I. For coding expressions of the language of arithmetic as numbers, we have proposed the following fourfold path:

First, associate a number with each symbol; this means we can think of an expression as a sequence of numbers.

\[ + \quad 0 \quad = \quad 1 \quad = \quad 2 \quad = \quad 3 \quad = \quad 4 \quad = \quad 5 \quad = \quad 6 \quad = \quad 7 \quad = \quad 8 \quad = \quad 9 \quad = \quad 10 \quad = \quad 11 \quad = \quad 12 \quad = \quad 13 \quad = \quad 14 \quad = \quad 15 \quad = \quad 16 \]

Second, write a finite sequence of numbers as a finite set of pairs of numbers, by treating the sequence \( <s_0, s_1, s_2, \ldots, s_n> \) as \{\(<0, s_0>\), \(<1, s_1>\), \(<2, s_2>\), \ldots, \(<n, s_n>\)\}.

Third, code each pair \( <i, j> \) by setting \( \text{Pair}(i, j) = \frac{1}{2}(i^2 + 2ij + j^2 + i + 3j) \). This codes an expression as a finite set of numbers.

Fourth, code a finite set of numbers as a single number by setting \( \text{Code}(F) = \sum_{i \in F} 2^i \).

(a) Write out the Arabic numeral for the code of the term “s0”.
(b) Write an anithmetical expression (it doesn’t have to be an Arabic numeral; you can use exponents) for the code of the sentence “0 = 0”.

II. Peano Arithmetic \( \text{(PA)} \) consists of the following axioms:

\( \text{(PA1)} \quad (\forall x) \neg sx = 0 \)
\( \text{(PA2)} \quad (\forall x)(\forall x')(sx = sx' \rightarrow x = x') \)
\( \text{(PA3)} \quad (\forall x)(x + 0) = x \)
\( \text{(PA4)} \quad (\forall x)(\forall x') (x + sx') = s(x + x') \)
\( \text{(PA5)} \quad (\forall x) (x \times 0) = 0 \)
\( \text{(PA6)} \quad (\forall x)(\forall x') (x \times sx') = ((x \times x') + x) \)

All sentences obtained from the following induction axiom schema by substituting a formula for “Rx,” then prefixing universal quantifiers:

\( \text{(PA7)} \quad ((R0 \land (\forall x)(Rx \rightarrow Rsx))) \rightarrow (\forall x)Rx \)

In class, we gave an informal proof that the commutative law of addition (“\((\forall x)(\forall x') (x + x') = (x' + x)\)”) is a theorem of \( \text{PA} \). We also defined “\( x < x' \equiv_{\text{def}} (\exists x)'' (x + sx'') = x' \)”, and we proved \( (\forall x) \rightarrow x < 0 \)” and \( (\forall x)(\forall x')(x < sx' \rightarrow (x < x' \lor x = x')) \). Please give informal proofs of the following theorems:

(a) The associative law of addition: \( (\forall x)(\forall x')(\forall x'') ((x + x') + x'') = (x + (x' + x'')) \)
(b) The commutative law of multiplication: \( (\forall x)(\forall x') (x \times x') = (x' \times x) \)
(c) “\(<\)” is antireflexive: \( (\forall x) \rightarrow x < x \)