Complex Number and Geometry Formula Sheet

Peng Shi *

Life is complex. It has real and imaginary components.

1 The Complex Number

Representations of a complex number: $z = x + iy = Ae^{i\theta} = A(\cos\theta + i\sin\theta)$ where $A = |z| = \sqrt{x^2+y^2}$ (A is called the magnitude of z) and $\theta = \arg(z)$ (θ is called the argument of z). θ is the counter-clockwise angle between the positive x axis and the vector (x, y). The $Re(z) = x$ is called the real part of z, and $Im(z) = y$ is the imaginary part.

Complex conjugate: The complex conjugate of $z = a + ib$ is $\overline{z} = a - ib = ce^{-i\theta}$. Arithmetic operations commute with the conjugate sign. (*i.e.* $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{-a} = -\overline{a}$, $\overline{ab} = \overline{a}\overline{b}$, $\overline{a/b} = \overline{a}/\overline{b}$.)

Primitive roots: The nth primitive root of unity is $z = e^{i\frac{2\pi}{n}}$. The powers of z–$1, z, z^2 \cdots z^{n-1}$–are all the roots of $x^n - 1 = 0$. The sum of these roots, $\sum_{k=0}^{n-1} z^k = 0$.

In a geometry problem, instead of representing a point by Cartesian co-ordinates (x, y), we use the complex number $z = x + iy$. Complex numbers allows us to easily describe translations, rotations and reflections. In almost all cases, we can reduce the given geometry problem to a polynomial identity, which can always be proven by expanding and collecting terms. Other bash methods such as coordinates or trigonometry sometimes require us to use clever factoring or identities, so we sometimes can get stuck; complex number bash always works. In the following theorems, we use capital letters to denote points and lowercase letters to denote the corresponding complex number.

Mathematics and magic are the only systems where you can mix a bunch of imaginary things together and have a pie come out.

2 Useful Geometry theorems

Key fact: z is real iff $z = \overline{z}$; z is imaginary iff $z = -\overline{z}$. This fact makes manipulations with z and $\overline{z}$ nice. Moreover, on the unit circle, $\overline{z} = 1/z$, which makes manipulations on the unit circle even nicer. Thus, complex number bash is fairly manageable in problems involving one circle.

Theorem 2.1. *Let A, B, C, Z be arbitrary points,*

1. *Z is on the line going through C and parallel to AB iff $\frac{z-c}{b-a}$ is real.* i.e. $\frac{z-c}{b-a} = \frac{\bar{z}-\bar{c}}{\bar{b}-\bar{a}}$
2. *Z is on the line passing through C perpendicular to AB iff $\frac{z-c}{b-a}$ is imaginary.* i.e. $\frac{z-c}{b-a} = -\frac{\bar{z}-\bar{c}}{\bar{b}-\bar{a}}$

Corollary 2.2. *Let A and B be points on the unit circle, and let C and Z be arbitrary points*

*Mostly copied from Yi Sun's MOP 2007 notes, Marko Radovanovic's notes in the IMO Compendium, and Kun Y. Li's notes in Mathematical Excalibur 9(1)

1. Z is on the line AB iff $z + ab\overline{z} = a + b$.
2. Z is on the line tangent to the circle at A iff $z + a^2\overline{z} = 2a$.
3. Z is on the line passing through C, perpendicular to AB iff $z - ab\overline{z} = c - ab\overline{c}$.
4. Z is on the line passing through C, perpendicular to the tangent at A iff $z - a^2\overline{z} = c - a^2\overline{c}$.

Theorem 2.3. *Let A, B, C, X, Y, Z be points in the plane,*

1. $\angle ABC = \angle XYZ$ iff $\frac{a-b}{c-b}/\frac{x-y}{z-y}$ is real.
2. $\triangle ABC \sim \triangle XYZ$ in this orientation iff $\frac{a-b}{c-b} = \frac{x-y}{z-y}$.

Corollary 2.4. *If A, B, C, D are points in the plane, then they are cyclic iff $\frac{a-b}{c-b}/\frac{a-d}{c-d}$ is real. Here, "cyclic" includes the degenerate case in which they are all co-linear.*

Theorem 2.5. *If A, B, C, D are points on the unit circle, Z is an arbitrary point,*

1. $\frac{a-b}{\overline{a}-\overline{b}} = -ab$
2. The projection of Z onto the line AB is $p = (a + b + c - ab\overline{z})/2$.
3. The intersection of chords AB and CD is $p = \frac{ab(c+d)-cd(a+b)}{ab-cd}$.
4. The intersection of the tangents at A and B is $\frac{2ab}{a+b}$.

Theorem 2.6. *All theorems about vectors carry over to complex numbers:*

1. The centroid G of $\triangle ABC$ is $g = \frac{a+b+c}{3}$.
2. Euler's Line: The orthocenter H and circumcenter O of $\triangle ABC$ satisfy $h + 2o = 3g = a + b + c$. The center N of the nine-point circle satisfies $2n + o = 3g = a + b + c$. If you conveniently set the circumcentre to be the origin, then $h = a + b + c$, $n = (a + b + c)/2$.
3. Point C on AB divides segment AB in the signed ratio $AC/CB = \lambda$ iff $c = \frac{a+\lambda b}{1+\lambda}$.
4. Dot product: $\vec{A} \cdot \vec{B} = Re(a\overline{b}) = (a\overline{b} + \overline{a}b)/2$.
5. Cross product: $|\vec{A} \times \vec{B}| = |Im(a\overline{b})| = |(a\overline{b} - \overline{a}b)/2|$.
6. Using the cross product, we get that the area of triangle ABC is

$$p = |\frac{i}{4}(a\overline{b} + b\overline{c} + c\overline{a} - \overline{a}b - \overline{b}c - \overline{c}a)|$$

.

Here's a useful substitution when we need the midpoints of arcs or the incentre.

Theorem 2.7. *For $\triangle ABC$, there exists u, v, w such that $a = u^2$, $b = u^2$, $c = w^2$,*

1. $-uv$, $-uw$, $-vw$ are the midpoints of arcs AB, AC, BC that don't contain C, B and A respectively.
2. The incentre I of $\triangle ABC$ is $i = -(uv + vw + wu)$.

We're sorry. You have reached an imaginary number.
Please rotate your phone 90 degrees and try again...