Gauss Maps of Complete Minimal Surfaces in $\mathbb{R}^3$

Yuzhou Gu

1 Introduction

Let M be a complete minimal surface in $\mathbb{R}^3$. Then it is naturally equipped with a Gauss map $g : M \to S^2$ which sends a point to the unit normal vector at this point. It is interesting to study properties of g. When M is flat, g is a constant function, thus im g contains a single point. However, when M is non-flat, it turns out that im g is dense in S^2. This was conjectured by Nirenberg and was proved by Osserman [Oss59]. This is Osserman's theorem, which says that the image of the Gauss map of a complete non-flat minimal surface is dense in S^2. Later, Xavier [Xav81] drastically improved the result and proved that the Gauss map can omit at most 6 points. Fujimoto [Fuj88] then proved the optimal result that the Gauss map can omit at most 4 points. Fujimoto's result is optimal in the sense that the value 4 is taken when M is a Scherk surface. Actually, Voss [Vos64] showed that given any $q \leq 4$ points on S^2, there is a complete minimal surface whose Gauss map omits exactly those q points.

There has been stronger results conditioned on the total curvature of M. Osserman [Oss64] proved that if M is non-flat and has finite total curvature, then its Gauss map can omit at most 3 points. Weitsman and Xavier [WX87] proved that if such M has total curvature $> -16\pi$, then its Gauss map can omit at most 2 points. Fang [Fan93] improved Weitsman and Xavier's result to complete non-flat minimal surfaces with total curvature $> -20\pi$. It is still open whether a complete non-flat minimal surface with finite total curvature can have Gauss map omit exactly 3 points. In infinite total curvature case, Mo and Osserman [MO90] proved that if M has infinite total curvature, then its Gauss map can take at most 4 points a finite number of times.

In this expository paper we will review Osserman's proof of Nirenberg conjecture, Fujimoto's theorem, and Voss' theorem.

2 Preliminaries

A (regular) minimal surface M is an isometric immersion $I : \Omega \to \mathbb{R}^3$ where Ω is a Riemann surface, and I is harmonic in each coordinate. Some authors use a definition where I is not necessarily an immersion. We assume I is an immersion so that M is regular.

2.1 Enneper-Weierstrass representation

Theorem 1 *(Enneper-Weierstrass) A simply connected minimal surface M determines, and is uniquely (up to translation in $\mathbb{R}^3$) determined by the Enneper-Weierstrass representation (f, g), where:*

- *$f : \Omega \to \mathbb{C}$ is holomorphic;*

- *$g : \Omega \to \mathbb{C}$ is meromorphic;*
- *when $g(z) \neq \infty$, $f(z) \neq 0$.*
- *when g has a pole of order $k \geq 0$ at $z \in \Omega$, then f has zero of order exactly $2k$ at z.*

(The domain Ω is implicit in the Enneper-Weierstrass representation.)

Proof Assume given f, g satisfying the conditions. Let $\phi_1 = \frac{1}{2}(1-g^2)f$, $\phi_2 = \frac{i}{2}(1+g^2)f$, $\phi_3 = fg$. Choose some point $(x_1, x_2, x_3) \in \mathbb{R}^3$. Let $I_i = x_i + \mathrm{Re}\int_0^z \phi_i$. Then we can check that $I = (I_1, I_2, I_3)$ gives a conformal harmonic immersion, i.e. a minimal surface. Note that for I to be a immersion, we need $\sum_{1 \leq i \leq 3} |\phi_i|^2 \neq 0$, which is equivalent to the last two conditions on (f, g) in the statement of the theorem.

Given M, we can use I_1, I_2, I_3 to uniquely recover ϕ_1, ϕ_2, ϕ_3. Let $f = \phi_1 - i\phi_2$, $g = \frac{\phi_3}{\phi_1 - i\phi_2}$. We can check that (f, g) satisfies the requirements.

Also, we can check that the above two maps are inverse to each other, and finish the proof.□

The Enneper-Weierstrass representation has many interesting properties. We can derive immediately from the definitions the following properties.

Proposition 2 *Let M be a simply connected minimal surface and (f, g) be its Enneper-Weierstrass representation. Then*

(1) g is just the Gauss map if we identify $\overline{C}$ and S^2 using the stereographic projection.

(2) the metric on M is given by $ds = \lambda|dz|$, where $\lambda = \frac{1}{2}|f|(1+|g|^2)$.

Proof Direct calculation. Omitted. □

2.2 Completeness

A complete surface, roughly speaking, is a surface without boundary. There are two equivalent definitions.

Definition 3 *A complete surface is a surface S where any parametrized geodesic $\gamma : [0, \epsilon) \to S$ can be extended to a paremetrized geodesic $\bar{\gamma} : \mathbb{R} \to S$ defined on $\mathbb{R}$.*

Definition 4 *A divergent curve is a curve $\gamma : [0, a) \to S$ such that for every compact subset $K \subseteq S$, there exists $t_0 \in (0, a)$ such that for all $t \in (t_0, a)$, $\gamma(t) \notin K$. A complete surface is a surface S on which every divergent curve has unbounded length.*

Definition 3 agrees with intuition, and Definition 4 is usually easier to use in proofs. In the following we will use Definition 4 more often.

Some examples of complete surfaces are $\mathbb{R}^n$, S^n, the catenoid. Some examples of non-complete surfaces are $\mathbb{R}^n - \{0\}$, $D^n = \{|z| < 1 : z \in \mathbb{R}^n\}$. These examples agree with the intuition.

2.3 Reduce to simply connected case

Let $M := I : \Omega \to \mathbb{R}^3$ be a minimal surface. Let $\pi : \hat{\Omega} \to \Omega$ be the universal covering of Ω. Then $\hat{M} := I \circ \pi : \hat{\Omega} \to \mathbb{R}^3$ is a simply connected minimal surface. Universal covering preserves many good properties. In particular we have the following proposition.

Proposition 5 *M is complete if and only if $\hat{M}$ is complete. The image of the Gauss map of M equals the image of Gauss map of $\hat{M}$.*

Proof M is locally indistinguishable from $\hat{M}$. The Gauss map is defined locally, so $g(\pi(x)) = g(x)$ for all $x \in M$. π is surjective, so $g(M) = g(\hat{M})$.

Now consider completeness. If M is complete, then for any curve $\gamma : [0, a) \to \hat{M}$, $\pi \circ \gamma$ can be extended to a curve on M defined over $\mathbb{R}$. Universal covering has path lifting property, so this gives an extension of γ to a curve on $\hat{M}$ defined over $\mathbb{R}$. So $\hat{M}$ is complete.

Conversely, assume $\hat{M}$ is complete, then any curve $\gamma : [0, a) \to M$ can be lifted to $\hat{M}$, extended to a curve on $\hat{M}$ defined over $\mathbb{R}$, and then projected back to M. So M is complete. □

As we focus on Gauss maps of complete minimal surfaces, we can assume that our surface is simply connected, so that we can apply the uniformization theorem. We have the following proposition.

Proposition 6 *Let $M := I : \Omega \to \mathbb{R}^3$ be a simply connected minimal surface. Then Ω is conformally equivalent to the complex plane $\mathbb{C}$ or the open unit disk D.*

Proof By the uniformization theorem, Ω is conformally equivalent to $\mathbb{C}$, D, or the Riemann sphere $\overline{C}$. For the sake of contrary, assume Ω is conformally equivalent to $\overline{C}$. Then Ω is compact. However there is no compact minimal surface in $\mathbb{R}^3$. □

3 Osserman's Theorem

In this section we prove Osserman's theorem, which says that the Gauss map of a complete minimal surface cannot omit a nonempty open set in S^2.

By definition of completeness and properties of Enneper-Weierstrass representation, we see that for any divergent curve γ on a complete minimal surface M with Enneper-Weierstrass representation (f, g), we have $\int_\gamma |f|(1 + |g|^2)|dz| = \infty$. However, if the factor $1 + |g|^2$ is removed, this is not necessarily true.

Lemma 7 *Let $f : D \to \mathbb{C}$ be a holomorphic function that omits 0 (where D is the open unit disk). Then there exists a divergent curve γ such that $\int_\gamma |f||dz| < \infty$.*

Proof Define $F(z) = \int_0^z f(\zeta)d\zeta$. For $\theta \in [0, 2\pi)$, define γ_θ to be the path that lifts the path $t \mapsto te^{i\theta}$, i.e. $\gamma_\theta(0) = 0$ and $F(\gamma_\theta(t)) = te^{i\theta}$. This lifting exists because f omits 0. Let t_θ be the maximum value such that γ_θ is defined on $[0, t_\theta)$.

We prove that there exists some θ such that $t_\theta < \infty$. Assume that such θ does not exist. Then we can define $G : \mathbb{C} \to D$ as $G(0) = 0$, $G(te^{i\theta}) = \gamma_\theta(t)$. Then $F(G(te^{i\theta})) = F(\gamma_\theta(t)) = te^{i\theta}$. So $F \circ G = \text{id}$. By inverse function theorem, G is a holomorphic function. However, $G : \mathbb{C} \to D$ is bounded, so is constant by Liouville's theorem. This contradicts with that $F \circ G = \text{id}$.

So there exists some θ such that $t_\theta < \infty$. Fix one such θ. Then $\gamma_\theta : [0, t_\theta) \to D$ is a divergent curve because $\lim_{t \to t_\theta} |\gamma_\theta(t)| = 1$. Then

$$\int_{\gamma_\theta} |f||dz| = \int_0^{t_\theta} |f(t)\gamma_\theta'(t)|dt = \int_0^{t_\theta} |(F \circ \gamma_\theta)'(t)|dt = t_\theta < \infty.$$

We are done. □

Now we prove Osserman's theorem.

Theorem 8 (Osserman) *The Gauss map of a complete non-flat minimal surface cannot omit an nonempty open set.*

Proof Let $M = I : \Omega \to \mathbb{R}^3$ be a complete non-flat minimal surface and with Enneper-Weierstrass representation (f, g) such that g omits a nonempty open set in S^2. We can assume that $g(M)$ omits a neighborhood of ∞. Then g is bounded.

By the discussion in section 2, we can assume that M is simply connected. By Proposition 6, we can assume that $\Omega = \mathbb{C}$ or $\Omega = D$.

When $\Omega = \mathbb{C}$, $g(M)$ is bounded. However $g \circ I : \mathbb{C} \to \mathbb{C}$ is holomorphic. By Picard theroem, $g(M)$ must be constant, which means that M is flat. Contradiction.

Now assume $\Omega = D$. $g(M)$ is bounded, so f is nowhere zero (becauze g does not have pole). By completeness, for any divergent curve γ, we have $\int_\gamma |f|(1 + |g|^2)|dz| = \infty$. $|g|$ is bounded, so this means that $\int_\gamma |f||dz| = \infty$. However, this contradicts with Lemma 7. $\square$

Remark We assume throughout the paper that M is regular. In non-regular case, Osserman's theorem no longer holds. Actually, for arbitrary small open set U in S^2, we can construct a non-regular complete non-flat minimal surface whose Gauss map has image in U. For the construction, see Dierkes, Hildebrandt, and Sauvigny [DHS10], pp. 193.

Remark Osserman's theorem easily implies Bernstein's theorem.

Theorem 9 (Bernstein) *A minimal graph over $\mathbb{R}^2$ in $\mathbb{R}^3$ is flat.*

Proof Such a minimal graph must be complete. The image of the Gauss map is contained in half of S^2, so cannot be dense. Using 8 we get the desired result. $\square$

4 Fujimoto's Theorem

In this section we prove Fujimoto's theorem. We will follow the proof in Dierkes, Hildebrandt, and Sauvigny [DHS10], section 3.7.

For the case $\Omega = \mathbb{C}$, the proof is the same. For the case $\Omega = D$, we construct a divergent curve with finite length. The construction is in some sense similar to the construction in Osserman's theorem. The following lemma is the core lemma which will be used to prove the constructed curve has finite length.

Lemma 10 *Let D_R be the open disk of radius R. For any ϵ, η satisfying $0 < 4\eta < \epsilon < 1$, there exists $B > 0$ such that for any holomorphic function $g : D_R \to \mathbb{C}$ which omits $\alpha_1, \ldots, \alpha_4$, we have*

$$(1 + |g|^2)^{\frac{3-\epsilon}{2}} \prod_{1 \le j \le 4} |g - \alpha_j|^{\eta - 1} |g'| \le \frac{BR}{R^2 - |z|^2}.$$

Proof Let $\Omega = \mathbb{C} \backslash \{\alpha_1, \ldots, \alpha_4\}$. Let $\hat{\Omega}$ be its universal cover. Then $\hat{\Omega}$ is a Riemann surface conformally equivalent to the open unit disk D. By pulling back the Poincaré metric on $\hat{\Omega}$, we get a metric $ds = \rho|dw|$ on Ω whose Gauss curvature is -1.

Then we know that as $w \to \alpha_j$ we have

$$\rho(w) \sim \frac{C_j}{|w - \alpha_j| \log |w - \alpha_j|}$$

for some nonzero constant C. Also, as $w \to \infty$, we have

$$\rho(w) \sim \frac{C_0}{|w| \log |w|}$$

for some nonzero constant C. (See Nevanlinna [1] pp. 259-260 and 250.)

Now consider the function $\psi : \Omega \to \mathbb{C}$ be defined as

$$\psi(w) = (1 + |w|^2)^{\frac{3-\epsilon}{2}} \rho(w)^{-1} \prod_{1 \le j \le 4} |w - \alpha_j|^{\eta - 1}.$$

From the previous approximations of ρ, we see that as $w \to \alpha_j$,

$$\psi(w) \sim C_j' |w - \alpha_j|^{\eta} \log |w - \alpha_j| \sim 0,$$

and when $w \to \infty$,

$$\psi(w) \sim C_0' |w|^{4\eta - \epsilon} \log |w| \sim 0.$$

So ψ is bounded on some neighborhood of ∞ and $\alpha_1, \ldots, \alpha_4$. However, Ω minus these neighborhood is compact, so ψ is bounded. Say $\psi \le C$.

Now consider $g : D_R \to \Omega$. There exists a function $G : D_R \to D$ such that the composition

$$D_R \xrightarrow{G} D \to \hat{\Omega} \xrightarrow{\pi} \Omega$$

equals g (where the second map is the map defining a conformal equivalence between D and $\hat{\Omega}$). Now consider the map $G \circ \tau : D \to D$ where $\tau : D \to D_R$ is the conformal rescaling map. Applying Schwartz-Pick theorem, we get that

$$\rho(g(z)) |g'(z)| \le \frac{2R}{R^2 - |z|^2}.$$

Multiplying this inequality by the bound on ψ, we get the desired result. □

Theorem 11 (Fujimoto) *The Gauss map of a complete non-flat minimal surface cannot omit 5 points.*

Proof Let $M := I : \Omega \to \mathbb{R}^3$ be a non-flat complete minimal surface with Enneper-Weierstrass representation (f, g) where g omits (at least) 5 points in S^2. As before, we can assume $\Omega = \mathbb{C}$ or $\Omega = D$.

When $\Omega = \mathbb{C}$, g is a holomorphic function $\mathbb{C} \to \overline{\mathbb{C}}$ which omits (at least) 5 points. Picard theorem says that g must be constant, and therefore M is flat. So we can assume $\Omega = D$.

Now assume g omits distinct points $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ where $q = 5$. By changing coordinates we can assume that $\alpha_q = \infty$. We will construct a divergent curve with finite length on M, and derive a contradiction with that M is complete.

Let $\Sigma = \{g'(z) = 0 \in \Omega\}$. Choose numbers ϵ, η such that $0 < 4\eta < \epsilon < 1$. Let $p = \frac{2}{3-\epsilon}$. Define $u : \Omega \backslash \Sigma \to \mathbb{C}$ as

$$u(w) = (\frac{1}{2} f(w))^{\frac{1}{1-p}} \prod_{1 \le j \le 4} (g(w) - \alpha_j)^{\frac{p(1-\eta)}{1-p}} (g'(w))^{-\frac{p}{1-p}}.$$

(This u has similar function as f in Lemma 7. However here f is already used for the first component of the Enneper-Weierstrass representation, so we call this function u.)

Let $\hat{B}$ be the universal covering of $B = \Omega\backslash\Sigma$ with projection map $\pi : \hat{B} \to B$. Define $\hat{F} : \hat{\Omega} \to \mathbb{C}$ as $\hat{F}(w) = \int_0^w u(\pi(\zeta))d\zeta$ (where 0 is a point we pick on $\hat{\Omega}$). Then we can find a largest R such that we have a inverse $\hat{G} : D_R \to \hat{B}$ of $\hat{G}$, i.e. $\hat{F} \circ \hat{G} = \text{id}$. By Liouville's theorem R must be finite. There exists a singular point $z_0 \in \partial D_R$ of $\hat{G}$. Define $G : D_R \to B$ as $G = \pi \circ \hat{G}$.

From the definitions, we can see that $u(G(z)) = \frac{1}{G'(z)}$. So for $w = G(z)$, we have

$$\begin{aligned}
|u(w)| &= \frac{1}{2}|f(w)| \prod_{1\le j\le 4} |g(w) - \alpha_j|^{p(1-\eta)}|g'(w)|^{-p}|u(w)|^p \\
&= \frac{1}{2}|f(G(z))| \prod_{1\le j\le 4} |g(G(z)) - \alpha_j|^{p(1-\eta)}|(g\circ G)'(z))|^{-p}|G'(z)|^p|u(w)|^p \\
&= \frac{1}{2}|f(G(z))| \prod_{1\le j\le 4} |g(G(z)) - \alpha_j|^{p(1-\eta)}|(g\circ G)'(z))|^{-p}
\end{aligned}$$

With this setup we can define our curve. Define $\gamma^* : [0,1) \to D_R$ as $\gamma^*(t) = tz_0$. Define $\gamma : [0,1) \to M$ as $\gamma = G \circ \gamma^*$. We first prove that γ has finite length.

$$\begin{aligned}
L(\gamma) &= \frac{1}{2}\int_\gamma |f|(1+|g|^2)|dw| \\
&= \frac{1}{2}\int_{\gamma^*} |f\circ G|(1+|g\circ G|^2)|\frac{dw}{dz}||dz| \\
&= \frac{1}{2}\int_{\gamma^*} |f\circ G|(1+|g\circ G|^2)|u(w)|^{-1}|dz| \\
&= \int_{\gamma^*} |1+|g\circ G|^2| \prod_{1\le j\le 4} |g\circ G - \alpha_j|^{p(\eta-1)}|(g\circ G)'|^p|dz|
\end{aligned}$$

Now apply Lemma 10 (using $g \circ G$ as g in lemma). We get that

$$L(\gamma) \le \int_{\gamma^*} (\frac{BR}{R^2 - |z|^2})^p|dz| < \infty.$$

(Recall that $p = \frac{2}{3-\epsilon} < \frac{2}{3}$.)

It remains to prove that γ is divergent. Assume the contrary. Then there exists a compact set $K \subseteq M$ such that there exists t_0 such that for $t > t_0$, $\gamma(t) \in K$. So there exists a sequence $\{t_n\}$ approaching 1 such that $\gamma(t)$ approaches some point $w_0 \in K$.

Assume we can choose w_0 such that $w_0 \in \Omega\backslash\Sigma$. Then $u(w_0) \neq 0$. (f is nowhere zero because g is holomorphic (and by properties of the Enneper-Weierstrass representation).) Choose a lifting $w_0' \in \hat{B}$ of w_0. Then there exists an open neighborhood of w_0' such that $\hat{F}$ is invertible. Say the invert is $\hat{G}'$. Then

$$\hat{F}(w_0') = \lim_{n\to\infty} \hat{F}(\hat{G}'(t_n z_0)) = \lim_{n\to\infty} t_n z_0 = z_0.$$

So we can use $\hat{G}'$ to extend $\hat{G}$ to a neighborhood of z_0. However this contradicts with the definition of z_0. So $w_0 \notin \Omega\backslash\Sigma$.

So all possible w_0 are in Σ. Σ is discrete, so there exists $w_0 \in \Sigma$ such that $\lim_{t\to 1}\gamma(t) = w_0$. $g'(w_0) = 0$, so the power expansion of g' at w_0 has terms of exponent ≥ 1. So the power expansion of $(g')^{\frac{p}{1-p}}$ at w_0 has terms of exponent > 2. ($\frac{p}{1-p} = \frac{2}{1-\epsilon} > 2$.) So we have

$$R = \int_{\gamma^*} |dz| = \int_\gamma |u(w)||dw| > C \int_\gamma |w - w_0|^{-2}|dw| = \infty$$

where C is some positive constant. However we know that $R < \infty$. So there is contradiction.

We are done with the proof. $\square$

Remark Essentially the same proof shows that given a (not necessarily) minimal surface M whose Gauss map omits at most 5 points $\alpha_1, \ldots, \alpha_5$, then there exists a positive constant C depending only on $\alpha_1, \ldots, \alpha_5$ such that $K(p) \le \frac{C}{d(p)^2}$, where $K(p)$ is the Gauss curvature at p and $d(p)$ is the distance from p to the boundary of M.

Remark The proof of Theorem 11 is used in the proof of Mo and Osserman's theorem [MO90], which says that a complete non-flat minimal surface whose Gauss map takes 5 points finitely many times has finite curvature.

Assume such M exists, then we can find a nonempty compact set D such that the Gauss map restricted to D omits 5 points. In the proof of Theorem 11 we essentially defined a new metric on $B = \Omega \backslash \Sigma$ and proved that B is complete with this metric. Replacing B with $(\Omega\backslash\Sigma) \cup D$, we can prove that $(\Omega\backslash\Sigma) \cup D$ is complete with the defined metric. This then helps us bound the total curvature on $(\Omega\backslash\Sigma) \cup D$ and then on M.

5 Voss' Theorem

In this section we prove Voss' theorem on constructing complete minimal surfaces whose Gauss map omits given points.

Theorem 12 (Voss) *Given any $q \le 4$ distinct points $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$, there exists a complete minimal surface whose Gauss map omits exactly $\alpha_1, \ldots, \alpha_q$.*

Proof By changing coordinate we can assume $\alpha_q = \infty$. Let $\Omega = \mathbb{C}\backslash\{\alpha_1, \ldots, \alpha_q\}$, $\pi : \hat{\Omega} \to \Omega$ be the universal covering of Ω. Define $f : \hat{\Omega} \to \mathbb{C}$ as

$$f(z) = \frac{\pi'(z)}{\prod_{1 \le i \le q-1} |\pi(z) - \alpha_i|}.$$

Let $g : \hat{\Omega} \to \mathbb{C}$ as $g(z) = \pi(z)$. f is nowhere zero and g is holomorphic, so (f, g) satisfies the properties in Proposition 1. So (f, g) determines (up to translation in $\mathbb{R}^3$) a minimal surface $\hat{M}$. Clearly, by definition, the Gauss map of $\hat{M}$ (i.e. g) omits exactly the points $\alpha_1, \ldots, \alpha_q$. We only need to prove that $\hat{M}$ is complete.

One thing to note that is for $x \in \hat{\Omega}$, $f(x)$ and $g(x)$ only depends on $\pi(x)$. So f and g can be restricted to Ω. Let us denote them as $f_\Omega : \Omega \to \mathbb{C}$ and $g_\Omega : \Omega \to \mathbb{C}$. Then

$$f_\Omega(z) = \frac{1}{\prod_{1 \le i \le q-1} |z - \alpha_i|}$$

and $g_\Omega(z) = z$. This (f_Ω, g_Ω) gives a metric on Ω defined as $ds^2 = \frac{1}{4}|f_\Omega|^2(1 + |g_\Omega|^2)^2|dz|^2$. So this gives a Riemann surface M. (Note that this does not necessarily give a minimal surface structure on M.) The completeness part of Proposition 5 holds for non-minimal surfaces, so we only need to prove that M is complete.

Let $\gamma : [0, a) \to M$ be a divergent curve. Then

$$\begin{aligned} L(\gamma) &= \frac{1}{2} \int_\gamma |f_\Omega|(1 + |g_\Omega|^2)|dz| \\ &= \frac{1}{2} \int_\gamma \frac{1 + |z|^2}{\prod_{1 \le i \le q-1} |z - \alpha_i|} |dz| \end{aligned}$$

Now we consider the limit points of γ. Consider the compact set K defined as $\overline{C} - \bigcup_{1 \le i \le q} U_i$, where U_i is a small enough neighborhood of α_i such that $U_i \cap U_j = \emptyset$ for $i \ne J$. By definition of completeness, this means that there exists $t_0 \in [0, a)$ such that for $t \in (t_0, a)$, $\gamma(t) \notin K$. This means that $\gamma(t)$ will be restricted in some small neighborhood of a point in $\{\alpha_1, \ldots, \alpha_q\}$. By restricting the neighborhood to be smaller and smaller, we can see that that point is the unique limit point of γ, i.e. $\lim_{t \to a} \gamma(t) = \alpha_j$ for some $1 \le j \le q$.

If $j = q$, then $\lim_{t \to a} \gamma(t) = \infty$. Then

$$\frac{1 + |z|^2}{\prod_{1 \le i \le q-1} |z - \alpha_j|} \sim \frac{C}{|z|^{q-3}}$$

as $t \to a$, for some positive constant C. So $L(\gamma) = \infty$. (Note that $q \le 4$ is used crucially here.)

If $j < q$, then as $t \to a$, we have

$$\frac{1 + |z|^2}{\prod_{1 \le i \le q-1} |z - \alpha_j|} \sim \frac{C}{|z - \alpha_j|}$$

for some positive constant C. Again we get $L(\gamma) = \infty$.

So any divergent curve on M has infinite length. So M is complete. By the above discussion, $\overline{M}$ is a complete minimal surface whose Gauss map omits the given points $\alpha_1, \ldots, \alpha_q$. □

References

[DHS10] Ulrich Dierkes, Stefan Hildebrandt, and Friedrich Sauvigny. *Minimal surfaces*. Springer, 2010.

[Fan93] Yi Fang. On the Gauss map of complete minimal surfaces with finite total curvature. *Indiana University Mathematics Journal*, 42(4):1389–1411, 1993.

[Fuj88] Hirotaka Fujimoto. On the number of exceptional values of the Gauss maps of minimal surfaces. *J. Math. Soc. Japan*, 40(2):235–247, 04 1988.

[MO90] Xiaokang Mo and Robert Osserman. On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto's theorem. *Journal of Differential Geometry*, 31(2):343–355, 1990.

[Oss59] Robert Osserman. Proof of a conjecture of Nirenberg. *Communications on Pure and Applied Mathematics*, 12(2):229–232, 1959.

[Oss64] Robert Osserman. Global properties of minimal surfaces in E^3 and E^n. *Annals of Mathematics*, 80(2):340–364, 1964.

[Vos64] Konrad Voss. Uber vollständige minimalflächen. *L'Enseigment Math*, 10:316–317, 1964.

[WX87] Allen Weitsman and Frederico Xavier. Some function theoretic properties of the Gauss map for hyperbolic complete minimal surfaces. *The Michigan mathematical journal*, 34(2):275–283, 1987.

[Xav81] Frederico Xavier. The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere. *Annals of Mathematics*, 113(1):211–214, 1981.