6.962 Week 2

Topic: The Multiple Descriptions Problem

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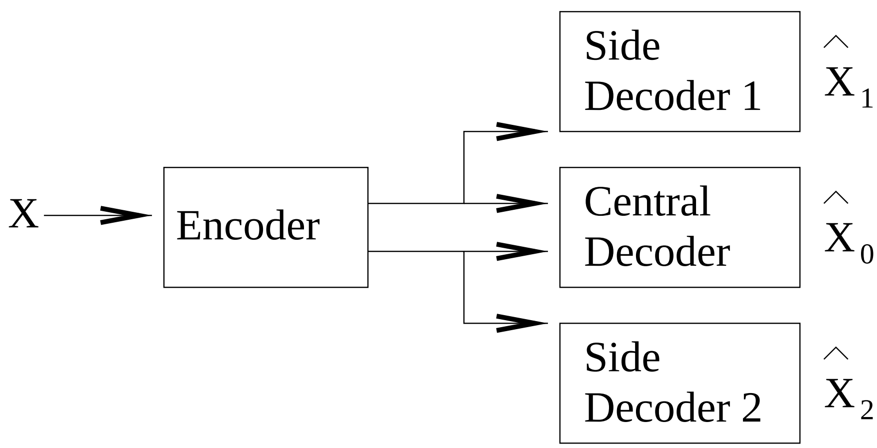
The Multiple Descriptions Problem

The multiple descriptions problem is often depicted as shown below:



A source X is encoded into two descriptions. Each description can be reconstructed into a coarse representation and the combination can be reconstructed into a higher quality representation. The theoretical question is what distortion triples and rate pairs are achievable.

Coding Theorem By El Gamal And Cover

El Gamal and Cover proved that a quintiple $(R_1, R_2, D_0, D_1, D_2)$ is achievable if there exists a pmf

$$p(\hat{x}_1, \hat{x}_2, \hat{x}_0 | x)$$

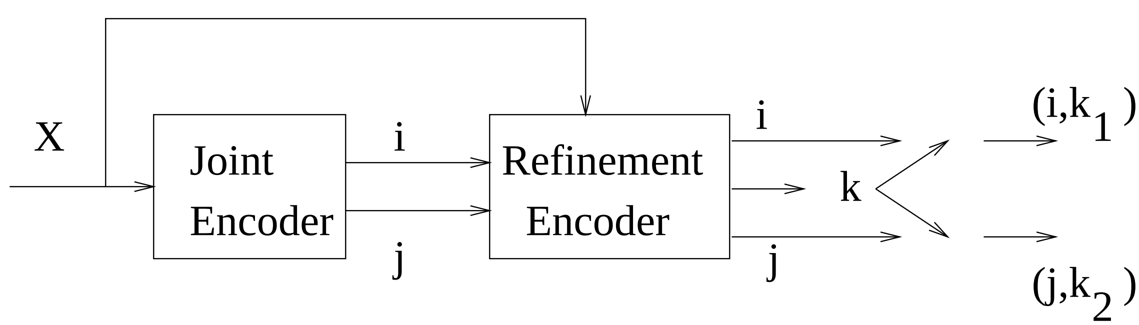
such that

$$\begin{aligned} D_m &> E[d(X, \hat{X}_m)] \quad \text{for } m \in \{0, 1, 2\} \\ R_1 &> I(X; \hat{X}_1) \\ R_2 &> I(X; \hat{X}_2) \\ R_1 + R_2 &> I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \end{aligned}$$

Encoding Procedure

The encoding procedure is to first jointly encode the source into two preliminary descriptions i and j of rates r_1 and r_2. Next a refinement is encoded into k of rate $r_\triangle = r_{\triangle,1} + r_{\triangle,2}$. The refinement is distributed arbitrarily among the two descriptions so the rates of the descriptions are

$$\begin{aligned} R_1 &= r_1 + r_{\triangle,1} \\ R_2 &= r_2 + r_{\triangle,2} \end{aligned}$$



Reconstructions

The reconstructions for the side decoders and the central decoder are

$$\begin{aligned}\hat{X}_1 &= \phi_1(i)\\ \hat{X}_2 &= \phi_2(j)\\ \hat{X}_0 &= \phi_0(i,j,k)\end{aligned}$$

Note that the side decoders do not use the refinement information which they have available.

Idea Of The Proof

1. Generate Random Codebooks

2. Typical Set Encoding

3. Show Probability Of Error Goes To 0

Generating Random Codebooks For The Joint Encoder

We first generate two random codebooks for the joint encoder:

Codebook	Number Of Codewords	pmf
$\mathcal{C}_{J,1}$	2^{nr_1}	$p(\hat{x}_1)$
$\mathcal{C}_{J,2}$	2^{nr_2}	$p(\hat{x}_2)$

Generating A Random Codebook For The Refinement Encoder

Next for each possible pair of values (i, j) from the previous codebooks, we generate a new random codebook $\mathcal{C}_R(i, j)$ with 2^{nr_Δ} codewords according to the distribution

$$p(\widehat{x}_0 | \widehat{x}_1(i), \widehat{x}_2(j))$$

Joint Encoder

First find a pair of codewords i and j which are each jointly typical with the source and also jointly typical with each other:

$$\begin{aligned}(\mathbf{X}, \mathbf{X}_1(i)) &\in T_\epsilon \\ (\mathbf{X}, \mathbf{X}_2(j)) &\in T_\epsilon \\ (\mathbf{X}, \mathbf{X}_1(i), \mathbf{X}_2(j)) &\in T_\epsilon\end{aligned}$$

The joint encoder outputs i and j if such codewords exist and declares an error otherwise.

Refinement Encoder

Next the refinement encoder finds a codeword k which is jointly typical with the source and the codewords from the previous encoder:

$$(\mathbf{X}, \mathbf{X}_1(i), \mathbf{X}_2(j), \mathbf{X}_0(k)) \in T_\epsilon$$

The refinement encoder declares an error if no such codeword exists. Otherwise the refinement encoder breaks the index k into k_1 and k_2 and outputs the two descriptions (i, k_1) and (j, k_2).

Distortion

Note that if the encoders manage to find a triple of codewords (i, j, k) jointly typical with the source, then by the property of typical sequences the distortion will satisfy

$$d_m(\mathbf{x}, \hat{\mathbf{x}}_m) \leq D_m, \quad m \in \{0, 1, 2\}$$

Probability Of Error

An overall error, E, occurs if any of the following occur:

$$
\begin{aligned}
E_0 &: \quad \mathbf{X} \notin T_\epsilon \\
E_1 &: \quad (\mathbf{X}, \hat{\mathbf{X}}_1(i)) \notin T_\epsilon \\
E_2 &: \quad (\mathbf{X}, \hat{\mathbf{X}}_2(j)) \notin T_\epsilon \\
E_3 &: \quad (\mathbf{X}, \hat{\mathbf{X}}_1(i), \hat{\mathbf{X}}_2(j)) \notin T_\epsilon \\
E_4 &: \quad (\mathbf{X}, \hat{\mathbf{X}}_1(i), \hat{\mathbf{X}}_2(j), \hat{\mathbf{X}}_0(i,j,k)) \notin T_\epsilon \text{ given } E_3
\end{aligned}
$$

We will show that $\Pr(E) \to 0$ where

$$E = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$$

Probabilities Of Error For E_0, E_1, E_2, and E_4

Based on the properties of typical sequences and known results from rate-distortion theory

$$
\begin{array}{rcl}
 & & \Pr(E_0) \to 0 \\
r_1 > I(X;\hat{X}_1) & \Rightarrow & \Pr(E_1 \cap E_0^c) \to 0 \\
r_2 > I(X;\hat{X}_2) & \Rightarrow & \Pr(E_2 \cap E_0^c) \to 0 \\
r_\Delta > I(X;\hat{X}_0|\hat{X}_1,\hat{X}_2) & \Rightarrow & \Pr(E_4 \cap E_0^c) \to 0
\end{array}
$$

Probability Of Error For E_3

The number of marginally typical codeword sequences and source sequences is roughly

$$2^{n(H(\hat{X}_1)+H(\hat{X}_2)+H(X))}$$

The number of jointly typical is roughly

$$2^{nH(\hat{X}_1,\hat{X}_2,X)}$$

Therefore the probability of a triple of sequences chosen uniformly from the marginally typical sets being jointly typical is

$$\begin{aligned}
p &= \frac{2^{nH(\hat{X}_1,\hat{X}_2,X)}}{2^{n(H(\hat{X}_1)+H(\hat{X}_2)+H(X))}} \\
p &= 2^{-n(H(\hat{X}_1)+H(\hat{X}_2)+H(X)-H(\hat{X}_1,\hat{X}_2)-H(X|\hat{X}_1,\hat{X}_2))} \\
p &= 2^{n(I(X;\hat{X}_1,\hat{X}_2)+I(\hat{X}_1;\hat{X}_2))}
\end{aligned}$$

Probability Of Error For E_3 Continued

The previous result combined with some technical manipulations implies that as long as their are at least

$$2^{n(I(X;\hat{X}_1,\hat{X}_2)+I(\hat{X}_1;\hat{X}_2))}$$

codewords then $\Pr(E_3) \to 0$.

Therefore

$$R_1+R_2 > I(X;\hat{X}_1,\hat{X}_2)+I(\hat{X}_1;\hat{X}_2) \Rightarrow \Pr(E_3) \to 0$$

Conclusion Of Proof

Since

$$\begin{aligned}\Pr(E) &\le \Pr(E_0)+\Pr(E_1\cap E_0^c)+\Pr(E_2\cap E_0^c)\\ &\quad+\Pr(E_3\cap E_0^c)+\Pr(E_4\cap E_0^c)\end{aligned}$$

$\Pr(E_0)\to 0$ and $\Pr(E_i\cap E_0^c)\to 0$ for $i\in\{0,1,2,3,4\}$ implies that $\Pr(E)\to 0$. This concludes the coding theorem of El Gamal and Cover.

A Converse Theorem

Sher And Feder proved a converse theorem in 1995 which states a rate pair (R_1, R_2) is achievable only if there exists a pmf

$$p(\hat{x}_1, \hat{x}_2, \hat{x}_0 | x)$$

such that

$$\begin{aligned} E[d(X, \hat{X}_m)] &< D_m \quad \text{for } m \in \{0, 1, 2\} \\ R_1 &> I(X; \hat{X}_1) \\ R_2 &> I(X; \hat{X}_2) \\ R_1 + R_2 &> I(X; \hat{X}_0 | \hat{X}_1, \hat{X}_2) + \\ & \quad I(X; \hat{X}_1) + I(X; \hat{X}_2) \end{aligned}$$

Comments On The Converse

Their proof relies on standard information inequalities to bound the excess rate required as follows:

$$
\begin{aligned}
r_{\triangle} &\geq \frac{1}{n} H(\widehat{X}_0^n | S_1, S_2) && (1)\\
&\geq \frac{1}{n} I(X^n; \widehat{X}_0^n | S_1, S_2) && (2)\\
&= \frac{1}{n} \sum_{k=1}^{n} I(X_k; \widehat{X}_0^n | S_1, S_2, X_1, ..., X_{k-1}) && (3)\\
&\geq \frac{1}{n} \sum_{k=1}^{n} I(X_k; \widehat{X}_{0k} | S_1, S_2, X_1, ..., X_{k-1}) && (4)\\
&= \frac{1}{n} \sum_{k=1}^{n} I(X_k; \widehat{X}_{0k} | u_k, v_k) && (5)
\end{aligned}
$$

Minimizing and single letterizing the last line yield the desired lower bound on $r_{\triangle}$.

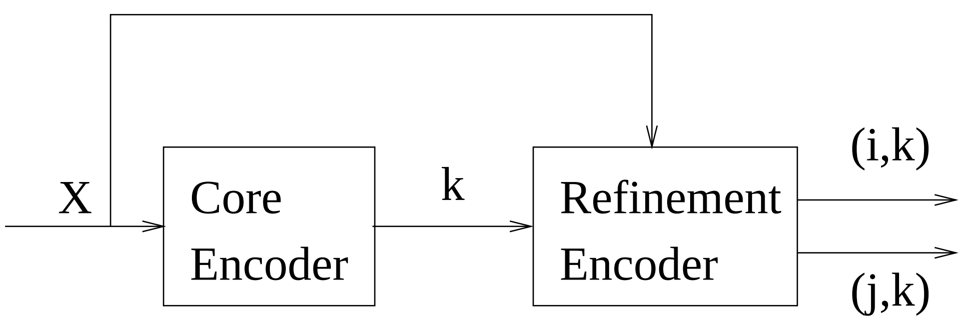
Tightness of EGC Theorem

In the "no excess rate" case where $R_1 + R_2 = R_0(D_0)$, Ahlswede has shown that it is sufficient to consider independent descriptions. In this case, the EGC theorem and the converse of Sher and Feder coincide.

Zhang and Berger have shown that in the "excess rate case" the EGC theorem is not tight by providing a counter example for a binary source with Hamming distortion.

Zhang And Berger's Encoder

Zhang and Berger's first extracts a common core which will be carried by both descriptions and then adds a refinement of the core for each description as shown in the figure below:



Zhang And Berger's Coding Theorem

Zhang and Berger prove a quintiple

$$(R_1, R_2, D_0, D_1, D_2)$$

is achievable if there exists a pmf

$$p(\hat{x}_1, \hat{x}_2, \hat{x}_0 | x)$$

and decoding functions ϕ_1, ϕ_2, and ϕ_0 such that

$$\begin{aligned}
D_m &> E[d(X, \phi_m(\hat{X}_0, \hat{X}_m))] \quad \text{for } m \in \{1, 2\} \\
D_0 &> E[d(X, \phi_0(\hat{X}_0, \hat{X}_1, \hat{X}_2))] \\
R_1 &> I(X; \hat{X}_1, \hat{X}_0) \\
R_2 &> I(X; \hat{X}_2, \hat{X}_0) \\
R_1 + R_2 &> 2I(X; \hat{X}_0) + I(\hat{X}_1; \hat{X}_2 | \hat{X}_0) + \\
&\quad I(X; \hat{X}_1, \hat{X}_2 | \hat{X}_0)
\end{aligned}$$

Idea Of The Proof

1. Generate Random Codebooks

2. Typical Set Encoding

3. Show Probability Of Error Goes To 0

Random Codebooks

First Generate a random codebook, $\mathcal{C}_C$ with 2^{nr_Δ} codewords according to the distribution $p(\hat{x}_0)$.

Next, for each codeword in the first encoder generate two random codebooks as shown below:

Codebook	Number Of Codewords	pmf
$\mathcal{C}_{R,1}$	2^{nr_1}	$p(\hat{x}_1\|\hat{x}_0)$
$\mathcal{C}_{R,2}$	2^{nr_2}	$p(\hat{x}_2\|\hat{x}_0)$

Encoding

To encode, first find a codeword $k \in \mathcal{C}_C$ jointly typical with the source. Next find a pair of codewords $i \in \mathcal{C}_{R,1}$ and $j \in \mathcal{C}_{R,2}$ jointly typical with each other and the source.

The two descriptions are (i, k) and (j, k).

Distortion and Probability Of Error

Note that if the encoders manage to find a triple of codewords (i, j, k) jointly typical with the source, then by the property of typical sequences the distortion will satisfy the required constraints.

Probability Of Error

An overall error, E, occurs if any of the following occur:

$$\begin{array}{ll}
E_* : & \mathbf{X} \notin T_\epsilon \\
E_0 : & (\mathbf{X}, \hat{\mathbf{X}}_0(k)) \notin T_\epsilon \\
E_1 : & (\mathbf{X}, \hat{\mathbf{X}}_1(j)|\hat{\mathbf{X}}_0(k)) \notin T_\epsilon \\
E_2 : & (\mathbf{X}, \hat{\mathbf{X}}_2(j)|\hat{\mathbf{X}}_0(k)) \notin T_\epsilon \\
E_{12} : & (\mathbf{X}, \hat{\mathbf{X}}_1(i), \hat{\mathbf{X}}_2(j)|\hat{\mathbf{X}}_0(k)) \notin T_\epsilon \text{ given } E_0
\end{array}$$

We will show that $\Pr(E) \to 0$ where

$$E = E_* \cup E_0 \cup E_1 \cup E_2 \cup E_{12}$$

Probabilities Of Error For E_*, E_0, E_1, and E_2

Based on the properties of typical sequences and known results from rate-distortion theory

$$
\begin{array}{rcl}
 & & \Pr(E_*) \to 0 \\
r_\Delta > I(X;\hat{X}_0) & \Rightarrow & \Pr(E_0 \cap E_*^c) \to 0 \\
r_1 + r_\Delta > I(X;\hat{X}_1,\hat{X}_0) & \Rightarrow & \Pr(E_1 \cap E_*^c) \to 0 \\
r_2 + r_\Delta > I(X;\hat{X}_2,\hat{X}_0) & \Rightarrow & \Pr(E_2 \cap E_*^c) \to 0
\end{array}
$$

Probability Of Error For E_{12}

The number of marginally typical source sequences and codeword sequences given $\hat{\mathbf{X}}_0(k)$ is roughly

$$2^{n(H(\hat{X}_1|\hat{X}_0)+H(\hat{X}_2|\hat{X}_0)+H(X))}$$

The number of jointly typical sequences is at least roughly

$$2^{n(H(\hat{X}_1,\hat{X}_2|X,\hat{X}_0)+H(X))}$$

Therefore the probability of a triple of sequences chosen uniformly from the marginally typical sets being jointly typical is

$$\begin{aligned}
p &\geq \frac{2^{nH(\hat{X}_1,\hat{X}_2|X,\hat{X}_0)+H(X)}}{2^{n(H(\hat{X}_1|\hat{X}_0)+H(\hat{X}_2|\hat{X}_0)+H(X))}} \\
p &\geq 2^{-n(H(\hat{X}_1|\hat{X}_0)+H(\hat{X}_2|\hat{X}_0)-H(\hat{X}_1,\hat{X}_2|X,\hat{X}_0))} \\
p &\geq 2^{-n(H(\hat{X}_1|\hat{X}_0)+H(\hat{X}_2|\hat{X}_0)-H(\hat{X}_1,\hat{X}_2|X,\hat{X}_0))} \\
&\quad \times 2^{-n(H(\hat{X}_1,\hat{X}_2|\hat{X}_0)+H(\hat{X}_1,\hat{X}_2|\hat{X}_0))} \\
p &\geq 2^{-n(I(X;\hat{X}_1,\hat{X}_2|\hat{X}_0)+I(\hat{X}_1;\hat{X}_2)|\hat{X}_0)}
\end{aligned}$$

Probability Of Error For E_{12} Continued

The previous result combined with some technical manipulations implies that as long as their are at least

$$2^{n(I(X;\hat{X}_1,\hat{X}_2|\hat{X}_0)+I(\hat{X}_1;\hat{X}_2|\hat{X}_0)}$$

codewords in $\mathcal{C}_{R,1}$ and $\mathcal{C}_{R,2}$ then $\Pr(E_3) \to 0$.

Therefore

$$r_1 + r_2 > I(X;\hat{X}_1,\hat{X}_2|\hat{X}_0) + I(\hat{X}_1;\hat{X}_2|\hat{X}_0)$$

implies $\Pr(E_{12}) \to 0$

Conclusion Of Proof

Since

$$\begin{aligned}\Pr(E) &\leq \Pr(E_*) + \Pr(E_0 \cap E_*^c) + \Pr(E_1 \cap E_*^c) \\ &\quad + \Pr(E_2 \cap E_*^c) + \Pr(E_{12} \cap E_*^c)\end{aligned}$$

$\Pr(E_*) \to 0$ and $\Pr(E_i \cap E_*^c)$ for $i \in \{0, 1, 2, 12\}$ implies that $\Pr(E) \to 0$.

The rates for the two descriptions are

$$\begin{aligned}R_1 &= r_1 + r_\Delta \\ R_2 &= r_2 + r_\Delta\end{aligned}$$

So the conditions of the theorem are proved.

Comparison Of Encoding Schemes

Zhang and Berger's scheme is almost the reverse of El Gamal And Cover's encoder. There are situations when the ZB achievable region is strictly larger than the EGC achievable region and vice versa.

This suggests that some combination of the two encoding schemes may be the right approach.

Possible New Encoding Scheme

Consider an encoding scheme where a common encoder first encodes the source into $\hat{X}_0$ at rate r_Δ. Then a joint encoder encodes $\hat{X}_1$ and $\hat{X}_2$ such that they can be decoded separately without $\hat{X}_0$. The encoding of $\hat{X}_0$ is distributed arbitrarily between the two descriptions and used only by the central decoder.

We conjecture that all rate pairs (R_1, R_2) which satisfy

$$\begin{aligned} R_1 &> I(X; \hat{X}_1 | \hat{X}_0) \\ R_2 &> I(X; \hat{X}_2 | \hat{X}_0) \\ R_1 + R_2 &> I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2 | \hat{X}_0) \end{aligned}$$

are achievable.