Subject 24.118. Paradox and Infinity. Homework due Thursday, Oct. 11.

I. Where $\mathbb{R}$ is the set of real numbers, define binary operations + and • on $\mathbb{R}\times\mathbb{R}$, as follows:

$$\langle a,b\rangle + \langle c,d\rangle = \langle a+c, b+d\rangle$$
$$\langle a,b\rangle \bullet \langle c,d\rangle = \langle ac - bd, ad + bc\rangle$$

(a) For what values of a and b do we have $\langle a,b\rangle + \langle a,b\rangle = \langle 1,0\rangle$? Explain your answer.
(b) For what values of a and b do we have $\langle a,b\rangle \bullet \langle a,b\rangle = \langle 1,0\rangle$? Explain your answer.
(c) For what values of a and b do we have $\langle a,b\rangle \bullet \langle a,b\rangle = \langle 0,1\rangle$? Explain your answer.
(d) Show that, for any a and b, not both zero, there exist c and d with $\langle a,b\rangle \bullet \langle c,d\rangle = \langle 1,0\rangle$.

II. Where ω is the set of natural numbers, let *Pair* be the bijection: $\omega\times\omega \to \omega$ described by the following table:

$y \backslash x$	0	1	2	3	4	5	6
0	0	1	3	6	10	15	21
1	2	4	7	11	16	22	29
2	5	8	12	17	23	30	38
3	9	13	18	24	31	39	48
4	14	19	25	32	40	49	59
5	20	26	33	41	50	60	71
6	27	34	42	51	61	72	84

Here *Pair*(x,y) is the number in the column labeled by x and he row labeled by y, so that, for example, *Pair*(2,4) = 25 and *Pair*(4,6) = 61. The function flows along successive diagonals in a southwesterly direction.

(a) What is *Pair*(7,3)?
(b) What is *Pair*(2,8)?
(c) Show that there is a polynomial p of the form $p(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ with $p(x,y) = Pair(x,y)$.

III. Define a bijection *Code* from the set of finite subsets of ω to ω by setting $Code(F) = \sum_{n\in F} 2^n$,

so that, for example, *Code*({0,2,4,5}) = $2^0 + 2^2 + 2^4 + 2^5 = 1 + 4 + 16 + 32 = 53$.

(a) What is *Code*({1,3,6})?
(b) What is *Code*({8,10})?
(c) What set F has *Code*(F) = 49?
(d) What set F has *Code*(F) = 69?

IV. Weiner defined ordered pairs by $\langle a,b\rangle_{\text{Weiner}} = \{\{\{a\},\varnothing\}, \{\{b\}\}\}$. Show that, if $\langle a,b\rangle_{\text{Weiner}} = \langle c,d\rangle_{\text{Weiner}}$, then $a = c$ and $b = d$.

V. Kuratowski defined ordered pairs by $\langle a,b\rangle_{\text{Kuratowski}} = \{\{a,b\},\{a\}\}$. Show that, if $\langle a,b\rangle_{\text{Kuratowski}} = \langle c,d\rangle_{\text{Kuratowski}}$, then $a = c$ and $b = d$.