Subject 24.118. Paradox and Infinity. Homework Due Thursday, November 1

Recall some ideas that were introduced in class. A *decision procedure* for a set S of numbers is a mechanical computation procedure that, for a given input n, gives the output 1 if $n \in S$ and 0 if $n \notin S$. S is *decidable* iff there is a decision procedure of S. A *proof procedure* for S is an algorithm that, given the input n, gives the output 1 iff $n \in S$; if $n \notin S$, the procedure gives some other output or no output at all. An *enumeration procedure* for S is an algorithm that lists the members of S. S is *effectively enumerable* iff there is an enumeration procedure for S. We showed that S is effectively enumerable iff there is a proof procedure for S, and that S is decidable iff there are proof procedures for S and for its complement. A *partial function* on the natural numbers is a function whose domain and range are both subsets of the set N of natural numbers. A *total function* is a partial function whose domain is the entire set of natural numbers. A partial funciton f is *calculable* iff there is an algorithm that, given input n, gives the output $f(n)$ if n is in the domain of f, and gives no output at all if n isn't in the domain of f. The same notions can also be applied to relations on the natural numbers, as well as sets of numbers, and to functions of more than one variable. For example, we showed that $\{<m,n>$: given input m, the nth computer program (in some standardized listing of computer programs) gives an output$\}$ is effectively enumerable but not decidable.

1. Show that a partial function f is calculable if and only if $\{Pair(m,n): m \in \text{Domain}(f) \wedge f(m) = n\}$ is effectively enumerable. (We worked with *Pair* last week; $Pair(i,j) = ½(i^2 + 2ij + j^2 + i + 3j)$.)

2. Show that a total function f is calculable if and only if $\{Pair(m,n): f(m) = n\}$ is decidable.

3. Show that there is a set of numbers S such that neither S nor the complement of S is effectively enumerable.

4. Show that a set S is effectively enumerable if and only if either $S = \varnothing$ or S is the range of a calculable total function.

5. Show that a set S is effectively enumerable if and only if either S is finite or S is the range of a one-one calculable total function.

6. Define a total function h as follows: If there is a sequence of n or more consecutive "7"s in the decimal expansion of π, the $h(n)$th digit is the place where the first such sequence begins. If there is no sequence of n or more consecutive "7"s, $h(n) = 0$. Show that n is calculable.

7. Show that it's not the case that every calculable partial function can be extended to a calculable total function.