Subject 24.118. Paradox and Infinity. Homework due Thursday, Oct. 11.

I. Where \( \mathbb{R} \) is the set of real numbers, define binary operations + and \( \cdot \) on \( \mathbb{R} \times \mathbb{R} \), as follows:

\[
\langle a, b \rangle + \langle c, d \rangle = \langle a+c, b+d \rangle
\]
\[
\langle a, b \rangle \cdot \langle c, d \rangle = \langle ac - bd, ad + bc \rangle
\]

(a) For what values of \( a \) and \( b \) do we have \( \langle a, b \rangle + \langle a, b \rangle = \langle 1, 0 \rangle ? \) Explain your answer.
(b) For what values of \( a \) and \( b \) do we have \( \langle a, b \rangle \cdot \langle a, b \rangle = \langle 1, 0 \rangle ? \) Explain your answer.
(c) For what values of \( a \) and \( b \) do we have \( \langle a, b \rangle \cdot \langle a, b \rangle = \langle 0, 1 \rangle ? \) Explain your answer.
(d) Show that, for any \( a \) and \( b \), not both zero, there exist \( c \) and \( d \) with \( \langle a, b \rangle \cdot \langle c, d \rangle = \langle 1, 0 \rangle . \)

II. Where \( \omega \) is the set of natural numbers, let \( \text{Pair} \) be the bijection: \( \omega \times \omega \to \omega \) described by the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
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<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>16</td>
<td>22</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>17</td>
<td>23</td>
<td>30</td>
<td>38</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>13</td>
<td>18</td>
<td>24</td>
<td>31</td>
<td>39</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>19</td>
<td>25</td>
<td>32</td>
<td>40</td>
<td>49</td>
<td>59</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>26</td>
<td>33</td>
<td>41</td>
<td>50</td>
<td>60</td>
<td>71</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>34</td>
<td>42</td>
<td>51</td>
<td>61</td>
<td>72</td>
<td>84</td>
</tr>
</tbody>
</table>

Here \( \text{Pair}(x, y) \) is the number in the column labeled by \( x \) and the row labeled by \( y \), so that, for example, \( \text{Pair}(2, 4) = 25 \) and \( \text{Pair}(4, 6) = 61 \). The function flows along successive diagonals in a southwesterly direction.

(a) What is \( \text{Pair}(7, 3) ? \)
(b) What is \( \text{Pair}(2, 8) ? \)
(c) Show that there is a polynomial \( p \) of the form \( p(x, y) = ax^2 + bxy + cy^2 + dx + ey + f \) with \( p(x, y) = \text{Pair}(x, y) \).

III. Define a bijection \( \text{Code} \) from the set of finite subsets of \( \omega \) to \( \omega \) by setting \( \text{Code}(F) = \sum_{n \in F} 2^n \), so that, for example, \( \text{Code}([0, 2, 4, 5]) = 2^0 + 2^2 + 2^4 + 2^5 = 1 + 4 + 16 + 32 = 53 \).

(a) What is \( \text{Code}(\{1, 3, 6\}) \)?
(b) What is \( \text{Code}(\{8, 10\}) \)?
(c) What set \( F \) has \( \text{Code}(F) = 49 \)?
(d) What set \( F \) has \( \text{Code}(F) = 69 \)?

IV. Weiner defined ordered pairs by \( \langle a, b \rangle_{\text{Weiner}} = \{ \{ \{ a \}, \varnothing \}, \{ \{ b \} \} \} \). Show that, if \( \langle a, b \rangle_{\text{Weiner}} = \langle c, d \rangle_{\text{Weiner}}, \) then \( a = c \) and \( b = d \).

V. Kuratowski defined ordered pairs by \( \langle a, b \rangle_{\text{Kuratowski}} = \{ \{ a \}, \{ a, b \} \} \). Show that, if \( \langle a, b \rangle_{\text{Kuratowski}} = \langle c, d \rangle_{\text{Kuratowski}}, \) then \( a = c \) and \( b = d \).