Notes on the Decision Model.*
Part I:
Introduction,
Models of Psychoacoustic Paradigms,
and
Models of One Interval Experiments.

Louis D. Braida
and
Nathaniel I. Durlach

Sensory Communication Group
Research Laboratory of Electronics
and
Department of Electrical Engineering and Computer Science
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, MA 02139

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1 Introduction

The purpose of these notes is to introduce a decision model that is used extensively in psychophysics. This model is a special case of the “Law of Categorical Judgment” developed by the psychologist Thurstone, and is closely related to the theory of ideal processing considered in the field of Communications Theory. Some of its important properties are:

- It is probabilistic and takes account of the empirical fact that a listener will not always respond in the same way to the same stimulus.

- It explicitly incorporates the effects of judgmental factors (related to the instructions and information given the listener by the experimenter and to various types of personal bias).

- It can be applied to a wide variety of experimental paradigms and be used to relate a wide variety of experimental results.

- It has had a strong influence on the design of experiments.

- It is frequently used in the literature (so that one often has a difficult time understanding the literature if one does not understand it).

The reader is referred to Appendix A for an introduction to random variables, to Appendix B for an introduction to the statistics of sampling, and to Appendix C for an introduction to Gaussian random variables.
2 General Model

A typical psychoacoustic experiment consists of a sequence of trials, that are identical except for some aspect of the sound presented to the listener. In the simplest experiment, there are only two variants of the sound, denoted by $S_1$ and $S_2$. For example, the listener may be exposed to a continuous background of noise, on each trial a light flashes, and a tone may or may not ($S_1$) or may (not ($S_2$) be presented when the light flashes, or on each trial the listener may be presented with a tone burst of rms pressure $P$ ($S_1$) or a tone burst of rms pressure $P + \Delta P$ ($S_2$).

The task of the listener is to determine whether the sound presented was $S_1$ or $S_2$. If the listener concludes that the sound was $S_1$, the listener is to respond $R_1$, otherwise to respond $R_2$.

2.1 Payoff Structure

\[
\begin{array}{c|cc}
 & R_1 & R_2 \\
\hline
S_1 & V_{11} & V_{12} \\
S_2 & V_{21} & V_{22} \\
\end{array}
\]

Table 1: The payoff matrix. $V_{ij}$ is the payoff for responding $R_j$ when the stimulus is $S_i$.

We assume that the experimenter has imposed a payoff structure on the experiment. If the listener responds $R_j$ when sound $S_i$ is presented, the listener receives a reward equal to $V_{ij}$. This is conveniently summarized in the payoff matrix for the experiment (Table 1).

At the end of the experiment it is determined that there have been $N$ total trials, with sound $S_1$ presented on $N_1$ of the trials and sound $S_2$ presented on $N_2 = N - N_1$ of the trials. On the $N_1$ trials on which the sound $S_1$ is presented, it is determined that the listener has responded $R_1$ on $N_{11}$ trials and $R_2$ on $N_{12} = N_1 - N_{11}$ trials. Similarly on the $N_2$ trials on which the sound $S_2$ is presented, it is determined that the listener has responded $R_2$ on $N_{22}$ trials and $R_1$ on $N_{21} = N_2 - N_{22}$ trials. The total payoff $V$ received by the listener is then

\[
V = \sum_{i=1}^{2} \sum_{j=1}^{2} V_{ij}N_{ij}
\]  

2.2 Presentation Process

We assume that a random (or pseudo-random)$^1$ mechanism determines which variant is presented on a given trial and that the mechanism operates identically and statistically

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$^1$All that is required is that the subject has no knowledge of the stimulus sequence in advance.
independently from trial to trial. The probability that sound $S_1$ is presented on a given trial is

$$P_1 = \Pr(S_1) = 1 - \Pr(S_2) = 1 - P_2.$$ 

\[ \begin{array}{cc}
R_1 & R_2 \\
S_1 & P_{11} & P_{12} \\
S_2 & P_{21} & P_{22}
\end{array} \]

Table 2: The conditional probability matrix. $P_{ij}$ is the probability that the listener responds $R_j$ when the stimulus is $S_i$.

2.3 Decision Process

We model the decision process used by the listener probabilistically. In particular, we assume that when stimulus $S_i$ is presented $P_{ij} = \Pr(R_j|S_i)$ specifies the conditional probability (see Appendix A) that the response is $R_j$ (see Table 2). The four conditional probabilities $\Pr(R_j|S_i)$ are not independent quantities: but must satisfy

$$P_{11} + P_{12} = 1$$
$$P_{21} + P_{22} = 1$$

2.4 Maximizing the Payoff

One constraint on the quantities $\Pr(R_1|S_1)$ and $\Pr(R_1|S_2)$ can be derived under the assumption that the listener attempts to maximize the payoff for participating in the experiment. Since the listener does not know the stimulus sequence in advance, only the expected payoff\(^2\), $E[V]$ can be maximized, where

$$E[V] = \sum_{i=1}^{2} \sum_{j=1}^{2} V_{ij} E[N_{ij}]$$

(2)

Since $E[N_{ij}] = N \Pr(R_j|S_i) \Pr(S_i)$, it is possible to express the expected value of the payoff per trial $E[V]/N$ as

$$E[V]/N = V_{11}P_{11}P_1 + V_{12}P_{12}P_1 + V_{21}P_{21}P_2 + V_{22}P_{22}P_2.$$ 

The above equation is equivalent to

$$E[V]/N = V_{12}P_1 + V_{22}P_2 + (V_{11} - V_{12}) P_{11}P_1 - (V_{22} - V_{21}) P_{21}P_2$$

\(^2\)This corresponds roughly to the total payoff per experiment when the listener participates in a large number of statistically identical experiments.
In a *rational* experiment, both
\[
V_{11} - V_{12} \geq 0 \quad (3)
\]
\[
V_{22} - V_{21} \geq 0 \quad (4)
\]
so that the listener must strive to make the quantity \((V_{11} - V_{12}) \Pr(S_1) \Pr(R_1|S_1)\) as large as possible, while keeping the quantity \((V_{22} - V_{21}) \Pr(S_2) \Pr(R_1|S_2)\) as small as possible.

### 2.5 Sensations

We assume that the presentation of sound \(S_i\) elicits a vector \(X\) of *sensations* \((X_1, X_2, \ldots, X_K)\) in a probabilistic fashion. (e.g., Fig. 1) where \(X_1\) might be the pitch of the sound, \(X_2\) the loudness of the sound, \(X_3\) the subjective duration of the sound . . .

![Figure 1: The observation space.](image)

During training, the observer becomes familiar with the pattern of occurrence of samples of the multidimensional percept conditioned on the presentation of each stimulus (\(S_1\) or \(S_2\)) (Fig. 2). We assume that this is equivalent to the observer’s having learned the conditional probability density function for each stimulus, \(p_X(X_0|S_1)\) and \(p_X(X_0|S_2)\).

When the experimenter presents a sound during the experiment, the listener’s response to the sound is determined by the specific value of \(X\) elicited by the sound. The strategy used by the listener is to divide the space of all possible sensations into two complementary regions, which we associate with the responses. If \(X \in \mathcal{R}_1\) the response is \(R_1\), otherwise the \(X \in \mathcal{R}_2\) and the response is \(R_2\).
These probabilities can readily be expressed in terms of the conditional probability density functions

\[ P_{11} = \int_{R_1} p_{X|S_1} dX_0 \]  
\[ P_{21} = \int_{R_1} p_{X|S_2} dX_0 \]

where the integrals are over the hypervolumes mapped to response \( R_1 \), e.g. Fig. 3.

The expected value of the payoff, \( V \), can now be expressed in terms of the conditional densities \( p_{X|S_i} \)

\[ E[V]/N = V_{12}P_1 + V_{22}P_2 + \int_{R_1} [(V_{11} - V_{12}) P_{1}p_{X|S_1} - (V_{22} - V_{21}) P_{2}p_{X|S_2}] dX_0. \]  

(7)

Now, to maximize the expected value of the payoff, we should assign to \( R_1 \) all those \( X \) for which the integrand is positive:

\[ (V_{11} - V_{12}) P_{1}p_{X|S_1} - (V_{22} - V_{21}) P_{2}p_{X|S_2} \geq 0 \]

This is equivalent to the decision rule: use the response \( R_1 \) if and only if

\[ \frac{p_{X|S_1}}{p_{X|S_2}} \geq \frac{(V_{22} - V_{21}) P_2}{(V_{11} - V_{12}) P_1} \]

(8)

This is a remarkable result. The left hand side of this inequality contains the observations. The right hand side contains the experimenter defined parameters (\( a \ priori \) probabilities and payoffs).
Figure 3: The decision regions in the observation space.

The left hand side of this inequality is called the likelihood ratio, $\lambda (X_0)$

$$\lambda (X_0) = \frac{p_X (X_0|S_1)}{p_X (X_0|S_2)}$$  \hspace{1cm} (9)

The likelihood ratio is itself a random variable. This can be seen by noting that $\lambda (X)$ is a deterministic function on the $X$ space. If $dR$ is that region of $X$ space that maps to $\lambda_0$ (Fig. 4)

$$p_\lambda (\lambda_0|S_1) d\lambda = \int_{dR} p_X (X_0|S_1) dX_0$$

$$p_\lambda (\lambda_0|S_2) d\lambda = \int_{dR} p_X (X_0|S_2) dX_0$$

This description of psychoacoustic experiments has several noteworthy characteristics.

1. Two aspects of the experiment and the decision making process are assumed to be random rather than deterministic: the method for selecting the stimulus to be presented on a given trial and the transformation from the stimulus to the vector of sensations.

2. The listener is assumed to use an optimal rule for determining the response appropriate for a given vector of sensations. The rule maximizes the expected value of the payoff $V$.

3. The decision process involves a comparison of the value of a function, the likelihood ratio, of the vector of sensations on each trial with a fixed criterion. All relevant
aspects of the vector of sensations are expressed in the value of the likelihood ratio, and this is unaffected by the a priori presentation probabilities and payoff structure of the experiment. All experimenter controlled aspects of the experiment (the a priori presentation probabilities and the payoff structure) affect only the value of the criterion used in making the decision.

4. The need for practice and training on the part of the listener is explicit in this formulation: to enable the listener to determine the value of the likelihood ratio for each of the various possible values of the vector of sensations.

5. The formulation assumes that there are no intertrial effects and that the listener’s state is constant throughout the experiment. Some aspects of this idealization deserve further scrutiny. Compare two experiments: in one the probability of presenting a tone signal \( S_1 \) in a continuous noise background is 0.5, in the other 0.01. Is it likely that the listener would be able to maintain knowledge of \( p(X|S_1) \) throughout the course of the experiment as well in the second experiment as in the first? If not, the the separation of the decision process discussed in item 3 above is not a good representation of the experiment.

6. Independent of the complexity of the set of sensations elicited by a given pair of sounds, the decision process can be characterized by a 1-dimensional variable, the likelihood ratio. This variable takes on values with one probability distribution when the stimulus is \( S_1 \) and another distribution when the stimulus is \( S_2 \). The density function \( p(\lambda_0|S_1) \) can be computed \( p(X|S_1) \) over that region of sensation space for which \( \lambda_0 \leq \lambda(X) < \lambda_0 + d\lambda_0 \).

Figure 4: Relevant to the interpretation of the likelihood ratio.
3 One-Interval, 2AFC, Paradigm

Consider an experiment in which

- There are two admissible signal sources \( S_1 \) and \( S_2 \);
- There are two admissible responses \( R_1 \) and \( R_2 \);
- On each trial, the experimenter presents \( S_1 \) or \( S_2 \) randomly with a priori probabilities \( \Pr(S_1) \) and \( \Pr(S_2) = 1 - \Pr(S_1) \);
- The subject is instructed to respond \( R_1 \) when the signal arises from source \( S_1 \) and \( R_2 \) when the signal arises from source \( S_2 \);
- The experimenter “pays off” the subject for responding \( R_j \) when the stimulus is \( S_i \) with payoff \( V_{ij} \), where \( V_{ii} > 0 \) (a reward) for \( i = 1, 2 \) and \( V_{ij} \leq 0 \) (a punishment) for \( i \neq j \).

We will refer to this type of experiment as a one-interval, two-alternative-forced choice (2AFC), experiment. The term “one-interval” refers to the fact that on each trial the subject is presented with \( S_1 \) or \( S_2 \), but not both [as would be the case for example, in a two-interval experiment where the subject is required to distinguish between the two temporally ordered presentations \((S_1, S_2)\) and \((S_2, S_1)\)].

The term “two-alternative-forced-choice” refers to the fact that there are only two allowable responses and that the subject is forced to choose one of them. (If he is uncertain as to whether the presentation is \( S_1 \) or \( S_2 \), he must guess). In many cases, the subject is informed after making his response whether or not his response was correct (or, equivalently, which of \( S_1 \) and \( S_2 \) was actually presented). In these cases, the experimental method is referred to as one-interval, 2AFC, plus feedback.

In a simple discrimination experiment, the sources are \( S_1 \) and \( S_2 \). In an absolute detection experiment, \( S_1 \) will be identified with the no signal, and \( S_2 \) with the signal to be detected. In a noise-masked detection experiment (where the task is to detect a signal \( S \) in a background of noise \( N \)), \( S_1 \) will be identified with the noise \( N \) and \( S_2 \) with the signal-plus-noise \( S + N \).

If we assume that the trials of the experiment are statistically independent\(^3\), and the only response property considered is the identity of the response (i.e., no attention is given to properties such as response latency), then the results of such an experiment can be summarized by a \( 2 \times 2 \) matrix whose entries \( f_{ij} \) are the relative frequencies of responding \( R_i \) to \( S_j \):

\[
 f_{ij} = \frac{N(R_j | S_i)}{N(R_1 | S_i) + N(R_2 | S_i)} = \frac{N_{ij}}{N_{i1} + N_{i2}} \tag{10}
\]

where \( N(R_i | S_j) \) is the number of times the subject responded \( R_i \) to \( S_j \), and \( N(R_1 | S_j) \) + \( N(R_2 | S_j) \) is the number of times \( S_j \) was presented. Furthermore, since \( f_{1j} + f_{2j} = 1 \), the

\(^3\)The assumption of statistical independence is almost certainly not correct during training. The extent to which it is true during data collection remains to be explored.
matrix can be specified by any one of the four pairs

\[
[f(R_2|S_2), f(R_2|S_1)] \\
[f(R_1|S_2), f(R_1|S_1)] \\
[f(R_2|S_2), f(R_1|S_1)] \\
[f(R_1|S_2), f(R_2|S_1)].
\]

For the sake of uniformity, we shall consistently use the pair \([f(R_2|S_2), f(R_2|S_1)]\). Also, as a carry-over from the special case in which the task is one of detection, we shall call \(f(R_2|S_2)\) the relative frequency of detection and \(f(R_2|S_1)\) the relative frequency of false alarm.

The extent to which \(f(R_2|S_2)\) and \(f(R_2|S_1)\) have different values measures the extent to which the subject has demonstrated a sensitivity to the differences between \(S_2\) and \(S_1\). The extent to which both \(f(R_2|S_2)\) and \(f(R_2|S_1)\) are close to unity measures the extent to which the subject has demonstrated a bias to respond \(R_2\) rather than \(R_1\). Note, also, that if \(f(R_2|S_2)\) is significantly less than \(f(R_2|S_1)\), then the subject has demonstrated a sensitivity to the differences between \(S_2\) and \(S_1\), but has employed the wrong response coding.

### 3.1 Decision Model for the One-Interval, 2AFC, Paradigm

The axioms of the decision model for the One-Interval 2AFC paradigm are:

1. There exists a real random variable \(X\) (the “decision axis”) with the property that each signal presentation determines a value \(X\).
2. There exists a fixed cut-off value \(C\) (the “criterion”) on the \(X\) axis.
3. The subject responds \(R_1\) if and only if \(X < C\), and \(R_2\) if and only if \(X \geq C\).
4. The statistics of \(X\) are independent of all aspects of the experiment except \(S_1\) and \(S_2\), and are described completely by the conditional probability density functions \(p_X(X_0|S_1)\) and \(p_X(X_0|S_2)\). In particular, the statistics are independent of the \textit{a priori} probabilities and payoffs, and the trials of the experiment are statistically independent.

The model is illustrated schematically in Fig. 5.

The conditional response \(\Pr(R_1|S_j)\) are given by:

\[
\Pr(R_1|S_1) = \int_{-\infty}^{C} p_X(X_0|S_1) dX_0
\]
Figure 5: The decision model for one-interval two-alternative forced-choice experiments.

\[
\begin{align*}
\Pr(R_2|S_1) & = \int_{-\infty}^{+\infty} p_X(X_0|S_1) dX_0 \\
\Pr(R_1|S_2) & = \int_{-\infty}^{C} p_X(X_0|S_2) dX_0 \\
\Pr(R_2|S_2) & = \int_{C}^{+\infty} p_X(X_0|S_2) dX_0
\end{align*}
\]

and are related to each other by

\[\Pr(R_1|S_j) + \Pr(R_2|S_j) = 1\]

for \(j = 1, 2\).

As with the response frequencies \(f(R_i|S_j)\), we shall specify response probabilities by the pair \([\Pr(R_2|S_2), \Pr(R_2|S_1)]\) and refer to \(\Pr(R_2|S_2)\) as the probability of detection, denoted \(P_D\) and to \(\Pr(R_2|S_1)\) as the probability of false alarm, denoted \(P_F\):

\[
\begin{align*}
P_D & = \Pr(R_2|S_2) \\
P_F & = \Pr(R_2|S_1)
\end{align*}
\]
According to this model, the sensitivity (or resolution) of the system is determined by the extent to which the densities $p_X(X_0|S_1)$ and $p_X(X_0|S_2)$ are nonoverlapping, and the bias of the system by the criterion $C$. In general, the value of $C$ is presumed to depend on the a priori probabilities $\Pr(S_j)$ and the payoffs $V(R_i|S_j)$. For example, if the subject knows that $\Pr(S_1) \gg \Pr(S_2)$ and therefore expects $S_1$ to occur much more frequently than $S_2$ he will probably be biased toward responding $R_1$ rather than $R_2$ and choose a relatively large value of $C$. Similarly, if $v_{22} \gg v_{11}$ i.e., the reward for responding $R_2$ to $S_2$ is much greater than for responding $R_1$ to $S_1$, and $-v_{21} \gg -v_{12}$ (i.e., the punishment for responding $R_1$ to $S_2$, is is much greater than for responding $R_2$ to $S_1$) then the subject will probably be biased toward responding $R_2$ rather than $R_1$ and choose a relatively small value of $C$.

As a carry-over from the detection case (in which $S_2$ contains the signal to be detected), a large value of $C$ is often referred to as a “conservative” criterion and a small value as a “liberal” criterion. A more liberal criterion produces larger values of both $P_D$ and $P_F$.

In addition to the general features of the model mentioned in the introduction (in particular, that it is probabilistic and that it incorporates the effects of judgmental factors on performance), the specific axioms stated in this section lead to three further important features.

1. The influence of the signals on performance (characterized by the properties of the densities $p_X(X_0|S_1)$ and $p_X(X_0|S_2)$) and the influence of the a priori probabilities and payoffs on performance (characterized by the value of $C$) are completely separable. A change in the a priori probabilities or payoffs does not produce a change in the densities.

2. On each trial, the influence of the signal presentation on the subject’s response is completely summarized by the value of a single real number. In particular, the decision space $X$ is assumed to be unidimensional.

3. The decision sets on the $X$ axis are composed of single intervals $X < C$ and $C \leq X$ (which are fixed throughout the experiment). Thus, for example, the axioms rule out the possibility that there exist two criteria $C_1$ and $C_2$ with $R_1$ corresponding to $C_1 \leq X < C_2$ and $R_2$ corresponding to $(X < C_1) \cup (C_2 \leq X)$, or the possibility that the criterion $C$, as well as the decision variable $X$, is a random variable and changes from trial to trial.

Finally, it is important to note that in order to develop the decision model into a complete theory of performance, it is necessary to develop a model of sensitivity [for relating the densities $p_x(X_0|S_1)$ and $p_x(X_0|S_2)$ to the physical properties of the stimuli and the subject’s auditory system], and a model of bias (for relating the criterion $C$ to the a priori probabilities, the payoffs, and the subject’s decision-making apparatus). In these notes, we shall be concerned primarily with sensitivity and will ignore the problem of constructing a model for bias. Also, at this stage, we will ignore the difference between the response probabilities $\Pr(R_i|S_j)$ and the relative response frequencies $f(R_i|S_j)$ [which are assumed to converge to $\Pr(R_i|S_j)$ as the number of trials becomes infinite].
3.2 ROC for the One-Interval, 2AFC, Paradigm

![ROC Curve](image)

Figure 6: ROC for one-interval 2AFC experiments. Note that the ROC passes through the points (0, 0) and (1, 1) by definition.

Consider now the equations relating the response probabilities $P_D$ and $P_F$ to the densities $p_X (X_0 | S_2)$ and $p_X (X_0 | S_1)$ and the criterion $C$:

\[
P_D = \int_{C}^{+\infty} p_X (X_0 | S_2) dX_0 \quad (15)
\]

\[
P_F = \int_{C}^{+\infty} p_X (X_0 | S_1) dX_0 \quad (16)
\]

According to these equations, if we know the densities and the criterion, we can determine $P_D$ and $P_F$ merely by integration. Suppose, however, we estimate $P_D$ experimentally [from $f (R_2 | S_2)$] and $P_F$ experimentally [from $f (R_2 | S_1)$] and want to determine the densities $p_X (X_0 | S_2)$ and $p_X (X_0 | S_1)$. In general, this is an unsolvable problem because the constraints on the densities imposed by assigning values to $P_D$ and $P_F$ are too weak. For example, one could choose any density whatsoever for $p_X (X_0 | S_2)$, then choose $C$ to satisfy the equation for $P_D$, and then choose $p_X (X_0 | S_1)$ to be any density for which the integral from $C$ to $\infty$
equals \( P_F \). Suppose, however, that we perform a series of experiments that are identical except for the values of the a priori probabilities and/or the payoffs, and that these factors are varied in such a way that the subject changes his criterion \( C \) between experiments. We can then obtain estimates of a sequence of pairs

\[
[P_D(C_1), P_F(C_1)], [P_D(C_2), P_F(C_2)], [P_D(C_3), P_F(C_3)],
\]

etc. If we could determine the functions \( P_D(C) \) and \( P_F(C) \), then we could determine the densities merely by differentiating Eqs. 15 and 16.

\[
\frac{dP_D}{dC} = -p_X(C|S_2)
\]
\[
\frac{dP_F}{dC} = -p_X(C|S_1)
\]

Unfortunately, however, although we can vary \( C \) and measure \( P_D \) and \( P_F \) as \( C \) is varied, we cannot determine the functions \( P_D(C) \) and \( P_F(C) \) because we have no way of determining the value of \( C \). On the other hand, since the value of \( C \) is the same for \( P_D \) and \( P_F \) (whatever it is), we can eliminate \( C \) and determine \( P_D \) as a function of \( P_F \). This function, which is obtained empirically for a fixed pair of densities (i.e., fixed \( S_2 \) and \( S_1 \)) by causing the subject to vary the criterion between experiments [and which is referred to as the “isosensitivity curve” or “receiver operating characteristic” (ROC)], summarizes all the knowledge about the densities that can be determined from the response probabilities \( P_D(C) \) and \( P_F(C) \).

The question of what properties of the densities are determined by the ROC is considered in Sec. 9. A schematic illustration of an ROC is shown in Fig. 6.

Since

\[
\frac{dP_D}{dP_F} = \frac{p_X(C|S_2)}{p_X(C|S_1)} \geq 0 \tag{17}
\]

and

\[
P_D(-\infty) = P_F(-\infty) = 1 \tag{18}
\]
\[
P_D(+\infty) = P_F(+\infty) = 0 \tag{19}
\]

the model predicts that all ROC’s are nondecreasing functions which pass through the points \((0,0)\) and \((1,1)\).
4 Other Paradigms

In the previous paragraphs, our discussion of the decision process has been restricted to the one-interval, 2AFC, paradigm outlined in Sec. 3, and the model we have constructed is limited to this paradigm only. With minor modifications, however, the model may be extended to a variety of other paradigms. In fact, as suggested in the introduction, one of the main advantages of the ideas that we have been considering is that they can be used to relate results from widely different paradigms. We will now extend the model to a “confidence-rating” paradigm and to a “two-interval” paradigm.

4.1 Confidence-Rating Paradigm

In the confidence-rating paradigm,
- There are two admissible signal sources $S_1$ and $S_2$;
- There are $M$ admissible responses $R_m$, $m = 1, \ldots, M$, (where $M$ is an even integer);
- On each trial, the experimenter presents $S_1$ or $S_2$ randomly with a priori probabilities $Pr(S_1)$ and $Pr(S_2) = 1 - Pr(S_1)$;

Figure 7: Decision model for confidence-rating experiments.
The subject is instructed to use the responses $R_m$ not only to indicate which of $S_1$ or $S_2$ occurred, but also to express his/her degree of confidence in his decision (i.e., $R_1$ for very confident that $S_1$ occurred, $R_2$ for moderately confident that $S_1$ occurred, ..., $R_{M/2}$ for very slightly confident that $S_1$ occurred, $R_{1+M/2}$ for very slightly confident that $S_2$ occurred, ..., $R_{M-1}$ for moderately confident that $S_2$ occurred, $R_M$ for very confident that $S_2$ occurred).

The subject is instructed to use all available responses, to use them in a consistent manner, and to maximize the number of correct responses (where any of $R_1, \ldots, R_{M/2}$ is correct when $S_1$ is presented, and any of $R_{1+M/2}, \ldots, R_M$ is correct when $S_2$ is presented.)

In terms of the decision model, this paradigm is essentially the same as the one-interval, 2AFC, paradigm, except for the number of criteria. Whereas in the one-interval paradigm, the subject is required to maintain only one criterion at a time (and the ROC is obtained by conducting a series of experiments in which the criterion is varied between the experiments by changing the a priori probabilities and/or payoffs), in this paradigm, the subject is required to maintain many criteria simultaneously (and the ROC is obtained in a single experiment). Also, since the experimenter has no objective means for evaluating the relative correctness of the responses in the classes $R_1, \ldots, R_{M/2}$ and $R_{M/2+1}, \ldots, R_M$, the feedback in this paradigm is restricted to telling the subject whether his response was in the correct class (i.e., which of $S_1$ or $S_2$ was presented).

The axioms of the decision model for the One-Interval Confidence-Rating paradigm are:

1. There exists a real random variable $X$ with the property that each signal presentation determines a value of $X$.

2. There exist $M + 1$ criteria

   \[ C_0 = -\infty < C_1 < \cdots < C_{M-1} < C_M = +\infty \]

   on the decision axis;

3. The subject responds $R_m$ if and only if $C_{m-1} \leq X < C_m$ for $m = 1, \ldots, M$.

4. The statistics of $X$ are independent of all aspects of the experiment except $S_1$ and $S_2$, and are described completely by the conditional probability density functions $p_X (X_0 | S_1)$ and $p_X (X_0 | S_2)$. In particular, the statistics are independent of the a priori probabilities and payoffs, and the trials of the experiment are statistically independent.

Note that the decision model for the Confidence-Rating paradigm, differs from that for the 2AFC paradigm only with respect to axioms (2) and (3). A schematic illustration of the model for the Confidence-Rating paradigm is shown in Fig. 7.
As before, the statistics of $X$ are assumed to depend only on $S_1$ and $S_2$ (so that they should be identical to those for the one-interval, 2AFC, paradigm, provided only that $S_1$ and $S_2$ are the same), and the criteria are assumed to be constant throughout the experiment.

The conditional response probabilities $Pr(R_m | S_j)$ are given by

$$Pr(R_m | S_j) = \frac{C_m}{C_{m-1}} \int_{C_{m-1}} p_X(X_0 | S_j) dX_0.$$  

(20)

These probabilities are often summarized in a table, such as that shown in Tbl. 3: The

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$\cdots$</th>
<th>$R_{M-1}$</th>
<th>$R_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$P_{11}$</td>
<td>$P_{12}$</td>
<td>$\cdots$</td>
<td>$P_{1M-1}$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$P_{21}$</td>
<td>$P_{22}$</td>
<td>$\cdots$</td>
<td>$P_{2M-1}$</td>
</tr>
</tbody>
</table>

Table 3: The Stimulus-Response matrix for the confidence rating experiment.

The cumulative response probabilities $Pr(R \geq R_m | S_j)$ are given by

$$Pr(R \geq R_m | S_j) = \int_{C_{m-1}}^{+\infty} p_X(X_0 | S_j) dX_0 = \sum_{k=m}^{M} Pr(R_k | S_j).$$  

(21)

The ROC is obtained by plotting $Pr(R \geq R_m | S_2)$ as a function of $Pr(R \geq R_m | S_1)$. A single point on the ROC is obtained for each of the values $m = 1, 2, \ldots, M$. If the subject makes use of all the available responses (as instructed), then the experiment will produce $M - 1$ distinct nontrivial points on the ROC. [The case $m = 1$ leads to the trivial point $(1,1)$]. Since the density functions are assumed to be the same in this paradigm as in the one-interval, 2AFC, paradigm, the ROC obtained from this paradigm is predicted to be identical to that obtained from the one-interval, 2AFC, paradigm.

4.2 Two-Interval, 2AFC, Paradigm

In the two-interval paradigm,

- There are two admissible presentations, $U_1 = (S_2, S_1)$ and $U_2 = (S_1, S_2)$, each of which is a temporally-ordered pair.

- There are two admissible responses $R_1$ and $R_2$.

- On each trial, the experimenter presents $U_1$ or $U_2$ randomly with a priori probabilities $Pr(U_1)$ and $Pr(U_2) = 1 - Pr(U_1)$.

- The subject is instructed to respond $R_1$ when $U_1$ is presented and $R_2$ when $U_2$ is presented.
Figure 8: The decision model for two-interval 2AFC experiments.

- The experimenter “pays off” the subject for responding $R_j$ when the stimulus is $U_i$ with payoff $V_{ij}$, where $V_{ii} > 0$ (a reward) for $i = 1, 2$ and $V_{ij} \leq 0$ (a punishment) for $i \neq j$.

The axioms of the decision model for the Two-Interval 2AFC paradigm are:

1. There exists a real random variable $X$ with the property that each signal presentation determines a value of $X$.

2. Each stimulus presentation (of the form $U_1 = (S_2, S_1)$ or $U_2 = (S_1, S_2)$, determines an ordered pair of values $X_1, X_2$ where $X_1$ is the value of $X$ determined from the first member of the pair ($S_2$ in the case of $U_1$, $S_1$ in the case of $U_2$), $X_2$ is the value of $X$ determined from the second member of the pair ($S_1$ in the case of $U_1$ and $S_2$ in the case of $U_2$), and $X_1$ and $X_2$ are statistically independent.

3. The subject observes the pair $(X_1, X_2)$ and forms the decision variable $Y = X_2 - X_1$.

4. There exists a fixed cut-off value $C$ (the criterion) on the $Y$ axis.

5. The subject responds $R_1$ if and only if $Y < C$, and $R_2$ if and only if $Y \geq C$. 


6. The statistics of \( X \) are independent of all aspects of the experiment except \( S_1 \) and \( S_2 \), and are described completely by the conditional probability density functions \( p_X(X_0|S_1) \) and \( p_X(X_0|S_2) \). In particular, the statistics are independent of the a priori probabilities and payoffs, and the trials of the experiment are statistically independent.

Note that the decision model for the Two-Interval 2AFC paradigm, differs from that for the One-Interval 2AFC paradigm only with respect to axioms (2) and (3) of that model. A schematic illustration of the model for the Two-Interval paradigm is shown in Fig. 8. In particular, since the statistics of \( X \) are assumed to depend only on \( S_1 \) and \( S_2 \), they should be identical to those for the one-interval, 2AFC, paradigm and the confidence-rating paradigm, provided only that \( S_1 \) and \( S_2 \) are the same.

If we ignore, for the moment, the way in which \( Y \) is generated from \( X_1 \) and \( X_2 \), this model is identical to the model for the one-interval, 2AFC, paradigm, and all the statements for that paradigm can be made valid for the present paradigm merely by substituting \( Y \) for \( X \), \( U_1 \) for \( S_1 \), and \( U_2 \) for \( S_2 \). Thus, for example, if one makes these substitutions, Equations 11, and 12) provide the expressions for the conditional response probabilities \( \Pr(R_i|U_j) \). Similarly, the ROC is obtained by plotting \( P_D(C) = \Pr(R_2|U_2) \) as a function of \( P_F(C) = \Pr(R_2|U_1) \), and the previous comments on the ROC are all equally valid for the ROC obtained from this paradigm.

If, on the other hand, we make use of the assumptions that \( Y = X_2 - X_1 \), that \( X_1 \) and \( X_2 \) are statistically independent, and that the density functions \( p_X(X_0|S_1) \) and \( p_X(X_0|S_2) \) are the same in this paradigm as in the previous ones, we can derive a number of further results. For example, because \( Y = X_2 - X_1 \), and \( X_1 \) and \( X_2 \) are statistically independent, the densities \( p_Y(Y_0|U_j) \) are given by the convolutions:

\[
p_Y(Y_0|U_1) = \int_{-\infty}^{+\infty} p_X(Y_0 + X_0|S_1) p_X(X_0|S_2) dX_0 \tag{22}
\]

\[
p_Y(Y_0|U_2) = \int_{-\infty}^{+\infty} p_X(Y_0 + X_0|S_2) p_X(X_0|S_1) dX_0 \tag{23}
\]

Second, by substituting \( X_1 = X_0 + Y_0 \) in the above integrals, one can show that

\[
p_Y(Y_0|U_2) = p_Y(-Y_0|U_1) \tag{24}
\]

Thus, the densities \( p_Y(Y_0|U_j) \) are symmetric about \( Y_0 = 0 \) as shown schematically in Fig. 8. This symmetry, in turn, implies that

\[
P_D(C) = 1 - P_F(-C) \tag{25}
\]

\[
P_D(-C) = 1 - P_F(C) \tag{26}
\]

so that the ROC is symmetric about the negative diagonal (as illustrated in Fig. 9).
Figure 9: The ROC for two-interval 2AFC experiments.

Finally, making use of the assumption that $p_X (X_0 | S_j)$ is the same in the various paradigms, one can show that if the a priori probabilities are equal and the subject is "unbiased" in the two-interval paradigm, i.e. that $\Pr(U_1) = \Pr(U_2) = 1/2$ and $C = 0$, then the probability of making a correct response $Q$ in the two-interval paradigm is equal to the area under the ROC for the one-interval paradigm. Specifically,

$$Q = \Pr(U_1) \Pr(R_1 | U_1) + \Pr(U_2) \Pr(R_2 | U_2)$$

(27)

thus

$$Q = \Pr(R_2 | U_2) = \Pr(Y \geq 0 | (S_1, S_2))$$

but

$$\Pr(Y \geq 0 | (S_1, S_2)) = \Pr(X_2 \geq X_1 | X_1 \text{ from } S_1 \text{ and } X_2 \text{ from } S_2)$$

so that

$$\Pr(Y \geq 0 | (S_1, S_2)) = \int_{-\infty}^{+\infty} p_X (X_1 | S_1) dX_1 \int_{X_1}^{+\infty} p_X (X_2 | S_2) dX_2$$

Thus

$$Q = \int_{-\infty}^{+\infty} P_D (X_1) \frac{-dP_F (X_1)}{dX_1} dX_1$$
By changing the variable of integration from $X_1$ to $P_F$ one then has

$$Q = \int_0^1 P_D (P_F) dP_F$$

(28)

Thus the probability of being correct in a two-interval two-alternative forced-choice experiment is equal to the area under the ROC curve for the same stimuli in a one-interval two-alternative forced-choice experiment. This result, which is known as Green’s Theorem\(^4\) is one of the few general theorems, not tied to assumptions about the form of the decision densities, relating performance in different paradigms.

---

\(^4\)After David M. Green, who first proved it.
Problems for Notes on the Decision Model. I.

**Problem 1**

a. In some objective experiments there are no explicit payoffs, but the listener is instructed to achieve the highest number of correct responses. Can such an experiment be modelled by a payoff structure? If so, identify the equivalent payoff structure.

b. Can this model be applied usefully to a subjective experiment, such as a loudness rating experiment?

**Problem 2**

The observation space in a hypothetical experiment is the unit square, with $X_1$ representing the pitch of sounds and $X_2$ representing the loudness of sounds. Suppose

- $X = (X_1, X_2)$,
- for $0 < X_{10} < 1$ and $0 < X_{20} < 1$
  
  \[
  p_{X_1,X_2}(X_{10}, X_{20}|S_1) = 2X_{10} \\
  p_{X_1,X_2}(X_{10}, X_{20}|S_2) = 2X_{20}
  \]

- $V_{11} = 2, V_{12} = 1, V_{21} = 3, V_{22} = 5,$
- $\Pr (S_1) = 0.7 = 1 - \Pr (S_2)$.

Assume that an observer who knows the set of the $V_{ij}$, the two $\Pr (S_k)$, and the two functions $p_{X_1,X_2}(X_{10}, X_{20}|S_k)$ attempts to maximize the expected payoff .

a. Determine the response region that corresponds to response $R_1$.

b. Determine the probability density functions $p_\lambda (\lambda_0|S_1)$ and $p_\lambda (\lambda_0|S_2)$. Hint: the probability density functions can be derived by differentiating expressions for $\Pr (\lambda \leq \lambda_0|S_i)$.

c. Determine the probabilities $P_{ij}$.

d. What is the probability that the listener responds correctly to sound $S_1$? to sound $S_2$?

e. Determine $E [V] / N$.

**Problem 3**

For the densities shown in Fig. 34, with $B = 0.70, D = 0.80$
Figure 34: Densities for Problem 3.

a. Compute $P_D$ and $P_F$ for $C = 0.0, 0.5, 1.0$.

b. Plot the ROC for these densities. Scale and label the axes carefully. Indicate the coordinate values at any breakpoints and/or intercepts.

Problem 4

Let $Q$ be the probability of a correct response.

a. Determine an expression for $Q$ in terms of the a priori stimulus presentation probabilities $\Pr(S_1)$ and $\Pr(S_2)$ and the detection and hit probabilities $P_D$ and $P_F$.

b. Prove that the contours of constant $Q$ in the $(P_D, P_F)$ plane are straight lines with slope $\Pr(S_1) / \Pr(S_2)$ and intercept $[Q - \Pr(S_1)] / \Pr(S_2)$.

c. Consider all the values of $Q$ along a single ROC and prove that if $C = C_0$ is a criterion value for which the derivatives of $P_D(C)$ and $P_F(C)$ exist, and

$$\left[ \frac{dP_F}{dC} \right]_{C=C_0} \neq 0,$$

then

$$\left[ \frac{dQ}{dC} \right]_{C=C_0} = 0,$$

[so that $Q$ is a maximum, minimum, or point of inflection] if and only if the slope of the ROC at $C_0$ is

$$\frac{\Pr(S_1)}{\Pr(S_2)}.$$
Problem 5

For the decision density functions shown in Fig. 37:

a. Compute the conditional mean $M_j = E[X | S_j]$.

b. Compute the conditional variance

$$\sigma_j^2 = E[(X - M_j)^2 | S_j]$$

.  

c. Prove that the ROC generated by $p_X(X_0 | S_1)$ and $p_X(X_0 | S_2)$ depends only on

$$d' = \frac{M_2 - M_1}{\sigma}.$$  

d. Plot ROC’s corresponding to $d' = 0, 1, 2, 4$.  

Figure 35: Densities for Problem 5.
Problem 6

For the decision density functions shown in Fig. 36:

a. Compute the conditional mean \( M_j = E[X | S_j] \).

b. Compute the conditional variance \( \sigma_j^2 = E[(X - M_j)^2 | S_j] \).

c. Prove that the ROC generated by \( p_X (X_0 | S_1) \) and \( p_X (X_0 | S_2) \) depends only on

\[
d' = \frac{M_2 - M_1}{\sigma}.\]

d. Plot ROC’s for \( d' = 0, 1, 2, 4 \).
Problems for Notes on the Decision Model.

II.

Problem 7
Suppose you performed an experiment and obtained an ROC that had a negative slope at some value of $P_F$. How might you revise the model to account for such a result?

Problem 8
In a classic experiment concerning the ability to distinguish between lifted weights, Bressler (1933) had subjects lift two weights in sequence. The first weight, called the standard, always weighed 100 grams. The weight $W_C$, of the second, or comparison, varied from trial to trial. Subjects were instructed to report whether the comparison appeared heavier than, equal to, or lighter than the standard. Bressler’s results (expressed as proportions) are presented in Table 33 below.

<table>
<thead>
<tr>
<th>Judgment</th>
<th>$W_C$ (grams)</th>
<th>88</th>
<th>92</th>
<th>96</th>
<th>100</th>
<th>104</th>
<th>108</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greater</td>
<td></td>
<td>0.04</td>
<td>0.18</td>
<td>0.38</td>
<td>0.62</td>
<td>0.77</td>
<td>0.92</td>
</tr>
<tr>
<td>Equal</td>
<td></td>
<td>0.14</td>
<td>0.27</td>
<td>0.30</td>
<td>0.23</td>
<td>0.13</td>
<td>0.04</td>
</tr>
<tr>
<td>Smaller</td>
<td></td>
<td>0.82</td>
<td>0.55</td>
<td>0.32</td>
<td>0.15</td>
<td>0.10</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 33: Results of a three-category weight-lifting experiment (Bressler, 1933).

a. On a common set of coordinates, plot graphs of the proportion of Greater, Equal, and Smaller judgments as a function of $W_C$.

b. The value of $W_C$ corresponding to equal proportions of Greater and Smaller judgments is usually referred to as the point of subjective equality (PSE). Estimate the value of the PSE for these data.

c. The value of $W_C$ corresponding to 50% judgments Greater (Smaller) defines the upper (lower) discrimination threshold $T_+$ ($T_-$). Estimate the value of $T_+$ and $T_-$ for these data.

d. The difference limen is defined as $\frac{T_+-T_-}{2} = DL$. Estimate the values of the $DL$ for these data.
e. What does this analysis suggest about the ability to discriminate increases in weight as opposed to decreases?

The Decision Model can be applied to Bressler’s data.

f. Describe how to analyze the measurements given above in terms of the Decision Model. Hint: treat the different responses as different category ratings.

g. The Decision Model can be used to estimate whether an observer finds it easier to discriminate increases in weight than decreases. An observer would be equally sensitive to increases as decreases if the ROC curve corresponding to the discrimination \{100, 100 + X\} grams were the same as the curve corresponding \{100 − X, 100\} grams. Construct some ROC curves to determine whether the observer is more sensitive to increases or decreases in weight.

**Problem 9**

Consider the densities \( p_{\lambda}(\lambda_0 | S_j) \) determined in Problem 2.

a. Plot the ROC curve corresponding to these densities assuming that the likelihood ratio is the decision variable.

b. Compute the area under the ROC obtained from these densities;

c. Compute the corresponding two-interval densities \( p_Y(Y_0 | U_j) \).

d. Plot the ROC corresponding to \( p_Y(Y_0 | U_j) \).

e. Compute the probability of a correct response \( Q \) for the two-interval case under the assumption that \( \Pr(U_1) = \Pr(U_2) \) and \( C = 0 \).

f. Compare the results to the area under the ROC for the densities in Fig. 34.

**Problem 10**

Prove the assertion that the ROC curve for a 2I2AFC experiment is symmetric about the negative diagonal.

**Problem 11**

This problem is concerned with the predicted probability of a correct response in a two interval experiment.

Suppose that in a one-interval experiment an observer performs with \( P_F = p \) and \( P_D = q \), where \( q \geq p \). Many ROC curves are consistent with this observation. One, a fairly pessimistic assumption is that the ROC extends linearly from \((0, 0)\) to \((p, q)\) and then extends linearly to the point \((1, 1)\).
a) Determine the predicted probability of a correct response in a two-interval experiment for this assumed shape of the ROC curve.

b) Specialize this result to the case where \( p = 1 - q \), so that the probability of correct response in the one-interval experiment \( Q_1 = q \). How does the probability of correct response in the corresponding two-interval experiment, \( Q_2 \) relate to \( Q_1 \)?

Now attempt to generalize your result. Assume that the one interval ROC consists of three straight line segments: one from \((0, 0)\) to \((P, Q)\), one from \((P, Q)\) to \((P + a, Q + a)\), and one from \((P + a, Q + a)\) to \((1, 1)\).

c) What is \( Q_2 \) in this case as a function of \( P, Q, \) and \( a \)?

d) How does your result compare to the result obtained in part (a) of this problem when \( P = p - a/2 \) and \( Q = q - a/2 \)? How does this result compare to the result obtained in part (b) when \( P + a = 1 - Q \)?

**Problem 12**

This problem is concerned with the relationship between \( Q_2 \) and \( Q_1 \).

Specifically, you are to assume that in the 1I2AFC experiment

\[
P(S_1) = P(S_2)
\]

and the value of \( C \) is chosen to maximize \( Q_1 \).

In the 2I2AFC experiment, you are to assume that

\[
P(U_1) = P(U_2)
\]

and that the observer bases his decision on the random variable \( Y = X_2 - X_1 \), where \( X_i \) is the value of \( X \) determined from the first member of the pair \( (S_2 \text{ in the case of } U_1, S_1 \text{ in the case of } U_2) \), \( X_2 \) is the value of \( X \) determined from the second member of the pair \( (S_1 \text{ in the case of } U_1 \text{ and } S_2 \text{ in the case of } U_2) \), and \( X_1 \) and \( X_2 \) are statistically independent. Further assume that \( C = 0 \).

Determine three pairs of densities \( p_X (X_0|S_1) \) and \( p_X (X_0|S_2) \) such that each pair has one of the following properties:

1. \( Q_2 > Q_1 \).
2. \( Q_2 = Q_1 \).
3. \( Q_2 < Q_1 \).

You should be able to do this subject to the constraint

\[
p_X (X_0|S_1) = p_X (-X_0|S_2).
\]
Problem 13

Consider the pairs of one-interval densities \( p_X (X_0 | S_j) \) given in Fig. 37 for \( d' = 0, 1, 2, \) and 4.

a. Compute the area under the ROC obtained from these densities;

b. Determine the corresponding two-interval densities \( p_Y (Y_0 | U_j) \).

c. Plot the four ROCs corresponding to \( p_Y (Y_0 | U_j) \) for which the one-interval \( d' = 0, 1, 2, \) and 4.

d. Compute the probability of a correct response \( Q \) for the two-interval case under the assumption that \( \Pr (U_1) = \Pr (U_2) \) and \( C = 0 \).

e. Compare the results to the area under the ROC for the densities in Fig. 37.

Problem 14

Consider the densities \( p_X (X_0 | S_j) \) given in Problem 6.

a. Compute the area under the ROC obtained from these densities;

b. Compute the corresponding two-interval densities \( p_Y (Y_0 | U_j) \).

c. Plot the ROC corresponding to \( p_Y (Y_0 | U_j) \).
d. Compute the probability of a correct response $Q$ for the two-interval case under the assumption that $\Pr(U_1) = \Pr(U_2)$ and $C = 0$.

e. Compare the results to the area under the ROC for the densities in Fig. 34.

**Problem 15**

Four experiments (E1-4) that used the same stimuli $S_1$ and $S_2$ yielded the data shown in Tbl. 34. In each matrix the entry in row $j$ column $i$ is the number of times stimulus $S_j$ elicited response $R_i$.

a. Plot six points on an ROC from these data.

b. For all six of the points, plot ± one standard deviation bounds about the data points of part (a).

<table>
<thead>
<tr>
<th></th>
<th>1 Response</th>
<th>2 Response</th>
<th>3 Response</th>
<th>4 Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stimulus</td>
<td>$R_1$</td>
<td>$R_2$</td>
<td>$R_1$</td>
<td>$R_2$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>10</td>
<td>64</td>
<td>61</td>
<td>19</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>79</td>
<td>12</td>
<td>63</td>
</tr>
</tbody>
</table>

Table 34: The results of four hypothetical two-stimulus two-response experiments.

**Problem 16**

Data from a confidence-rating experiment is given in Table 35. The matrix entry in row $i$ column $j$ is the number of times stimulus $S_i$ elicited response $R_j$. $R_1$ corresponds to the response “most confident that $S_1$ was presented”, and $R_6$ to “most confident that $S_2$ was presented.”

a. Determine maximum likelihood estimates of the probability of using each response conditioned on the presentation of each stimulus.

b. Plot the ROC curve that best corresponds to these data.
Table 35: Stimulus - Response matrix from a hypothetical confidence rating experiment.

<table>
<thead>
<tr>
<th>Stimulus</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
<th>$R_5$</th>
<th>$R_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>37</td>
<td>39</td>
<td>59</td>
<td>46</td>
<td>51</td>
<td>13</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>6</td>
<td>10</td>
<td>19</td>
<td>67</td>
<td>106</td>
</tr>
</tbody>
</table>

Now consider the quantities

\[
P_F = \Pr (R_4 \cup R_5 \cup R_6 | S_1) \quad (95)\]
\[
P_D = \Pr (R_4 \cup R_5 \cup R_6 | S_2)\]

Determine the standard deviation of your estimates of $P_F$ and $P_D$.

**Problem 17**

a. Determine the function $\zeta (p)$ for the densities of Problem 5.

b. Determine the function $\zeta (p)$ for the densities of Problem 6.

**Problem 18**

Consider the densities described in Problem 6.

a. Determine the transformation $\zeta (p)$ such that the transformed ROCs $(\zeta_F, \zeta_D)$ are straight lines.

b. Determine the transformed ROCs for the cases $d' = 0, 1, 2, 4$. 
Problems for Notes on
the Decision Model.

III.

In the next six problems assume (unless stated otherwise) that the density functions are Gaussian and of equal variance, as specified by Eq. 92). One sheet of Normal-Deviate Graph Paper is attached; additional sheets can be obtained from the teaching staff.

Problem 19

Contours of

- constant sensitivity \(d'\),
- constant bias, \(\beta\),
- constant probability of correct response \(Q\).

generally look different when linear-linear and normal-normal coordinates are used.

a. Plot contours on normal-normal coordinates for the values

- \(d' = 0, 1, 2, 3\);
- \(\beta = 0, \pm 0.5, \pm 1\);
- \(Q = 0.5, 0.7, 0.9\).

b. Plot contours on linear-linear coordinates for

- \(d' = 0, 2\);
- \(\beta = 0, \pm 1\);
- \(Q = 0.5, 0.7, 0.9\).

In plotting the \(Q\) contours, assume that the a priori probabilities satisfy \(\Pr(S_1) = \Pr(S_2) = 0.5\).

Although there are 19 contours to be plotted, this should not take too much time. Use of normal-deviate paper should make it unnecessary to resort to tables of the normal integral for translating between \((P_D, P_F)\) and \((Z_D, Z_F)\).

Normal-normal and linear-linear graph sheets are attached at the end of this problem set.
Problem 20

In a two-alternative forced-choice experiment, when stimulus $S_1$ is presented, $p_X (X_0|S_1) = g_X (0, 1)$; when stimulus $S_1$ is presented, $p_X (X_0|S_2) = g_X (m, s^2)$, where

$$g_X (\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
$$

a. If $h(x)$ is monotonic non-decreasing function of $x$, then if $x_1 \leq x_2$ $h(x_1) \leq h(x_2)$. Assume that $m > 0$ and $s = 1$. Show that under these conditions the likelihood ratio $\lambda(x)$ is a monotonic non-decreasing function of $x$, where

$$\lambda(x) = \frac{p_X (X_0|S_2)}{p_X (X_0|S_1)}.
$$

b. Does this result apply if $m = 0$ and $s > 1$?

c. Does this result apply if $m > 0$ and $s > 1$?

d. Can you think of another pair of densities for which the likelihood ratio is a monotonic non-decreasing function of $x$?

The case considered in part (a) of this problem has certain special properties:

- The densities are shifts of one another.
- The densities are symmetric about their means.
- The densities are monotonic decreasing about their means.

In the following parts of this problem, you will consider which of these properties are sufficient for the likelihood ratio to be a monotonic function of $x$.

e. Now assume that $p_X (X_0|S_1)$ and $p_X (X_0|S_2)$ are not necessarily Gaussian, but are shifts of a common density $f$:

$$p_X (X_0|S_i) = f (x - a_i).
$$

Is it true that the likelihood ratio $\lambda(x)$ is always a monotonic function of $x$ in this case?

f. Now assume both that $p_X (X_0|S_1)$ and $p_X (X_0|S_2)$ are shifts of a common density $f$:

$$p_X (X_0|S_i) = f (x - a_i)
$$

and are symmetric about their means, i.e

$$f (x - m) = f (m - x)
$$

Does this imply that the likelihood ratio $\lambda(x)$ is
g. Now assume both that \( p_X(X_0|S_1) \) and \( p_X(X_0|S_2) \) are shifts of a common density \( f \):

\[
p_X(X_0|S_i) = f(x - a_i)
\]

and are symmetric about their mean, i.e.

\[
f(x - m) = f(m - x)
\]

and are monotonic decreasing about their means, i.e. if \( x_1 - m \leq x_2 - m > 0 \) then

\[
f(x_2 - m) \leq f(x_1 - m).
\]

Does this imply that the likelihood ratio \( \lambda(x) \) is always a monotonic function of \( x \)?

**Problem 21**

Consider the ROC plotted on linear paper in Problem 19 for \( d' = 2 \). For the case \( \Pr(S_1) = 0.5, 0.3, 0.1 \) find the point on the ROC which maximizes \( Q \), and find the the value of \( Q_{\text{MAX}} \).

Also, measure the slope of the ROC at this point and check the result you established in Problem 4.

**Problem 22**

a. Replot the ROC’s from Problem 3 on normal-deviate paper.

b. Replot the ROC’s from Problem 5 on normal-deviate paper. Consider only the cases \( d' = 0, 1, \text{and} 2 \).

c. Replot the ROC’s from Problem 6 on normal-deviate paper. Consider only the cases \( d' = 0, 1, \text{and} 2 \).

**Problem 23**

In a one-interval two-alternative forced-choice experiment, when stimulus \( S_1 \) is presented, \( p_X(X_0|S_1) = g_X(0, 1) \); when stimulus \( S_1 \) is presented, \( p_X(X_0|S_2) = g_X(m, s^2) \), where

\[
g_X(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

a. A listener is assumed to respond \( R_1 \) when the decision variable \( X \leq C \) and to respond \( R_2 \) when \( X > C \). After a very large number of stimulus presentations, the experimenter estimates that

\[
\Pr(R_2|S_1) = 0.09
\]

\[
\Pr(R_2|S_2) = 0.76
\]

If \( s = 1.0 \), what value of \( m \) is consistent with this data
b. In a separate experiment the listener responds $R_1$ when he/she is confident that stimulus $S_1$ was presented, $R_2$ when confident that $S_2$ was presented, or $U$ when not sure which stimulus was presented. After a very large number of stimulus presentations, the experimenter estimates that

$$
\begin{align*}
\Pr(R_2|S_1) &= 0.09 \\
\Pr(U|S_1) &= 0.32 \\
\Pr(R_2|S_2) &= 0.76 \\
\Pr(U|S_2) &= 0.15
\end{align*}$$

What values of $m$ and $s$ are consistent with this data?

**Problem 24**

According to the decision model, the probability of responding correctly in a one-interval two-alternative forced-choice experiment is bounded by

$$q_1 = \Phi \left( \frac{d'}{2} \right).$$

while the probability of responding correctly in a one-interval two-alternative forced-choice experiment is bounded by

$$q_2 = \Phi \left( \frac{\sqrt{2}d'}{2} \right).$$

Obviously for $d' = 0$, both $q_1$ and $q_2$ are equal to 0.5, while both $q_1$ and $q_2$ approach 1.0 as $d' \to \infty$.

What value of $d'$ maximizes the difference between $q_2$ and $q_1$?

**Problem 25**

In these notes it is assumed that all the variability is in the decision variable, $X$ and none is in the criterion $C$. This problem is concerned with the effects of criterion variability.

1. Sketch an ROC curve based on Gaussian densities on linear-linear coordinates assuming $d' = 2.0$. Assume that an observer adopts a criterion $C$. Indicate on your sketch the values of $P_F$ and $P_D$ corresponing to this value of $C$.

(a) Assume that on a fraction $f_1$ of the trials, the subject decides to use criterion $C_1 > C$ and on a fraction $f_2 = 1 - f_1$ of the trials, the subject uses criterion $C_2 < C$. Sketch the values $(P_{F1}, P_{D1})$ and $(P_{F2}, P_{D2})$ corresponing to these two values of the criterion. Also determine and sketch the location of the average values of the detection and false alarm

$$\begin{align*}
\bar{P}_F &= f_1P_{F1} + f_2P_{F2} \\
\bar{P}_D &= f_1P_{D1} + f_2P_{D2}
\end{align*}$$
(b) Assume that on each trial of an experiment the subject flips a biased coin and decides to use criterion \( C_1 > C \) with probability \( P_1 \) and criterion \( C_2 < C \) with probability \( P_2 = 1 - P_1 \). Unaware of the coin flipping, what we measure in an experiment is

\[
\begin{align*}
P_F &= P_1 P_{F1} + P_2 P_{F2} \\
P_D &= P_1 P_{D1} + P_2 P_{D2}
\end{align*}
\]

Graph the locus of points on which \((P_F, P_D)\) must lie when the subject adopts the coin-flipping strategy. Indicate on your graph the points corresponding to \( P_1 = 0 \) and to \( P_1 = 1 \).

2. Now suppose that in our decision model for the one-interval, 2AFC, paradigm, we had assumed that

- \( C \), as well as \( X \), varies randomly from trial to trial.
- \( C \) is statistically independent of \( X \).
- The density \( p_C(C_0) \) is Gaussian with mean \( M_C \) and variance \( \sigma_C^2 \).
- \( M_C \) is a function of the a priori probabilities and payoffs, but \( \sigma_C \) is independent of these factors.

Under these assumptions, how will the resulting ROC appear on normal-deviate paper? Do you see any way of distinguishing between randomness in \( C \) and randomness in \( X \)?
Problem 26

Consider a “same-different” paradigm in which the two presentations to be distinguished are

- \( U_1 = (S_1, S_1) \) and
- \( U_2 = (S_1, S_2) \).

Making use of the same axioms as those given for the two-interval paradigm in Section 4, including the equal-variance assumption, show that the densities \( p_Y(Y_0|U_j) \) have equal variance and conditional means

\[
E[Y|U_1] = M_1 = 0 \\
E[Y|U_2] = M_2 = M_2 - M_1 \\
V[Y|U_j] = \sigma_{SD}^2 = 2\sigma^2.
\]

Thus, if the densities of \( X_i \) are conditionally Gaussian

\[
d'_{SD} = \frac{M_2 - M_1}{\sigma_{SD}} = \frac{d'}{\sqrt{2}}.
\]

According to the result \( d'_{SD} = d'/\sqrt{2} \), the listener’s performance in this task is generally worse than if the signal in the first interval (which formally contains no information) is ignored and the presentation is treated like a one-interval paradigm. What do you think actually happens? The results depend on features of the experiment that are not considered in the above model. In answering this question, distinguish between two cases:

1. \( S_1 \) and \( S_2 \) are fixed stimuli, e.g. \( S_1 \) is a 60 dB SPL tone and \( S_2 \) is a 61 dB SPL tone.
2. \( S_1 \) and \( S_2 \) are variable stimuli, e.g. \( S_1 \) is an \( X \) dB SPL tone and \( S_2 \) is an \( X + 1 \) dB SPL tone, where \( X \) varies randomly from trial to trial over the range \( 40 \leq X \leq 80 \).
Problem 27

In this problem you are to complete the analysis of the weight discrimination data from Bressler (1933). Assume that for each comparison weight, $W_i$, the conditional density function of the decision variable is

$$p_X(X_0|W_i) = g_X(m_i, 1)$$

where

$$g_X(m_i, \sigma^2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(X-m_i)^2}{2\sigma^2}}$$

Treating the responses as confidence ratings, you are to analyze Bressler’s discrimination data by assuming

$$\Pr(\text{"Greater"}|W_i) = \int_{C_1}^{\infty} g_X(m_i, 1) dX$$

and

$$\Pr(\text{"Greater" or \"Equal\"}|W_i) = \int_{C_2}^{\infty} g_X(m_i, 1) dX$$

For simplicity, assume that $m_4 = E[X|W_4 = 100] = 0$.

a. Show how two estimates of the value of each the other $m_i$ can be determined from Bressler’s data. Derive a single estimate $\hat{m}_i$ by averaging these two estimates for each of the other weights.

b. Plot a graph of $\hat{m}_i$ vs. $W_i$ and fit a straight line to the estimates. Make sure the straight line passes through the point $(100, 0)$ [parenthetically, why?].

Now suppose that the Two-interval 2AFC paradigm (Sec. 4.2) is used for the weight discrimination experiment. The presentations are $U_1 = (S, W)$ and $U_2 = (W, S)$, where $S = 100$ grams. Assume that the observer’s criterion $C = 0$.

c. What value of $W$ will result in the $Q = 0.75$, i.e., the probability of answering correctly is 0.75? How does this result compare with the value of $DL$ determined using the analysis of Bressler’s experiment in Problem 8?