“Bashing Geometry with Complex Numbers” Problem Set

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Reality may be a line, but a little imagination makes it a plane!

1 Slick Bashing

These problems are perfect for complex number solutions. Exploit the power of complex numbers in representing translations, rotations, and reflections, and use the nice formula for centroid and orthocenter.

Problem 1. *(Yug MO 1990) Let O and H be the circumcentre and orthocentre of $\triangle ABC$ respectively. Let Q be the reflection of H across O. Let T_1, T_2, T_3 be the centroids of $\triangle BCQ$, $\triangle CAQ$ and $\triangle ABQ$. Prove that $AT_1 = BT_2 = CT_3 = \frac{4}{3}OA$.*

Problem 2. *(BMO 1984) Let $ABCD$ be an cyclic quadrilateral and let H_A, H_B, H_C, H_D be the orthocentres of triangles BCD, CDA, DAB, and ABC respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.*

Problem 3. *Consider triangle ABC and its circumcircle S. Reflect the circle with respect to AB, AC, BC to get three new circles S_{AB}, S_{BC}, and S_{BC}. Show that these three circles intersect at a common point and identify this point.*

Problem 4. *(2000 St. Petersburg MO) The line T is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line T (K is the root of the perpendicular from the orthocenter to S). Let L be the midpoint of side AC. Show that the triangle BKL is isosceles.*

Problem 5. *(Yug TST 1992) The squares $BCDE$, $CAFG$, and $ABHI$ are constructed outside the triangle ABC. Let $GCDQ$ and $EBHP$ be parallelograms. Prove that APQ is isosceles and $\angle PAQ = \pi/2$.*

Problem 6. *(Yug MO 2003) Given triangle ABC, construct equilateral triangle ABC_1, BCA_1, CAB_1 on the outside of ABC. Let P, Q denote the midpoints of C_1A_1 and C_1B_1 respectively. Let R be the midpoint of AB. Prove that triangle PQR is isosceles.*

Problem 7. *(Napolean’s Triangle) Given triangle ABC, construct an equilateral triangle on the outside of each of the sides. Let P, Q, R be the centroids of these equilateral triangles, prove that triangle PQR is equilateral.*

2 Straightforward Bashing

These problems probably have other intended solutions, but they still can be solved by straightforward application of complex numbers.

*Mostly copied from Yi Sun’s MOP 2007 notes, Marko Radovanovic’s notes in the IMO Compendium, Kun Y. Li’s notes in Mathematical Excalibur 9(1), and Mathlinks.

Problem 8. *(Simson Line) Let Z be a point on the circumcircle of triangle ABC and P, Q, R be the feet of perpendiclars from Z to BC, AC, AB respectively. Prove that P, Q, R are collinear. (This line is called the* Simson line *of triangle ABC from Z.)*

Problem 9. *Given cyclic quadrilateral $ABCD$, let P and Q be the reflection of C across lies AB and AD respectively. Prove that PQ passes through the orthocentre of triangle ABD.*

Problem 10. *Let $W_1W_2W_3$ be a triangle with circumcircle S, and let A_1, A_2, A_3 be the midpoints of W_2W_3, W_1W_3, W_1W_2 respectively. From A_i drop a perpendicular to the line tangent to S at W_i. Prove that these perpendicular lines are concurrent and identify this point of concurrency.*

Problem 11. *Let $A_0A_1A_2A_3A_4A_5A_6$ be a regular 7-gon. Prove that*

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_3}$$

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Problem 12. *Given point P_0 in the plane of triangle $A_1A_2A_3$. Denote $A_s = A_{s-3}$, for $s > 3$. Construct points $P_1, P_2, \cdots$ sequentially such that point P_{k+1} is P_k rotated $120°$ counter-clockwise around A_{k+1}. Prove that if $P_{1986} = P_0$ then triangle $A_1A_2A_3$ is isosceles.*

Problem 13. *(MOP 2006) Point H is the orthocenter of triangle ABC. Points D, E and F lie on the circumcircle of triangle ABC such that $AD \| BE \| CF$. Points S, T, and U are the respective reflections of D, E, F across the lines BC, CA and AB. Prove that S, T, U, H are cyclic.*

3 Smart Bashing

Although almost all geometry problems are doable with complex number bash, you should always try to simplify the problem first by proving various properties of the given configuration. This will drastically reduce the time of bashing. Also, try to set up your bash in the simplest way possible, such as to define some key line as the real axis. The following problems are probably too complex to bash (no pun intended) unless you simplify the problem first.

Problem 14. *(2003 IMO, Problem 4) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ iff the bisectors of $\angle ABC$ and $\angle ADC$ meet on AC.*

Problem 15. *Let O be the circumcentre of triangle ABC. A line through O intersects sides AB and AC at M and N respectively. Let S and R be the midpoints of BN and CM, respectively. Prove that $\angle ROS = \angle BAC$.*

Problem 16. *(IMO Shortlist 1992) Let $ABCD$ be a convex quadrilateral for which $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral and pointing outward. Let O_1, O_2, O_3, O_4 be the centres of the triangles constructed on AB, BC, CD, and DA respectively. Prove that lines O_1O_3 and O_2O_4 are perpendicular.*

Problem 17. *(USA TST 2006, Problem 6) Let ABC be a triangle. Triangles PAB and QAC are constructed outside of ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcentre of triangle BCR. Prove that $AO \perp PQ$.*

In an actual contest, I recommend that you do not start complex number bash before absolutely exhausting all other approaches. This is because although complex number bash always works in the end, the algebraic manipulations might take a few hours. Nevertheless, knowing *how to* complex number bash is nice because even the hardest problems can be done this way if you just keep going. Moreover, if you have already significantly simplified the problem but can't finish it off, complex number bash is a handy tool. With this powerful, universal tool at your disposal, you can tackle hard geometry problems with confidence!