Cardinal Arithmetic: Definitions and Key Results

Most of these results are due to Georg Cantor, who single-handedly developed the arithmetic of infinite numbers.

Definitions. Given two sets A and B, the *Cartesian product* $A \times B$ is $\{<a,b>: a \in A \text{ and } b \in B\}$. A subset f of $A \times B$ is a *function*: $A \to B$ just in case, for each a in A, there is one and only one element b of B with $<a,b> \in f$. If $<a,b> \in f$, we write $f(a) = b$. f is *one-one* or *injective* just in case, whenever $f(a) = f(a^*)$, we have $a = b$. f is *onto* or *surjective* just in case, for each element b of B, there is an element a of A with $f(a) = b$. f is a *one-one correspondence* or *bijective* just in case, it is both one-one and onto; that is, f associates each member of A with a unique element of B, and it associates each member of B with a unique element of A. If f is a one-one correspondence: $A \to B$, the *inverse* of f, f^{-1} is the one-one correspondence: $B \to A$ given by $f^{-1} = \{<b,a>: <a,b> \in f\}$.

With each set A, we associate a *cardinal number* $\#(A)$ in such a way that $\#(A) = \#(B)$ iff (if and only if) there is a one-one correspondence from A to B. We say $\#(A) \leq \#(B)$ iff there is a one-one function: $A \to B$, so that A is in one-one correspondence with a subset of B. We say $\alpha < \beta$ iff $\alpha \leq \beta$ but not $\alpha = \beta$. We define 0 to be $\#(\varnothing)$, 1 to be $\#(\{\varnothing\})$, 2 to be $\#(\{0,1\})$, 3 to be $\#(\{0,1,2\})$, and so on. We have the following results, for any cardinals α, β, and γ:

$0 \leq \alpha$.
$0 = \alpha$ iff $\alpha = \varnothing$.
$\alpha \leq \alpha$.
If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.
If $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$ (Schroeder-Bernstein Theorem).
Either $\alpha \leq \beta$ or $\beta \leq \alpha$ (Zermelo).

The cardinals are well-ordered. That is, they are ordered in such a way that every nonempty collection of cardinals has a least element. Equivalently, there is no infinite descending sequence of cardinal numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$, with $\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3 > \ldots$ (Zermelo).

Definition. Given cardinal numbers α and β, $\alpha + \beta$ = the unique cardinal number γ such that there exist sets A and B with $\#(A) = \alpha$, $\#(B) = \beta$, $A \cap B = \varnothing$, and $\#(A \cup B) = \gamma$.

For any sets A and B, we have $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$.

We have the following results, for any cardinal numbers α, β, and γ:

$\alpha + 0 = \alpha$.
$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
$\alpha + \beta = \beta + \alpha$.

Definition. Given cardinal numbers α and β, $\alpha \bullet \beta$ = the unique cardinal number γ such that there exist sets A and B with $\#(A) = \alpha$, $\#(B) = \beta$, and $\#(A \times B) = \gamma$.

We have the following results, for any cardinal numbers α, β, and γ:

$\alpha \bullet 0 = 0$.
$\alpha \bullet 1 = \alpha$.
$\alpha \bullet 2 = \alpha + \alpha$.
$\alpha \bullet 3 = \alpha + \alpha + \alpha$, and so on.
$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)$.
$\alpha \bullet \beta = \beta \bullet \alpha$.
$\alpha \bullet (\beta + \gamma) = (\alpha \bullet \beta) + (\alpha \bullet \gamma)$.

Definition. Given cardinal numbers α and β, α^β, also written $\alpha E \beta$ = the unique cardinal number γ such that there exist sets A and B with $\#(A) = \alpha$, $\#(B) = \beta$, and $\#(\{\text{functions: } B \to A\}) = \gamma$.

We have the following results, for any cardinal numbers α, β, and γ:

$\alpha^0 = 1$.
$\alpha^1 = \alpha$.
$\alpha^2 = \alpha \bullet \alpha$.
$\alpha^3 = \alpha \bullet \alpha \bullet \alpha$, and so on.
For $\beta \neq 0$, $0^\beta = 0$.
$1^\beta = 1$.
$\beta < 2^\beta$ (Cantor).
$\alpha^{(\beta+\gamma)} = (\alpha^\beta) \bullet (\alpha^\gamma)$.
$\alpha^{(\beta \bullet \gamma)} = (\alpha^\beta)^\gamma$.
$(\alpha \bullet \beta)^\gamma = (\alpha^\gamma) \bullet (\beta^\gamma)$.

For any set A, $2^{\#(A)} = \#(\wp(A))$, where $\wp(A)$, the *power set* of A, is $\{B: B \subseteq A\}$.

Definition. $\aleph_0$ = the least infinite cardinal number = #({natural numbers}. $\aleph_1$ = the least cardinal number > $\aleph_0$. $\aleph_2$ = the least cardinal number > $\aleph_2$, and so on.

$\aleph_0$ = #({natural numbers}) = #({integers}) = #({rational numbers}) = $\aleph_0 + \aleph_0 = \aleph_0 \bullet \aleph_0$.

$2^{\aleph_0}$ = #($\wp$({natural numbers}) = #({real numbers}) = #({complex numbers}) = #({functions from the natural numbers to the natural numbers}) = #({continuous functions from the real numbers to the real numbers}).

$2^{(2^{\aleph_0})}$ = #($\wp$({real numbers}) = #({functions from the real numbers to the real numbers}.

$(2^2)^{\aleph_0} = 4^{\aleph_0} = 2^{\aleph_0}$.

We know that $\aleph_1 \leq 2^{\aleph_0}$. Whether α_1 is equal to $2^{\aleph_0}$ is the *continuum problem*, the leading unsolved problem about the infinite cardinal arithmetic. It is know that the continuum problem

cannot be settled on the basis of currently accepted axioms of set theory.

If α and β are nonzero cardinal numbers and one or both of them are infinite, then $\alpha \bullet \beta = \alpha + \beta =$ the maximum of α and β (Tarski).

We can go on to talk about infinite sums and products of cardinal numbers. The neat thing is that infinite series and products always converge, and the answer doesn't depend on the order in which the terms appear. We won't go into this here.