Subject 24.118. Paradox and Infinity. Homework assignment due Thursday, October 25

NOTE: There were a couple of typos in the version that was handed out in class that have been corrected here. In part I, the sentence that begins "*Second,*" originally had "<s_n>" where it should have had "<n, s_n>". The sentence that begins "*Third*" had "*ij*" where it should have had "2*ij*". Sorry about that.

I. For coding expressions of the language of arithmetic as numbers, we have proposed the following fourfold path:

First, associate a number with each symbol; this means we can think of a expression as a sequence of numbers. I don't remember what association I wrote on the blackboard, but this will do:

⌜0⌝ = 1	⌜=⌝ = 5	⌜→⌝ = 9	⌜(⌝ = 13
⌜s⌝ = 2	⌜∨⌝ = 6	⌜↔⌝ = 10	⌜)⌝ = 14
⌜+⌝ = 3	⌜∧⌝ = 7	⌜∀⌝ = 11	⌜*x*⌝ = 15
⌜×⌝ = 4	⌜¬⌝ = 8	⌜∃⌝ = 12	⌜'⌝ = 16

Second, write a finite sequence of numbers as a finite set of pairs of numbers, by treating the sequence <s_0, s_1, s_2,..., s_n> as {<0,s_0>, <1,s_1>, <2,s_2>,..., <n,s_n>}.

Third, code each pair <i,j> by setting $Pair(i,j) = ½(i^2 + 2ij + j^2 + i + 3j)$. This codes an expression as a finite set of numbers.

Fourth, code a finite set of numbers as a single number by setting $Code(F) = \sum_{i \in F} 2^i$.

(a) Write out the Arabic numeral for the code of the term "s0".

(b) Write an anithmetical expression (it doesn't have to be an Arabic numeral; you can use exponents) for the code of the sentence "0 = 0".

II. *Peano Arithmetic* (*PA*) consists of the following axioms:

(PA1) (∀x)¬ sx = 0

(PA2) (∀x)(∀x')(sx = sx' → x = x')

(PA3) (∀x)(x + 0) = x

(PA4) (∀x)(∀x') (x + sx') = s(x + x')

(PA5) (∀x) (x × 0) = 0

(PA6) (∀x)(∀x') (x × sx') = ((x × x') + x)

All sentences obtained from the following *induction axiom schema* by substituting a formula for "Rx," then prefixing universal quantifiers:

(PA7) ((R0 ∧ (∀x)(Rx → Rsx)) → (∀x)Rx

In class, we gave an informal proof that the commutative law of addition ("(∀x)(∀x') (x + x') = (x' + x)") is a theorem of PA. We also defined "x < x' $=_{\text{Def}}$ (∃x'') (x + sx'') = x', and we proved (∀x) ¬ x < 0" and (∀x)(∀x')(x < sx' ↔ (x < x' ∨ x = x')). Please give informal proofs of the following theorems:

(a) The associative law of addition: (∀x)(∀x')(∀x'') ((x + x') + x'') = (x + (x' + x''))

(b) The commutative law of multiplication: (∀x)(∀x') (x × x') = (x' × x)

(c) "<" is antireflexive: (∀x) ¬ x < x