Subject 24.118. Paradox and Infinity. Homework due Thursday, November 15.

1. True or false? Explain your answers. “Set” means “set of natural numbers”:
   - (a) The union of two effectively enumerable sets is effectively enumerable.
   - (b) The intersection of two effectively enumerable sets is effectively enumerable.
   - (c) The complement of an effectively enumerable set is effectively enumerable.
   - (d) The union of an infinite sequence of effectively enumerable sets is effectively enumerable.
   - (e) For any effectively enumerable. sets A and B, there exist effectively enumerable sets $C \subseteq A$ and $D \subseteq B$ with $C \cup D = A \cup B$ and $C \cap D = \varnothing$.
   - (f) The calculable function p given by $p(n) = 1$ if n is the Gödel number of a sentence provable in PA, $p(n) = 0$ if n is the Gödel number of a sentence refutable in PA, and $p(n)$ is undefined otherwise can be extended to a calculable total function.
   - (g) If R is an effectively enumerable relation such that $(\forall x)(\exists y)R(x,y)$, then there is a calculable total function f such that $R(x,f(x))$, for every x. (See figure.)
   - (h) The set of Gödel numbers of sentences decidable (either provable or refutable) in PA is effectively enumerable but not decidable.
   - (i) $Bew_{\mathrm{PA}}(\ulcorner[2] + [2] = [4]\urcorner)$ is decidable in PA.
   - (j) $Bew_{\mathrm{PA}}(\ulcorner[2] + [2] = [5]\urcorner)$ is decidable in PA.

2. Goldbach’s Conjecture states that every even number greater than two is the sum of two primes. It is not known whether Goldbach’s Conjecture is true, and it is also not known whether it’s decidable in PA. Show, however, that if Goldbach’s Conjecture is undecidable in PA, then it’s true.