lambda < \frac{1}{n-1}$, there exists a constant $C > 0$ so that $(n-1+C)\lambda = 1$. Can you come up with an invariant depending on C?

16. Invariants are good for showing that any sequence of moves must terminate. Consider other techniques for uniqueness.

17. $\frac{1+\sqrt{5}}{2}$.

18. This game has three important properties: (1) the two players are essentially symmetric, (2) the game decomposes into similar, independent sub-games, and (3) on your turn, you may make a move in any *one* sub-game. These facts imply the game is in some sense equivalent to Nim, so start thinking about exclusive-or!

19. Yes, it is a bit of a stretch to call this an invariant problem, but there is a reason for it. Can you simplify $\cos(\alpha) \cdot \cos(2\alpha) \cdot \ldots \cdot \cos(2^t\alpha)$ in general?

20. This is another sort-of invariant problem. Is there a position for A where the problem is easy? Now what happens when you move A?

21. Consider the Klein group. This has 4 elements $\{e, a, b, c\}$ that can be multiplied according to the following rules. Let x, y, z be an ordering of a, b, c. Then, $e \cdot x = x \cdot e = x$, $x^2 = e$, and $x \cdot y = z$.