6.962: Week 2 Summary Of Discussion

Presenter: Emin Martinian
Date: September 20, 2000
Topic: The Multiple Descriptions Problem

1 Summary

The following problem of multiple descriptions was posed by Gersho, Witsenhausen, Wolf, Wyner, Ziv, and Ozarow at the September 1979 IEEE Information Theory Workshop. A random source, X_1^n, is observed by a transmitter that can send information over two separate channels. The goal is to create two descriptions, one for each channel. The encoding is such that either description serves as a coarse representation of the source while the combination serves as a higher quality description. This provides diversity in case one of the channels becomes unusable. Applications include computer networks such as the Internet, database storage, and others.

The multiple descriptions problem is often depicted as shown in Figure 1. The source X is

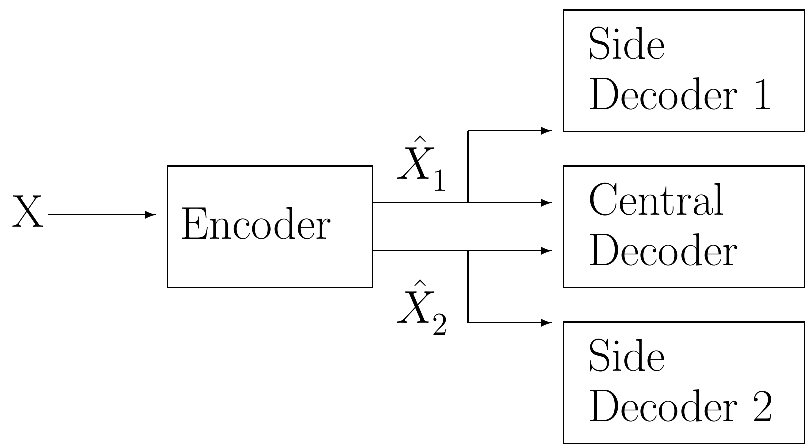

Figure 1: A diagram of the multiple descriptions problem.

encoded into two descriptions such that description i can be decoded by side decoder i for a coarse representation. In addition both descriptions can be decoded together by the central decoder for a high quality reconstruction.

The main theoretical question is which central and side distortion triples $\mathbf{D} = (D_0, D_1, D_2)$ are achievable for a given rate pair (R_1, R_2). Intuitively, if the two descriptions are each a good representation of the source, they will be highly correlated. Consequently, combining the two encodings will not significantly increase the fidelity of the central decoder. A variety of researchers have tackled this problem in [1]-[6] and obtained various results, but the general problem remains unsolved. We will focus on some on the results obtained to date.

1.1 Coding Theorem in [1]

El Gamal and Cover proved that distortions (D_0, D_1, D_2) are achievable if there exists a probability distribution $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ such that

$$\begin{aligned} E[d(X, \hat{X}_m)] &\leq D_m \quad \text{for } m \in \{0, 1, 2\} && (1) \\ R_1 &> I(X; \hat{X}_1) && (2) \\ R_2 &> I(X; \hat{X}_2) && (3) \\ R_1 + R_2 &> I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1, \hat{X}_2) && (4) \end{aligned}$$

This is usually referred to as the EGC theorem.

They use arguments similar to the proof of the classical rate-distortion theorem in [7] (.i.e. random codebooks, typical set encoding, etc.). Their encoding procedure effectively first encodes $\hat{X}_1$ and $\hat{X}_2$ to achieve the required side distortions. Then additional refinement information is created and shared among $\hat{X}_1$ and $\hat{X}_2$. The additional refinement information is effectively only used by the central decoder and ignored by the side decoders. This structure suggests that the EGC theorem might not be tight.

1.2 Alternate Coding Theorem In [4]

Zhang and Berger prove an alternate coding theorem which states that distortions D_0, D_1, D_2 are achievable if there exists a probability distribution $p(x_1, x_2, x_0|x)$ and decoding functions $\phi_1(\cdot)$, $\phi_2(\cdot)$, and $\phi_0(\cdot)$ such that

$$\begin{aligned} R_1 + R_2 &> 2I(X; \hat{X}_0) + I(\hat{X}_1; \hat{X}_2|\hat{X}_0) + I(X; \hat{X}_1, \hat{X}_2|\hat{X}_0) && (5) \\ R_1 &> I(X; \hat{X}_1, \hat{X}_0) && (6) \\ R_2 &> I(X; \hat{X}_2, \hat{X}_0) && (7) \\ E[d(X, \phi_t(\hat{X}_0, \hat{X}_t)] &\leq D_t, \ t \in \{1, 2\} && (8) \\ E[d(X, \phi_0(\hat{X}_0, \hat{X}_1, \hat{X}_2))] &\leq D_0 && (9) \end{aligned}$$

This alternate coding theorem is used to show that the EGC theorem is not tight for a binary source and Hamming distortion. In addition, the alternate coding theorem is also useful for the insight it provides about the multiple descriptions problem.

Zhang and Berger's scheme effectively first encodes the core information of the source in $\hat{X}_0$. Both descriptions carry this common core information. In addition, the descriptions $\hat{X}_1$ and $\hat{X}_2$ are created to supplement $\hat{X}_0$. This is almost the reverse of El Gamal and Cover's encoding scheme where the additional refinement information was generated last and used only by the central decoder. Since both descriptions carry exactly the same core, Zhang and Berger's theorem is not tight when the two descriptions should be independent.

1.3 Converse Theorem In [2]

Sher and Feder prove a converse theorem which states that the (D_0, D_1, D_2) are achievable only if there exists a probability distribution $p(x_1, x_2, x_0|x)$ such that

$$\begin{aligned} R_1 + R_2 &> I(X; \hat{X}_0|\hat{X}_1, \hat{X}_2) + I(\hat{X}; \hat{X}_1) + I(\hat{X}; \hat{X}_2) && (10) \\ R_1 &> I(X; \hat{X}_1) && (11) \\ R_2 &> I(X; \hat{X}_2) && (12) \\ E[d(X, \hat{X}_t)] &\leq D_t,\ t \in \{0, 1, 2\} && (13) \end{aligned}$$

The proof is based on standard techniques using auxiliary variables and information inequalities.

The difference between the converse theorem and the EGC theorem is a term of $I(\hat{X}_1; \hat{X}_2)$ in $R_1 + R_2$. Consequently, the converse and the EGC theorem coincide in the case where the two descriptions are independent. Ahlswede [6] has shown that in the "no excess rate" case it is sufficient to consider independent descriptions. Therefore the achievable region is known for this case.

1.4 The Gaussian Source, Bounds For MSE Distortion, and Practical Constructions

Ozarow solved the multiple descriptions problem for the Gaussian case with mean square error as the distortion metric [9]. He showed that for a Gaussian source with variance σ^2, the achievable region is

$$\begin{aligned} D_1 &\geq \sigma^2 \cdot 2^{-2R_1} && (14) \\ D_2 &\geq \sigma^2 \cdot 2^{-2R_2} && (15) \\ D_0 &\geq \frac{\sigma^2 \cdot 2^{-2(R_1+R_2)}}{1 - (|\sqrt{\pi} - \Delta|^+)^2} && (16) \\ \pi &= (1 - D_1/\sigma^2) && (17) \\ \Delta &= D_1 D_2/\sigma^4 - 2^{-2(R_1+R_2)} && (18) \end{aligned}$$

where

$$|x|^+ = \begin{cases} x, & \text{if } \ x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Zamir [3] used this result to obtain inner and outer bounds for the achievable distortion region for any source with MSE distortion based on the source variance and source entropy power. These results are similar to the well-known inner and outer bounds obtained for the classical rate-distortion function of a source based on variance and entropy power.

Various researchers have described methods to build practical multiple description source coders [10], [11]. Practical coders are usually built and evaluated for Gaussian sources.

References

[1] Thomas M. Cover Abbas A. El Gamal. Achievable rates for multiple descriptions. *IEEE Transactions On Information Theory*, IT-28(6):851–857, November 1982.

[2] Sharon Sher and Meir Feder. A converse theorem for the multiple description problem. In *Eighteenth Annual Convention Of Electronic And Electrical Engineers*, 1995.

[3] Ram Zamir. Gaussian codes and shannon bounds for multiple descriptions. *IEEE Transactions On Information Theory*, IT-45(7):2629–2636, November 1999.

[4] Zhen Zhang and Toby Berger. New results in binary multiple descriptions. *IEEE Transactions On Information Theory*, IT-41(33):502–521, July 1987.

[5] Zhen Zhang and Toby Berger. Multiple description source coding with no excess marginal rate. *IEEE Transactions On Information Theory*, IT-41(2):349–357, March 1995.

[6] Rudolf Ahlswede. The rate-distortion region for multiple descriptions without excess rate. *IEEE Transactions On Information Theory*, IT-31(6):721–726, November 1985.

[7] Thomas M. Cover; Joy A. Thomas. *Elements of Information Theory*. John Wiley and Sons, Inc., 1991.

[8] T. Berger. *Rate Distortion Theory: A Mathematical Basis For Data Compression*. Prentice-Hall, Englewood Cliffs, NJ, 1971.

[9] L. H. Ozarow. On a source coding problem with wo channels and three receivers. *Bell Systems Technical Journal*, 59(10):1909–1921, December 1980.

[10] V. A. Vaishampayan. Design of multiple description scalar quantizers. *IEEE Transactions On Information Theory*, 39(3):821–833, May 1993.

[11] Hamid Jafarkhani and Vahid Tarokh. Multiple description trellis coded quantization. In *International Conference On Image Processing*, 1998.