NUMERICAL CRITERIA FOR AMPLENESS AND NEFNESS

YUZHOU GU

Contents

1. Introduction

In this expository paper we discuss numerical criteria for ampleness and nefness, including Nakai-Moishezon criterion for ampleness, Kleiman's theorem on nefness, and Kleiman's criterion for ampleness. These criteria are based on intersection theoretic properties, which are invariant under numerical equivalences, thus are called "numerical criteria". The main references are Lazarsfeld [Laz04] Chapter 1 and Vakil [Vak] Chapter 20.

Throughout this paper, X is a projective scheme over a field k. Although almost all results hold for proper schemes in general, we focus on the projective case because the proofs are easier.

2. Basics of ampleness

We recall some basic definitions and properties.

Definition 2.1. A line bundle $\mathcal{L}$ on X is called

(1) **base-point-free** if it is generated by global sections.
(2) **very ample** if there exists a closed immersion $i : X \hookrightarrow \mathbb{P}^N$ for some positive integer N such that $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^N}(1)$.
(3) **ample** if $\mathcal{L}^{\otimes m}$ is very ample for some positive integer m.

The Cartan-Serre-Grothendieck theorem is a cohomological criteria for ampleness.

Theorem 2.2 (Cartan-Serre-Grothendieck theorem, [Laz04] Theorem 1.2.6)**.** *Let $\mathcal{L}$ be a line bundle on X. The following are equivalent:*

(1) *$\mathcal{L}$ is ample.*
(2) *For any coherent sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and m large enough.*

(3) *For any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is generated by global sections for m large enough.*
(4) *$\mathcal{L}^{\otimes m}$ is very ample for m large enough.*

The proof is omitted.

Proposition 2.3 ([Vak] Exercise 16.6.C)**.** *If $\mathcal{L}_1$ is very ample and $\mathcal{L}_2$ is base-point-free, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample.*

Proof. By assumption, we have a closed immersion $i : X \to \mathbb{P}^N$ such that $\mathcal{L}_1 = i^*\mathcal{O}_{\mathbb{P}^N}(1)$. $\mathcal{L}_2$ is generated by global sections, so we have a morphism $j : X \to \mathbb{P}^M$ (not necessarily an immersion) such that $\mathcal{L}_2 = j^*\mathcal{P}_{\mathbb{P}^N}(1)$. Let $S : \mathbb{P}^N \times \mathbb{P}^M \to \mathbb{P}^{NM+N+M}$ be the Segre embedding and $\Delta : X \to X \times X$ be the diagnoal map. Clearly $\mathcal{L}_1 \otimes \mathcal{L}_2 = (S \circ (i \times j) \circ \Delta)^*\mathcal{O}_{\mathbb{P}^{NM+N+M}}(1)$. We only need to prove that $S \circ (i \times j) \circ \Delta$ is a closed immersion. S is known to be a closed immersion. Let $\pi : \mathbb{P}^N \to \operatorname{Spec} k$ be the structure morphism. Then $(\mathrm{id} \times \pi) \circ (i \times j) \circ \Delta = i$, which is a closed immersion. $\mathrm{id} \times \pi$ is separated, so $(i \times j) \circ \Delta$ is a closed immersion. $\square$

Corollary 2.4 ([Vak] Exercise 16.6.E)**.** *If $\mathcal{L}_1$ is ample and $\mathcal{L}_2$ is any line bundle, then $\mathcal{L}_1^{\otimes m} \otimes \mathcal{L}_2$ is very ample for large enough m.*

Proof. Take some $n > 0$ such that $\mathcal{L}_1^{\otimes n}$ is very ample. By Theorem 2.2, $\mathcal{L}_1^{\otimes(m-n)} \otimes \mathcal{L}_2$ is base-point-free for m large enough. By Proposition 2.3, $\mathcal{L}_1^{\otimes m} \otimes \mathcal{L}_2$ is very ample for m large enough. $\square$

Corollary 2.5 ([Vak] Exercise 16.6.F)**.** *Any line bundle $\mathcal{L}$ can be written as $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ where $\mathcal{L}_1$ and $\mathcal{L}_2$ are very ample.*

Proof. Take any ample line bundle $\mathcal{L}_3$ and m large enough. Let $\mathcal{L}_2 = \mathcal{L}_3^{\otimes m}$ and $\mathcal{L}_1 = \mathcal{L}_2 \otimes \mathcal{L}$. By Corollary 2.4 and Theorem 2.2, when m is large enough, both $\mathcal{L}_1$ and $\mathcal{L}_2$ are very ample. $\square$

The following proposition helps us reduce to the case X is integral.

Proposition 2.6 ([Laz04] Proposition 1.2.16)**.** *Let $\mathcal{L}$ be a line bundle on X.*

(1) *$\mathcal{L}$ is ample on X iff $\mathcal{L}_{\mathrm{red}}$ is ample on X_{red}.*
(2) *$\mathcal{L}$ is ample on X iff for every irreducible component X_i of X, $\mathcal{L}|_{X_i}$ is ample on X_i.*

Proof. The only if parts are easy. We prove the if parts.

(1): Assume $\mathcal{L}_{\mathrm{red}}$ is ample on X_{red}. Let $\mathcal{F}$ be an arbitrary coherent sheaf on X. Let $\mathcal{N}$ be the nilradical of $\mathcal{O}_X$. Say $\mathcal{N}^r = 0$. We have a filtration

$$\mathcal{F} \supseteq \mathcal{N}\mathcal{F} \supseteq \cdots \supseteq \mathcal{N}^r\mathcal{F} = 0.$$

We perform induction on j from $j = r$ to $j = 0$ to prove that $H^i(X, \mathcal{N}^j\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and m large enough. The base case $j = r$ is trivial.

The quotients $\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}$ are coherent $\mathcal{O}_{X_{\mathrm{red}}}$-modules. $\mathcal{L}_{\mathrm{red}}$ is ample, so $H^i(X, (\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}) \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and m large enough. We have an exact sequence $0 \to \mathcal{N}^{j+1}\mathcal{F} \to \mathcal{N}^j\mathcal{F} \to \mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F} \to 0$. Tensoring with $\mathcal{L}^{\otimes m}$, taking H^i, and applying induction hypothesis, we see that $H^i(X, \mathcal{N}^j\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and m large enough.

(2): By (1), assume X is reduced. Let $X_1, \ldots, X_r$ be the irreducible components of X. Assume that $\mathcal{L}$ is ample on $\mathcal{L}_1, \cdots, \mathcal{L}_r$. We apply induction on r.

Let $\mathcal{I}$ be the ideal sheaf of X_1. Then we have an exact sequence

$$0 \to \mathcal{I}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}\mathcal{F} \to 0.$$

$\mathcal{I}\mathcal{F}$ and $\mathcal{F}/\mathcal{I}\mathcal{F}$ are supported on $X_2 \cup \cdots \cup X_r$. So $H^i(X, - \otimes \mathcal{L}^{\otimes m})$ of them are 0 for $i > 0$ and m large enough, by induction hypothesis. Applying H^i to the above short exact sequence, we see that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and m large enough. $\square$

3. The intersection product

Definition 3.1. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be line bundles on X and $\mathcal{F}$ be a coherent sheaf on X with $\dim \operatorname{Supp} \mathcal{F} \leq n$. Define the **intersection product of $\mathcal{L}_1, \ldots, \mathcal{L}_n$ with $\mathcal{F}$** to be

$$(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) = \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} \chi(X, \mathcal{F} \otimes \bigotimes_{i \in S} \mathcal{L}_i^\vee).$$

If $\mathcal{F} = \mathcal{O}_V$ is the structure sheaf some closed subscheme V, then we write $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot V)$. If all $\mathcal{L}_i$'s are the same, say $\mathcal{L}$, then we write $(\mathcal{L}^n \cdot \mathcal{F})$.

Proposition 3.2 ([Vak] Exercise 20.1.C)**.** *If D is an effective Cartier divisor that does not contain any associated point of $\mathcal{F}$, then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{O}(D) \cdot \mathcal{F}) = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}|_D).$$

Proof. Under the assumption, we have an exact sequence

$$0 \to \mathcal{F}(-D) \to \mathcal{F} \to \mathcal{F}|_D \to 0.$$

For any line bundle $\mathcal{L}$, by tensoring we get an exact sequence

$$0 \to \mathcal{F}(-D) \otimes \mathcal{L} \to \mathcal{F} \otimes \mathcal{L} \to \mathcal{F}|_D \otimes \mathcal{L} \to 0.$$

Therefore $\chi(\mathcal{F} \otimes \mathcal{L}) = \chi(\mathcal{F}(-D) \otimes \mathcal{L}) + \chi(\mathcal{F}|_D \otimes \mathcal{L})$. By expanding the definition of intersection products we get the desired result. $\square$

H^i of quasi-coherent sheaves are preserved under base field extension, so the intersection product is preserved under base field extension ([Vak] Exercise 20.1.D). Using this, we can always assume that we are working with k algebraic closed. In this case, for any very ample line bundle $\mathcal{L}$ and coherent sheaf $\mathcal{F}$, there is an effective Cartier divisor D such that $\mathcal{O}(D) = \mathcal{L}$ and D misses the associated points of $\mathcal{F}$ ([Vak] Exercise 18.6.A).

Proposition 3.3 ([Vak] Proposition 20.1.3)**.** *Fix n and coherent sheaf $\mathcal{F}$ such that $\dim \operatorname{Supp} \mathcal{F} \leq n$. The intersection product $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})$ is symmetric and multilinear in $\mathcal{L}_1, \ldots, \mathcal{L}_n$.*

Proof. Symmetricity is clear. For multilinearity, we perform induction on n. When $n = 0$ the result is trivial.

Now assume $n > 0$ and that multilinearity is true for $n' < n$. Consider the formula

$$\begin{aligned} A &= (\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) - ((\mathcal{L}_1 \otimes \mathcal{L}_1') \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) \\ &= (\mathcal{L}_1 \cdot \mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}). \end{aligned}$$

The equality is by expanding the definition of intersection products.

If $\mathcal{L}_n$ is very ample, then we can choose an effective Cartier divisor D such that $\mathcal{O}(D) = \mathcal{L}_n$ and D misses associated points of $\mathcal{F}$. In this case, by Proposition 3.2, $A = (\mathcal{L}_1 \cdot \mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}|_D)$. Therefore $A = 0$ by induction hypothesis.

By symmetricity, if $\mathcal{L}_1$ is very ample, we have $A = 0$. Now let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary very ample line bundles, and take $\mathcal{L}_1 = \mathcal{A}$, $\mathcal{L}_2 = \mathcal{B} \otimes \mathcal{A}^\vee$. We get

$$(\mathcal{A} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) + ((\mathcal{B} \otimes \mathcal{A}^\vee) \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{B} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) = 0.$$

The first term and the last term are linear in $\mathcal{L}_n$, so the second term is also linear in $\mathcal{L}_n$. By Corollary 2.5, any line bundle $\mathcal{L}$ can be written as $\mathcal{B} \otimes \mathcal{A}^\vee$. Therefore $(\mathcal{L} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$ is linear in $\mathcal{L}_n$. □

Remark 3.4. The proof of the proposition says that $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) = 0$ if $\dim \operatorname{Supp} \mathcal{F} < n$. Therefore the intersection product is interesting only when $n = \dim \operatorname{Supp} \mathcal{F}$.

Using intersection products we can define numerical equivalence.

Definition 3.5. Two line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ are **numerically equivalent** $(\mathcal{L}_1 \cdot C) = (\mathcal{L}_2 \cdot C)$ for every curve $C \subseteq X$.

Proposition 3.6 ([Vak] Proposition 20.1.4)**.** *The intersection product* $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})$ *only depends on the numerical equivalence classes of the* $\mathcal{L}_i$*'s.*

Proof. We prove that if $\mathcal{L}_1$ and $\mathcal{L}_1'$ are numerically equivalent, then

$$(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}).$$

By Proposition 3.3 and Corollary 2.5, we can assume $\mathcal{L}_2, \ldots, \mathcal{L}_n$ are very ample. By Proposition 3.2, we can remove one $\mathcal{L}_i$ at a time. Therefore we only to prove that $(\mathcal{L}_1 \cdot \mathcal{G}) = (\mathcal{L}_1' \cdot \mathcal{G})$ when $\dim \sup \mathcal{G} \le 1$. This clearly follows from that $\mathcal{L}_1$ and $\mathcal{L}_1'$ are numerically equivalent. □

Proposition 3.7 (Asymptotic Riemann-Roch, [Vak] Exercise 20.1.I)**.** *Let* $\mathcal{L}$ *be a line bundle and* $\mathcal{F}$ *be a coherent sheaf with* $\dim \operatorname{Supp} \mathcal{F} \le n$*. Then* $\chi(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F})$ *is a polynomial in* m *with degree* $\le n$ *and the coefficient of* m^n *is* $\frac{(\mathcal{L}^n \cdot \mathcal{F})}{n!}$.

Proof. By Remark 3.4, we have $(\mathcal{L}^{\vee(n+1)} \cdot (\mathcal{F} \otimes \mathcal{L}^{\otimes i})) = 0$ for all $i \ge 0$. Expanding the definition, we get

$$\sum_{0 \le k \le n+1} (-1)^k \binom{n+1}{k} \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes i+k}) = 0.$$

Let Δ be the operator that maps a function $f : \mathbb{N} \to \mathbb{Z}$ to $m \mapsto f(m+1) - f(m)$. Let $F : \mathbb{N} \to \mathbb{Z}$ be defined as $m \mapsto \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$. The above formula says that $\Delta^{n+1} F = 0$. Therefore $F(m)$ is a polynomial with degree $\le n$.

The coefficient of m^n in $F(m)$ is $\frac{(\Delta^n F)(0)}{n!}$. We have

$$(\Delta^n F)(0) = \sum_{0 \le k \le n} (-1)^{n-k} \binom{n}{k} \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes k}) = (-1)^n (\mathcal{L}^{\vee n} \cdot \mathcal{F}) = (\mathcal{L}^n \cdot \mathcal{F}).$$

□

Remark 3.8. When X is a complex variety, the intersection product $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot V)$ can be defined alternatively as follows.

Consider the analytifications X_{an} and $(\mathcal{L}_i)_{\mathrm{an}}$. We have the Chern class $c_1((\mathcal{L}_i)_{\mathrm{an}}) \in H^2(X_{\mathrm{an}}, \mathbb{Z})$. Taking cup product, we get $c_1((\mathcal{L}_1)_{\mathrm{an}}) \cdots c_1((\mathcal{L}_n)_{\mathrm{an}}) \in H^{2n}(X_{\mathrm{an}}, \mathbb{Z})$.

V gives rise to its fundamental class $[V] \in H_{2n}(X_{\mathrm{an}}, \mathbb{Z})$. Taking cap product, we get $(c_1((\mathcal{L}_1)_{\mathrm{an}}) \cdots c_1((\mathcal{L}_n)_{\mathrm{an}})) \cap [V] \in H_0(X_{\mathrm{an}}, \mathbb{Z}) = \mathbb{Z}$.

This is the definition of intersection products used in [Laz04].

4. Nakai-Moishezon criterion for ampleness

Nakai-Moishezon criterion is an important numerical criterion for ampleness. First we prove a proposition used in the proof of the Nakai-Moishezon criterion.

Proposition 4.1 ([Laz04] Corollary 1.2.15)**.** *Let $\mathcal{L}$ be a base-point-free line bundle and $i : X \to \mathbb{P}^N$ be the morphism defined by $\mathcal{L}$. The following are equivalent:*

(1) *$\mathcal{L}$ is ample.*
(2) *i is finite.*
(3) *$(\mathcal{L} \cdot C) > 0$ for every curve $C \subseteq X$.*

Proof. (1) $\Rightarrow$ (3): By linearity we can replace $\mathcal{L}$ with $\mathcal{L}^{\otimes m}$ for some m large enough and assume that $\mathcal{L}$ is very ample. Then $(\mathcal{L} \cdot C)$ is the degree of C in the embedding defined by $\mathcal{L}$ and is positive.

(2) $\Rightarrow$ (1): We have $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^N}(1)$ and pullbacks of ample line bundles along finite morphisms between projective schemes are ample.

(3) $\Rightarrow$ (2): If i is not finite, then some curve $C \subseteq X$ is mapped to a point by i, and we have $(\mathcal{L} \cdot C) = 0$. □

Theorem 4.2 (Nakai-Moishezon criterion, [Laz04] Theorem 1.2.23)**.** *A line bundle $\mathcal{L}$ on a projective k-scheme X is ample iff $(\mathcal{L}^{\dim V} \cdot V) > 0$ for every closed subvariety $V \subseteq X$.*

Proof. Assume $\mathcal{L}$ is ample. We prove that $(\mathcal{L}^{\dim V} \cdot V) > 0$ for every closed subvariety $V \subseteq X$. By multilinearity of the intersection product, we can assume $\mathcal{L}$ is very ample. Then $(\mathcal{L}^{\dim V} \cdot V)$ is the degree of V in the embedding defined by $\mathcal{L}$ and is positive.

Conversely, assume $(\mathcal{L}^{\dim V} \cdot V) > 0$ for every closed subvariety $V \subseteq X$. The proof is in several steps.

Step 1. By Proposition 2.6 we can assume that X is integral. By applying induction on dimension, we can assume that $\mathcal{L}$ is ample on any closed subvariety of X that is not equal to X.

Step 2. We prove that $H^0(X, \mathcal{L}^{\otimes m}) \neq 0$ for m large enough. Using Corollary 2.5, write $\mathcal{L} = \mathcal{A} \otimes \mathcal{B}^\vee$ where $\mathcal{A}$ and $\mathcal{B}$ are very ample. Write $\mathcal{A} = \mathcal{O}(A)$ and $\mathcal{B} = \mathcal{O}(B)$. From the exact sequence $0 \to \mathcal{O}(-A) \to \mathcal{O} \to \mathcal{O}|_A \to 0$, we have an exact sequence

$$0 \to \mathcal{L}^{\otimes m}(-B) \to \mathcal{L}^{\otimes(m+1)} \to \mathcal{L}^{\otimes(m+1)}|_A \to 0.$$

From the exact sequence $0 \to \mathcal{O}(-B) \to \mathcal{O} \to \mathcal{O}|_B \to 0$, we have an exact sequence

$$0 \to \mathcal{L}^{\otimes m}(-B) \to \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}|_B \to 0.$$

By induction hypothesis, $\mathcal{L}$ is proper on every closed subvariety of X not equal to X. So for m large enough, we have $H^i(X, \mathcal{L}^{\otimes(m+1)}|_{\mathcal{A}}) = 0$ and $H^i(X, \mathcal{L}^{\otimes m}|_{\mathcal{B}}) = 0$ for all $i \geq 1$. Taking H^i on the above two exact sequences, we get

$$H^i(X, \mathcal{L}^{\otimes(m+1)}) = H^i(X, \mathcal{L}^{\otimes m}(-B)) = H^i(X, \mathcal{L}^{\otimes m})$$

for all $i \geq 2$.

By asymptotic Riemann-Roch (Proposition 3.7), $\chi(X, \mathcal{L}^{\otimes m})$ is a polynomial in m with degree $\dim X$ and top degree coefficient $\frac{(\mathcal{L}^{\dim X} \cdot X)}{(\dim X)!} > 0$. So $\chi(X, \mathcal{L}^{\otimes m})$ goes to ∞ as m goes to ∞.

On the other hand, $\chi(X, \mathcal{L}^{\otimes m}) = h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) + \text{constant}$ for m large enough. So $h^0(X, \mathcal{L}^{\otimes m})$ goes to ∞ as m goes to ∞. We can replace $\mathcal{L}$ with $\mathcal{L}^{\otimes m}$ for some large enough m and assume that $\mathcal{L} = \mathcal{O}(D)$ where D is an effective Cartier divisor.

Step 3. We prove that $\mathcal{L}^{\otimes m}$ is base-point-free for m large enough. D is effective, so $\mathcal{L}^{\otimes m}$ is generated by global sections away from $\operatorname{Supp} D$. We only need to prove that $\mathcal{L}^{\otimes m}$ is generated by global sections in D.

Consider the exact sequence

$$0 \to \mathcal{L}^{\otimes(m-1)} \to \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}|_D \to 0.$$

By induction hypothesis, $\mathcal{L}^{\otimes m}|_D$ is ample. So $H^1(X, \mathcal{L}^{\otimes m}|_D) = 0$ for m large enough. Taking cohomology of the above exact sequence, we see that $H^1(X, \mathcal{L}^{\otimes(m-1)}) \to H^1(X, \mathcal{L}^{\otimes m})$ is surjective for m large enough. However, $H^1(X, \mathcal{L}^{\otimes m})$ is finite dimensional. So $H^1(X, \mathcal{L}^{\otimes(m-1)}) \to H^1(X, \mathcal{L}^{\otimes m})$ must be an isomorphism for m large enough.

Considering again the cohomology long exact sequence, we see that $H^0(X, \mathcal{L}^{\otimes m}) \to H^0(X, \mathcal{L}^{\otimes m}|_D)$ is surjective for m large enough. By induction hypothesis, $H^0(X, \mathcal{L}^{\otimes m}|_D)$ is base-point-free. So for every point in D, there is a global section of $\mathcal{L}^{\otimes m}$ that does not vanish at that point. Therefore $\mathcal{L}^{\otimes m}$ is generated by global sections in D.

Step 4. Apply Proposition 4.1. □

Corollary 4.3 ([Laz04] Corollary 1.2.24)**.** *Ampleness of line bundles depends only on the numerical equivalence class.*

Proof. By Proposition 3.6. □

5. $\mathbb{Q}$-line bundles and $\mathbb{R}$-line bundles

In the following, tensor products of line bundles (resp. $\mathbb{Q}$- or $\mathbb{R}$-line bundles) are often written as additions. For example, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is written as $\mathcal{L}_1 + \mathcal{L}_2$ and $\mathcal{L}^{\otimes m}$ is written as $m\mathcal{L}$.

Recall that $\operatorname{Pic}(X)$ is the abelian group of line bundles.

Definition 5.1. A **$\mathbb{Q}$-line bundle** is a finite $\mathbb{Q}$-linear combination of line bundles. The $\mathbb{Q}$-vector space of $\mathbb{Q}$-line bundles is $\operatorname{Pic}_{\mathbb{Q}}(X) = \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

An **$\mathbb{R}$-line bundle** is a finite $\mathbb{R}$-linear combination of line bundles. The $\mathbb{R}$-vector space of $\mathbb{R}$-line bundles is $\operatorname{Pic}_{\mathbb{R}}(X) = \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Recall that two line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ are called numerically equivalent if $(\mathcal{L}_1 \cdot C) = (\mathcal{L}_2 \cdot C)$ for every curve $C \subseteq X$. Numerical equivalence is preserved under addtion. Denote $N^1(X)$ to be the abelian group of numerical equivalence classes of line bundles.

Remark 5.2. In [Laz04], $N^1(X)$ is called the Néron-Severi group, which often refers to the abelian group of algebraic equivalence classes of line bundles.

Definition 5.3. We extend several notions for line bundles to $\mathbb{Q}$- and $\mathbb{R}$-line bundles.

(1) Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be $\mathbb{Q}$- (resp. $\mathbb{R}$-) line bundles and $\mathcal{F}$ be a coherent sheaf with $\dim \operatorname{Supp} \mathcal{F} \leq n$. The **intersection product** $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})$ is defined by extending by multilinearity.
(2) Two $\mathbb{Q}$- (resp. $\mathbb{R}$-) line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ are called **numerically equivalent** if $(\mathcal{L}_1 \cdot C) = (\mathcal{L}_2 \cdot C)$ for every curve $C \subseteq X$. Define $N^1_{\mathbb{Q}}(X)$ (resp. $N^1_{\mathbb{R}}(X)$) to be the $\mathbb{Q}$- (resp. $\mathbb{R}$-) vector space of numerical equivalence classes of $\mathbb{Q}$- (resp. $\mathbb{R}$-) line bundles. We have $N^1_{\mathbb{Q}}(X) = N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N^1_{\mathbb{R}}(X) = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.
(3) A $\mathbb{Q}$- (resp. $\mathbb{R}$-) line bundle is called **ample** if it can be written as a nonempty finite positive $\mathbb{Q}$- (resp. $\mathbb{R}$-) linear combination of ample line bundles.

With essentially the same proof as the proof of Proposition 3.6, we can prove that the intersection product for $\mathbb{Q}$- (resp. $\mathbb{R}$-) line bundles only depends on the numerical equivalence classes.

Sum of two ample line bundles is ample. If $n\mathcal{L}$ is ample then $\mathcal{L}$ is ample. Therefore the notion of ampleness for $\mathbb{Q}$-line bundles agrees with the notion of ampleness for line bundles. If a positive $\mathbb{R}$-linear combination of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ is a line bundle, then it is a positive $\mathbb{Q}$-linear combination of $\mathcal{L}_1, \ldots, \mathcal{L}_n$. Therefore the notion of ampleness for $\mathbb{R}$-line bundles agrees with the notion of ampleness for line bundles.

Recall that $N^1(X)$ has finite rank by the Néron-Severi theorem. Therefore $N^1_{\mathbb{Q}}(X)$ and $N^1_{\mathbb{R}}(X)$ have finite dimension. We can equip $N^1_{\mathbb{Q}}(X)$ (resp. $N^1_{\mathbb{R}}(X)$) with the natural topology induced from the usual topology on $\mathbb{Q}$ (resp. $\mathbb{R}$).

Proposition 5.4 (Nakai-Moishezon criterion for $\mathbb{Q}$-line bundles, [Laz04] Definition 1.3.6)**.** *A $\mathbb{Q}$-line bundle $\mathcal{L}$ is ample iff $(\mathcal{L}^{\dim V} \cdot V) > 0$ for every closed subvariety $V \subseteq X$.*

Proof. $m\mathcal{L}$ is a line bundle for some positive integer m. $\mathcal{L}$ is ample iff $m\mathcal{L}$ is ample. □

Corollary 5.5 ([Laz04] Definition 1.3.6)**.** *Ampleness of $\mathbb{Q}$-line bundles depends only on the numerical equivalence class.*

Therefore ampleness of $\mathbb{Q}$-line bundles is defined in $N^1_{\mathbb{Q}}(X)$. The set of numerical equivalence classes of ample $\mathbb{Q}$-line bundles is convex in $N^1_{\mathbb{Q}}(X)$ and is called the ample cone in $N^1_{\mathbb{Q}}(X)$.

Proposition 5.6 ([Laz04] Proposition 1.3.7)**.** *The ample cone in $N^1_{\mathbb{Q}}(X)$ is open.*

Proof. Let $\mathcal{A}$ be an ample $\mathbb{Q}$-line bundle and $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be arbitrary $\mathbb{Q}$-line bundles. We prove that $\mathcal{A} + \sum \epsilon_i \mathcal{L}_i$ is ample when $|\epsilon_i|$'s are sufficiently small.

We can assume that $\mathcal{A}$ and $\mathcal{L}_i$'s are line bundles. By Corollary 2.4, we can choose m such that $m\mathcal{A} \pm \mathcal{L}_i$ are ample.

When $|\epsilon_i|'s$ are sufficiently small,

$$\mathcal{A} + \sum \epsilon_i \mathcal{L}_i = (1 - \sum m\epsilon_i)\mathcal{A} + \sum (m\epsilon_i \mathcal{L} + \epsilon_i \mathcal{L}_i)$$

is a positive linear combination of ample $\mathbb{Q}$-line bundles, thus is ample. □

Remark 5.7. The Nakai-Moishezon criterion is also true for $\mathbb{R}$-line bundles. However, this is not immediate from the Theorem 4.2. See [Laz04] Remark 1.3.12.

Nevertheless, we can prove many properties of ample $\mathbb{R}$-line bundles without using the Nakai-Moishezon criterion for $\mathbb{R}$-line bundles.

Proposition 5.8 ([Laz04] Proposition 1.3.13)**.** *Ampleness of $\mathbb{R}$-line bundles depends only on the numerical equivalence class.*

Proof. We need to prove that if $\mathcal{A}$ is an ample $\mathbb{R}$-line bundle and $\mathcal{B}$ is a numerically trivial $\mathbb{R}$-line bundle, then $\mathcal{A}+\mathcal{B}$ is ample. $\mathcal{B}$ can be written as $\sum c_i\mathcal{L}_i$ where $c_i \in \mathbb{R}$ and $\mathcal{L}_i$'s are line bundles. Note that $(\mathcal{L}_i \cdot C)$ are integers. That $\mathcal{B}$ is numerically trivial means that $\sum c_i(\mathcal{L}_i \cdot C) = 0$ for all curves $C \subseteq X$. This is a system of linear equations in variables c_i with integral coefficients, so every solution can be written as $\mathbb{R}$-linear combinations of integral solutions. This means that $\mathcal{B}$ can be written as an $\mathbb{R}$-linear combination of numerically trivial line bundles. So we can assume $\mathcal{B} = r\mathcal{L}$ where $\mathcal{L}$ is a numerically trivial line bundle and $c \in \mathbb{R}$.

Now $\mathcal{A}$ can be written as $\sum c_i\mathcal{L}_i$ where $c_i > 0$ and $\mathcal{L}_i$'s are ample line bundles. To prove that $\mathcal{A} + \mathcal{B}$ is ample, we only need to choose some $d_i \in \mathbb{R}$ such that $\sum c_i d_i = r$ and prove that $\mathcal{L}_i + d_i\mathcal{L}$ is ample. So we can assume $\mathcal{A}$ is an ample line bundle.

Therefore we need to prove that $\mathcal{A} + r\mathcal{L}$ is ample where $\mathcal{A}$ is an ample line bundle, $r \in \mathbb{R}$ and $\mathcal{L}$ is a numerically trivial line bundle. Choose $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r < r_2$. By Corollary 5.5, $\mathcal{A} + r_1\mathcal{L}$ and $\mathcal{A} + r_2\mathcal{L}$ are ample. So $\mathcal{A} + r\mathcal{L}$ is also ample. $\square$

Therefore ampleness of $\mathbb{R}$-line bundles is defined in $N^1_{\mathbb{R}}(X)$. The set of numerical equivalence classes of $\mathbb{R}$-line bundles is convex in $N^1_{\mathbb{R}}(X)$ and is called the ample cone in $N^1_{\mathbb{R}}(X)$.

Proposition 5.9 ([Laz04] Example 1.3.14)**.** *The ample cone in $N^1_{\mathbb{R}}(X)$ is open.*

Proof. Let $\mathcal{A}$ be an ample $\mathbb{R}$-line bundle and $\mathcal{L}_1, \dots, \mathcal{L}_n$ be arbitrary $\mathbb{R}$-line bundles. We prove that $\mathcal{A}+\sum \epsilon_i\mathcal{L}_i$ is ample when $|\epsilon_i|$'s are sufficiently small. We can assume that ϵ_i's are rational.

Since $\mathcal{L}_i$'s are $\mathbb{R}$-linear combinations of line bundles, we can assume that $\mathcal{L}_i$ are line bundles. Write $\mathcal{A} = \sum c_i\mathcal{A}_i$ where $\mathcal{A}_i$ are ample line bundles and $c_i > 0$. Choose some $c \in \mathbb{Q}$ such that $0 < c < c_1$. Then

$$\mathcal{A} + \sum \epsilon_i\mathcal{L}_i = (c\mathcal{A}_1 + \sum \epsilon_i\mathcal{L}_i) + (c_1 - c)\mathcal{A}_1 + \sum_{i\geq 2} c_i\mathcal{A}_i.$$

$c\mathcal{A}_1 + \sum \epsilon_i\mathcal{L}_i$ is ample by the proof of Proposition 5.6. So RHS is a positive $\mathbb{R}$-linear combination of ample $\mathbb{R}$-line bundles, which is ample. $\square$

6. Nef line bundles and Kleiman's theorem

Definition 6.1. An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is called **numerically effective (nef)** if $(\mathcal{L} \cdot C) \geq 0$ for all irreducible curves $C \subseteq X$. The notion of nefness is defined in the same way for line bundles and $\mathbb{Q}$-line bundles.

Proposition 6.2 ([Laz04] Definition 1.4.1)**.** *Nefness of $\mathbb{R}$-line bundles (resp. $\mathbb{Q}$-line bundles, line bundles) only depends on the numerical equivalence class.*

Proof. By definition. $\square$

Therefore nefness is defined in $N^1_{\mathbb{R}}(X)$ (resp. $N^1_{\mathbb{Q}}(X)$, $N^1(X)$). The set of nef $\mathbb{R}$- (resp. $\mathbb{Q}$) line bundles is a convex subset of $N^1_{\mathbb{R}}(X)$ (resp. $N^1_{\mathbb{Q}}(X)$), and is called the nef cone in $N^1_{\mathbb{R}}(X)$ (resp. $N^1_{\mathbb{Q}}(X)$).

Theorem 6.3 (Kleiman's theorem, [Laz04] Theorem 1.4.9)**.** *A $\mathbb{Q}$-line bundle $\mathcal{L}$ on a projective k-scheme X is nef iff $(\mathcal{L}^{\dim V} \cdot V) \geq 0$ for every closed subvariety $V \subseteq X$.*

Proof. The if part is trivial. We prove the only if part.

Reduce to the case X is integral. Applying induction on $n = \dim X$, we can assume that $(\mathcal{L}^{\dim V} \cdot V) \geq 0$ for every closed subvariety $V \subseteq X$ not equal to X. We prove that $(\mathcal{L}^n \cdot X) \geq 0$.

Fix any very ample line bundle $\mathcal{A}$. Consider the function $P : \mathbb{R} \to \mathbb{R}$ defined as $t \mapsto ((\mathcal{L} + t\mathcal{A})^n \cdot X)$. Then

$$((\mathcal{L} + t\mathcal{A})^n \cdot X) = \sum_{0 \leq k \leq n} t^k \binom{n}{k} (\mathcal{L}^{n-k} \cdot \mathcal{A}^k \cdot X)$$

is a polynomial.

$(\mathcal{L}^n \cdot X)$ is the constant term of $P(t)$. For $k \geq 1$, the coefficient of t^k is $\binom{n}{k}(\mathcal{L}^{n-k} \cdot \mathcal{A}^k \cdot X)$. Choose any effective Cartier divisor D with $\mathcal{O}(D) = \mathcal{A}$. We get $(\mathcal{L}^{n-k} \cdot \mathcal{A}^k \cdot X) = (\mathcal{L}^{n-k} \cdot \mathcal{A}^{k-1} \cdot D)$, which is non-negative by induction hypothesis. Also, $(\mathcal{A}^n \cdot X) > 0$. So $P(t)$ is a polynomial with degree n and coefficients of non-constant terms are positive.

Assume for the sake of contrary that $(\mathcal{L}^n \cdot X) < 0$. Then there exists a unique $t_0 > 0$ such that $P(t_0) = 0$. Let $Q(t) = (\mathcal{L} \cdot (\mathcal{L} + t\mathcal{A})^{n-1} \cdot X)$ and $R(t) = (t\mathcal{A} \cdot (\mathcal{L} + t\mathcal{A})^{n-1} \cdot X)$. We have $P(t) = Q(t) + R(t)$.

$R(t)$ is a polynomial with non-negative coefficients. Coefficient of t^n is $(\mathcal{A}^n \cdot X) > 0$, so $R(t_0) > 0$.

Let $t_1 \in \mathbb{Q}$ and $t_1 > t_0$. We prove that $\mathcal{L} + t_1\mathcal{A}$ is ample. By Proposition 5.4, we only need to prove that $((\mathcal{L} + t_1\mathcal{A})^{\dim V} \cdot V) > 0$ for every closed subvariety $V \subseteq X$. When $V = X$, this is true because $t_1 > t_0$. Assume $V \neq X$. By induction hypothesis, $(\mathcal{L}^{\dim V} \cdot V) \geq 0$. So $((\mathcal{L} + t\mathcal{A})^{\dim V} \cdot V)$ is a polynomial in t with nonegative coefficients. Furthermore, the coefficient of $t^{\dim V}$ is $(\mathcal{A}^{\dim V} \cdot V)$, and is positive because $\mathcal{A}$ is very ample. Therefore $((\mathcal{L} + t_1\mathcal{A})^{\dim V} \cdot V) > 0$.

Now consider $Q(t_1) = (\mathcal{L} \cdot (\mathcal{L} + t_1\mathcal{A})^{n-1} \cdot X)$. $\mathcal{L} + t_1\mathcal{A}$ is ample, so $Q(t_1)$ is $c(\mathcal{L} \cdot C)$ for some $c > 0$ and C effective 1-cycle. So $Q(t_1) \geq 0$ for all $t_1 \in \mathbb{Q}$ with $t_1 > t_0$. By continuity, $Q(t_0) \geq 0$.

Then $P(t_0) = Q(t_0) + R(t_0) > 0$, which contradicts with the definition of t_0. $\square$

Corollary 6.4 (Kleiman's theorem for $\mathbb{R}$-line bundles, [Laz04] Theorem 1.4.9)**.** *An $\mathbb{R}$-line bundle $\mathcal{L}$ is nef iff $(\mathcal{L}^{\dim V} \cdot V) \geq 0$ for every closed subvariety $V \subseteq X$.*

Proof. Choose ample $\mathbb{Q}$-line bundles $\mathcal{L}_1, \cdots, \mathcal{L}_r$ such that they span $N^1_{\mathbb{R}}(X)$. This can be done because of Proposition 5.6. For $\epsilon_i > 0$, $\mathcal{L} + \sum \epsilon_i \mathcal{L}_i$ is sum of a nef $\mathbb{R}$-line bundle with several ample $\mathbb{Q}$-line bundles, which is nef.

The set of sums in the form $\mathcal{L} + \sum \epsilon_i \mathcal{L}_i$ contains a nonempty open subset of $N^1_{\mathbb{R}}(X)$ whose closure contains $\mathcal{L}$. Therefore we can find a sequence of nef $\mathbb{Q}$-line bundles that approaches $\mathcal{L}$ in $N^1_{\mathbb{R}}(X)$.

By Theorem 6.3 and continuity, we see that $(\mathcal{L}^{\dim V} \cdot V) \geq 0$ for every closed subvariety $V \subseteq X$. $\square$

Proposition 6.5 ([Laz04] Corollary 1.4.10)**.** *Let $\mathcal{L}$ be an $\mathbb{R}$-line bundle and $\mathcal{A}$ be an ample $\mathbb{R}$-line bundle. $\mathcal{L}$ is nef iff $\mathcal{L}+\epsilon\mathcal{A}$ is ample for sufficiently small $\epsilon > 0$.*

Proof. The if part is trivial by continuity. For the only if part, we only need to prove that $\mathcal{L}+\mathcal{A}$ is ample when $\mathcal{L}$ is nef and $\mathcal{A}$ is ample.

We first prove the case where $\mathcal{L}+\mathcal{A}$ is a $\mathbb{Q}$-line bundle. By Proposition 5.4, we only need to prove that $((\mathcal{L}+\mathcal{A})^{\dim V}\cdot V) \geq 0$ for every closed subvariety $V \subseteq X$. Write $m = \dim V$. We have

$$((\mathcal{L}+\mathcal{A})^m \cdot V) = \sum_{0\leq k\leq m} \binom{m}{k} (\mathcal{L}^{m-k}\cdot \mathcal{A}^k \cdot V).$$

$\mathcal{A}$ is a positive $\mathbb{R}$-linear combination of ample line bundles, so each $(\mathcal{L}^{m-k}\cdot\mathcal{A}^k\cdot V)$ is a positive $\mathbb{R}$-linear combination of $(\mathcal{L}^{m-k}\cdot V')$ for some effective $(m-k)$-cycle V'. So $(\mathcal{L}^{m-k}\cdot\mathcal{A}^k\cdot V)$ is nonnegative by Proposition 6.4. When $k=m$, we have a summand $(\mathcal{A}^m\cdot V)$ which is positive. Therefore $((\mathcal{L}+\mathcal{A})^{\dim V}\cdot V) > 0$.

Now we consider that case where $\mathcal{L}+\mathcal{A}$ is not necessarily a $\mathbb{Q}$-line bundle. This is similar to the proof of Proposition 6.4. Choose ample $\mathbb{Q}$-line bundles $\mathcal{L}_1,\ldots,\mathcal{L}_r$ such that they span $N^1_{\mathbb{R}}(X)$. For $\epsilon_i > 0$ with $|\epsilon_i|$ sufficiently small, $\mathcal{A}-\sum\epsilon_i\mathcal{L}_i$ is ample by Proposition 5.9.

The set of sums in the form $\mathcal{L}+\mathcal{A}-\sum\epsilon_i\mathcal{L}_i$ (with $|\epsilon_i|$ sufficiently small) contains a nonempty open subset of $N^1_{\mathbb{R}}(X)$. Therefore there exists some choice of ϵ_i such that $\mathcal{L}+\mathcal{A}-\sum\epsilon_i\mathcal{L}_i$ is a $\mathbb{Q}$-line bundle. We have proven that $\mathcal{L}+\mathcal{A}-\sum\epsilon_i\mathcal{L}_i$ is ample. So $\mathcal{L}+\mathcal{A} = (\mathcal{L}+\mathcal{A}-\sum\epsilon_i\mathcal{L}_i)+\sum\epsilon_i\mathcal{L}_i$ is also ample. □

Corollary 6.6 ([Laz04] Corollary 1.4.11)**.** *Let $\mathcal{A}$ be an ample $\mathbb{R}$-line bundle and $\mathcal{L}$ be an arbitrary $\mathbb{R}$-line bundle. Then $\mathcal{L}$ is ample iff there exists $\epsilon > 0$ such that $\frac{(\mathcal{L}\cdot C)}{(\mathcal{A}\cdot C)} \geq \epsilon$ for every curve $C \subseteq X$.*

Proof. The condition is equivalent to that $\mathcal{L}-\epsilon\mathcal{A}$ is nef. The if part is by Proposition 5.9. We prove the only if part. Note that the "sufficently-smallness" of ϵ in Proposition 6.5 only depends on $\mathcal{A}$. So we can choose ϵ small enough such that $\mathcal{L} = (\mathcal{L}-\epsilon\mathcal{A})+\epsilon\mathcal{A}$ is ample. □

7. Kleiman's criterion for ampleness

Denote $\mathrm{Amp}(X)$ to be the ample cone in $N^1_{\mathbb{R}}(X)$ and $\mathrm{Nef}(X)$ to be the nef cone in $N^1_{\mathbb{R}}(X)$.

Theorem 7.1 (Kleiman, [Laz04] Theorem 1.4.23)**.** *$\mathrm{Amp}(X)$ is the interior of $\mathrm{Nef}(X)$ and $\mathrm{Nef}(X)$ is the closure of $\mathrm{Amp}(X)$.*

Proof. $\mathrm{Nef}(X)$ is closed by definition and $\mathrm{Amp}(X)$ is open by Proposition 5.9. Clearly $\mathrm{Amp}(X) \subseteq \mathrm{Nef}(X)$.

By Proposition 6.5, any nef $\mathbb{R}$-line bundle is a limit of ample $\mathbb{R}$-line bundles. Therefore $\mathrm{Nef}(X)$ is the closure of $\mathrm{Amp}(X)$.

Now let $\mathcal{L}$ be an $\mathbb{R}$-line bundle in the interior of $\mathrm{Nef}(X)$. Fix an ample line bundle $\mathcal{A}$. For ϵ sufficiently small, $\mathcal{L}-\epsilon\mathcal{A}$ is nef. Therefore $\mathcal{L} = (\mathcal{L}-\epsilon\mathcal{A})+\epsilon\mathcal{A}$ is ample by Proposition 6.5. □

Definition 7.2. An **$\mathbb{R}$-1-cycle** is a finite $\mathbb{R}$-linear combination of curves in X. Define $Z_1(X)_{\mathbb{R}}$ to be the $\mathbb{R}$-vector space of $\mathbb{R}$-1-cycles on X.

The intersection product can be extended linearly to $\mathbb{R}$-1-cycles. Two $\mathbb{R}$-cycles C_1, C_2 are called **numerically equivalent** if $(\mathcal{L} \cdot C_1) = (\mathcal{L} \cdot C_2)$ for every $\mathcal{L} \in N^1_{\mathbb{R}}(X)$. Define $N_1(X)_{\mathbb{R}}$ to be the $\mathbb{R}$-vector space of numerical equivalence classes of $\mathbb{R}$-1-cycles.

By definition, we have a perfect paring $(- \cdot -) : N^1_{\mathbb{R}}(X) \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$. The set of all effective $\mathbb{R}$-1-cycles is a cone, and is called cone of curves $\mathrm{NE}(X)$. Denote $\overline{\mathrm{NE}}(X)$ to be the closure of $\mathrm{NE}(X)$ in $N_1(X)_{\mathbb{R}}$.

Proposition 7.3 ([Laz04] Proposition 1.4.28)**.** *$\overline{\mathrm{NE}}(X)$ is the set of $\mathbb{R}$-1-cycles C that satisfies $(\mathcal{L} \cdot C)$ for every $\mathcal{L} \in \mathrm{Nef}(X)$.*

Proof. $N_1(X)_{\mathbb{R}} = N^1_{\mathbb{R}}(X)^*$ via the perfect paring. By Proposition 6.4, $\mathrm{Nef}(X)$ is the dual of $\overline{\mathrm{NE}}(X)$. Therefore $\overline{\mathrm{NE}}(X)$ is the dual of $\mathrm{Nef}(X)$. □

Theorem 7.4 (Kleiman's criterion for ampleness, [Laz04] Theorem 1.4.29)**.** *Let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on a projective k-scheme X. The following are equivalent.*

1. *$\mathcal{L}$ is ample.*
2. *$(\mathcal{L} \cdot C) > 0$ for every $C \in \overline{\mathrm{NE}}(X)\backslash 0$.*
3. *Fix a norm $|| \; ||$ on $N_1(X)_{\mathbb{R}}$. There exists $\epsilon > 0$ such that $(\mathcal{L} \cdot C) \geq \epsilon ||C||$ for every $C \in \overline{\mathrm{NE}}(X)$.*
4. *Fix a norm $|| \; ||$ on $N_1(X)_{\mathbb{R}}$. There exists $\epsilon > 0$ such that $(\mathcal{L} \cdot C) \geq \epsilon ||C||$ for every curve $C \subseteq X$.*

Proof. (1) $\Rightarrow$ (2): Assume $\mathcal{L}$ is ample. Then $(\mathcal{L} \cdot C) > 0$ for every curve $C \subseteq X$. So $(\mathcal{L} \cdot C) \geq 0$ for $C \in \overline{\mathrm{NE}}(X)$ by continuity. Assume that $(\mathcal{L} \cdot C_0) = 0$ for some $C_0 \in \overline{\mathrm{NE}}(X)\backslash 0$. Choose an arbitrary line bundle $\mathcal{L}'$ such that $(\mathcal{L}' \cdot C_0) < 0$. Then for arbitrary small $\epsilon > 0$, $((\epsilon\mathcal{L}' + \mathcal{L}) \cdot C_0) < 0$. So $\epsilon\mathcal{L}' + \mathcal{L}$ is not ample, which contradicts with Proposition 5.9.

(2) $\Rightarrow$ (3): Let S be the unit sphere in $N_1(X)_{\mathbb{R}}$ under the norm $|| \; ||$. Then $\overline{\mathrm{NE}}(X) \cap S$ is compact. So there exists $\epsilon > 0$ such that $(\mathcal{L} \cdot C) \geq \epsilon$ for all $C \in \overline{\mathrm{NE}}(X) \cap S$.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Choose ample line bundles $\mathcal{A}_1, \ldots, \mathcal{A}_r$ which form a basis of $N^1_{\mathbb{R}}(X)$. Then we have a norm $|| \; ||_t$ defined as $||C||_t = \sum |(\mathcal{A}_i \cdot C)|$. $N_1(X)_{\mathbb{R}}$ is finite dimensional, so there exists some $c > 0$ such that $|| \; ||_t \leq c|| \; ||$. Take $\mathcal{A} = \sum \mathcal{A}_i$. Then $\mathcal{A}$ is ample and $(\mathcal{A}_i \cdot C) = ||C||_t \leq c||C||$ for every curve $C \subseteq X$. By assumption $\frac{(\mathcal{L} \cdot C)}{(\mathcal{C} \cdot C)} \geq \epsilon c^{-1}$ for every curve $C \subseteq X$. By Theorem 7.1 $\mathcal{L}$ is ample. □

References

[Laz04] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer Science & Business Media, 2004.

[Vak] Ravi Vakil. Foundations of algebraic geometry. December 29, 2015 draft.