Stability

$\Downarrow$?

SOS Lyapunov Function

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Lyapunov Analysis

Consider a **polynomial** vector field:

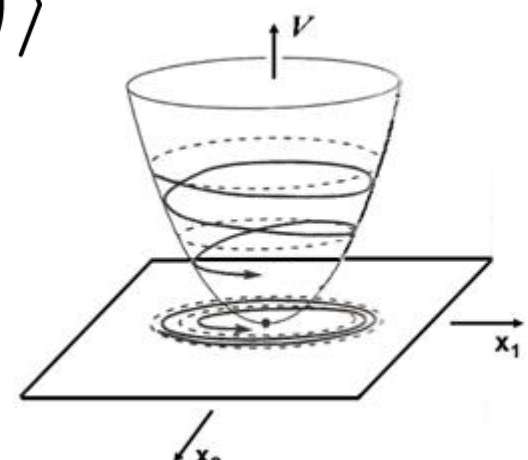
$$\dot{x} = f(x) \quad (f : \mathbb{R}^n \to \mathbb{R}^n)$$

Goal: prove global asymptotic stability (GAS)

Radially unbounded Lyapunov function

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

with derivative $$\dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle$$



[Chaos, Yorke]

$$V(x) > 0$$
$$-\dot{V}(x) > 0$$
$\Rightarrow$GAS

Lyapunov Analysis and Computation

- Classical converse Lyapunov theorem:
  - GAS $\Rightarrow$ C^1 Lyapunov function
  - But how to find one?
- Most common (and quite natural) to search for **polynomial Lyapunov functions**
  - Finitely parameterized for bounded degree
  - Existence *decidable* (for bounded degree) but **computationally intractable**
- A remedy: **SOS Lyapunov functions**
  - Exploits **tractability of semidefinite programming**

Nonnegativity and Sum of Squares (sos)

Defn. A polynomial $p(x) := p(x_1, \ldots, x_n)$ is *nonnegative* or *positive semidefinite* **(psd)** if $p(x) \geq 0 \;\; \forall x \in \mathbb{R}^n$.

Example. Decide if the following ternary quartic form is psd:

$$\begin{aligned} p(x) \;=\; & x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 \\ & -14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \end{aligned}$$

- Not so easy! (In fact, **NP-hard for degree ≥ 4**)
- But what if I told you:

$$\begin{aligned} p(x) \;=\; & (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 \\ & +(4x_2^2 - x_3^2)^2. \end{aligned}$$

Sum of Squares and Semidefinite Programming

Q. Is it any easier to decide sos?

- Yes!
  - Can be reduced to a **semidefinite program (SDP)**
  - Efficient numerical methods (e.g. interior point algorithms)
- Can also efficiently **search and optimize** over sos polynomials
- Numerous applications...
  - Our interest: algorithmic search for Lyapunov functions

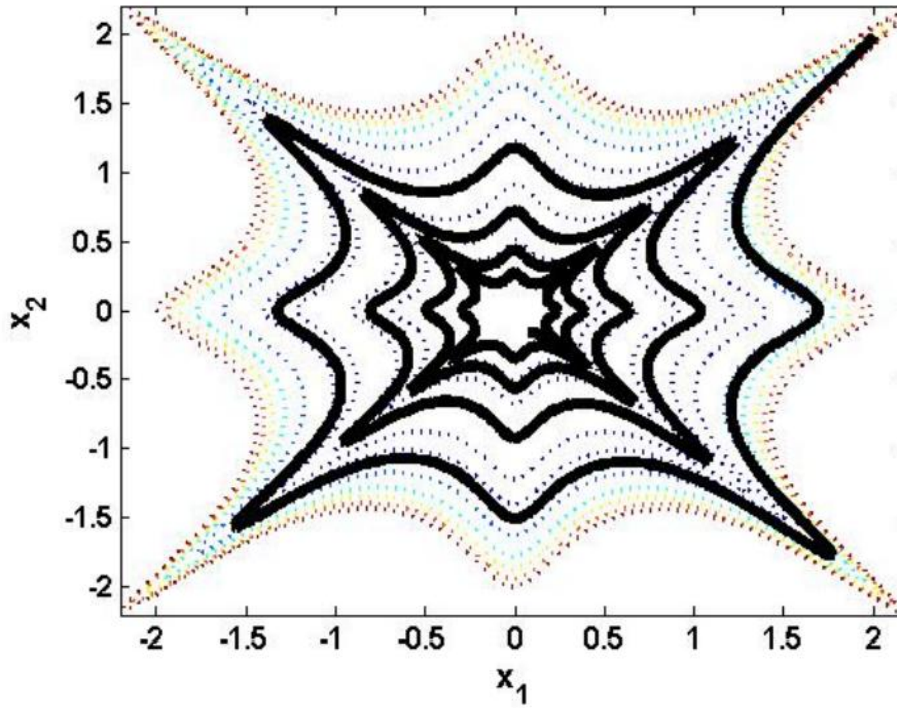
$$\begin{array}{l} V(x) \ \text{sos} \\ -\dot{V}(x) \ \text{sos} \end{array} \Rightarrow \begin{array}{l} V(x) > 0 \\ -\dot{V}(x) > 0 \end{array} \Rightarrow \text{GAS}$$

(various extensions...)

An Example

$$\dot{x}_1 = -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 + 14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7$$
$$\dot{x}_2 = -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 + 807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7$$

- SOS-program fails to find a Lyapunov function of degree 2, 4, 6
- SOS-program finds one of degree 8.



Output of SDP solver:

$$V = 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 - 3.517x_1^4x_2^4$$
$$+0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 - 0.015x_1x_2^7 + 0.02x_2^8.$$

Hilbert's 1888 Paper

psd=sos?

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

From Logicomix

Motzkin (1967):

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6$$

Robinson (1973):

$$\begin{aligned} R(x_1, x_2, x_3, x_4) \quad = \quad & x_1^2(x_1 - x_4)^2 + x_2^2(x_2 - x_4)^2 + x_3^2(x_3 - x_4)^2 \\ & + 2x_1 x_2 x_3 (x_1 + x_2 + x_3 - 2x_4) \end{aligned}$$

Example Revisited

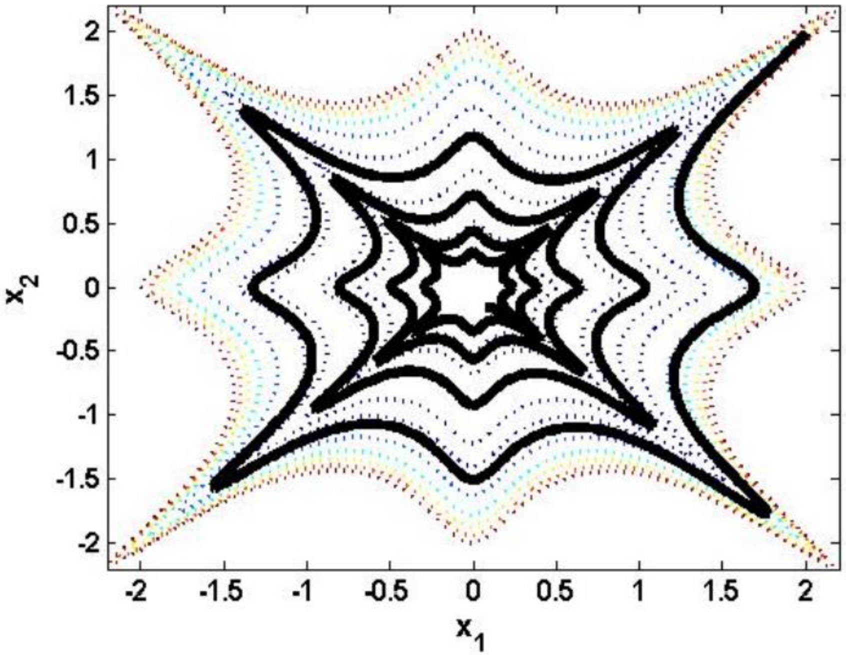
$$\dot{x}_1 = -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 + 14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7$$

$$\dot{x}_2 = -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 + 807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7$$

- Both V and $-\dot{V}$ are homogeneous polynomials of degree two $\Rightarrow$ psd=sos

- We now know no Lyapunov function of degree 2, 4, or 6 exists!

- SOS-program finds one of degree 8.



$$V = 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 - 3.517x_1^4x_2^4$$
$$+0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 - 0.015x_1x_2^7 + 0.02x_2^8.$$

Converse Questions

$$\begin{array}{ll} V(x) & \text{sos} \\ -\dot{V}(x) & \text{sos} \end{array} \Rightarrow \begin{array}{l} V(x) > 0 \\ -\dot{V}(x) > 0 \end{array} \Rightarrow \text{GAS}$$

$\overset{?}{\Leftarrow}$ **Focus of this talk**

$\overset{?}{\Leftarrow}$ ($V(x)$ polynomial)

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Converse SOS Questions

- We know there are psd polynomials that are not sos, but Lyapunov functions are not unique

- Perhaps within the set of all valid Lyapunov functions of a given degree, there is always one that satisfies the sos conditions (?)

- **Q1:** **NO**

polynomial Lyapunov function $\overset{?}{\Rightarrow}$ SOS Lyapunov function of *the same degree*?

- **Q2:** **YES, if homogeneous or planar**

polynomial Lyapunov function $\overset{?}{\Rightarrow}$ SOS Lyapunov function of *higher degree*?

Nonexistence of SOS Lyapunov functions

$$\begin{aligned}\dot{x}_1 &= -x_1^3x_2^2 + 2x_1^3x_2 - x_1^3 + 4x_1^2x_2^2 - 8x_1^2x_2 + 4x_1^2 - x_1x_2^4 + 4x_1x_2^3 - 4x_1 + 10x_2^2 \\ \dot{x}_2 &= -9x_1^2x_2 + 10x_1^2 + 2x_1x_2^3 - 8x_1x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2\end{aligned}$$

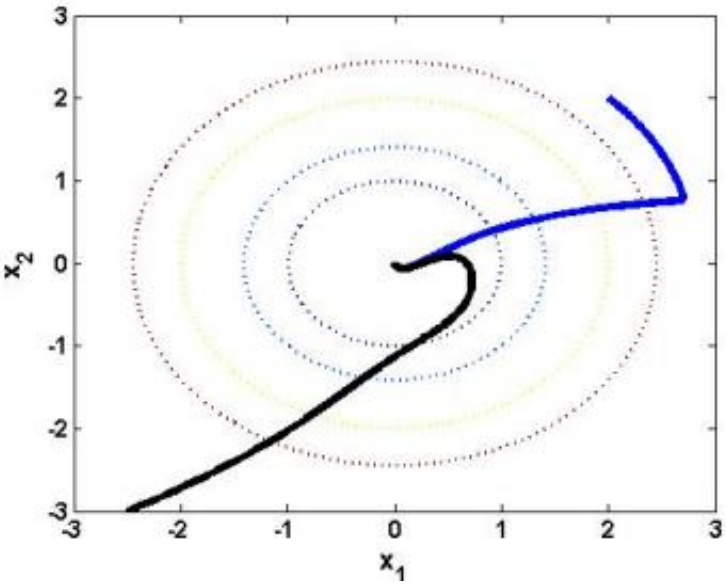
Claim 1:

No quadratic Lyapunov function

$$V(x) = c_1x_1^2 + c_2x_1x_2 + c_3x_2^2$$

can satisfy $V(x)$ sos

$-\dot{V}(x)$ sos



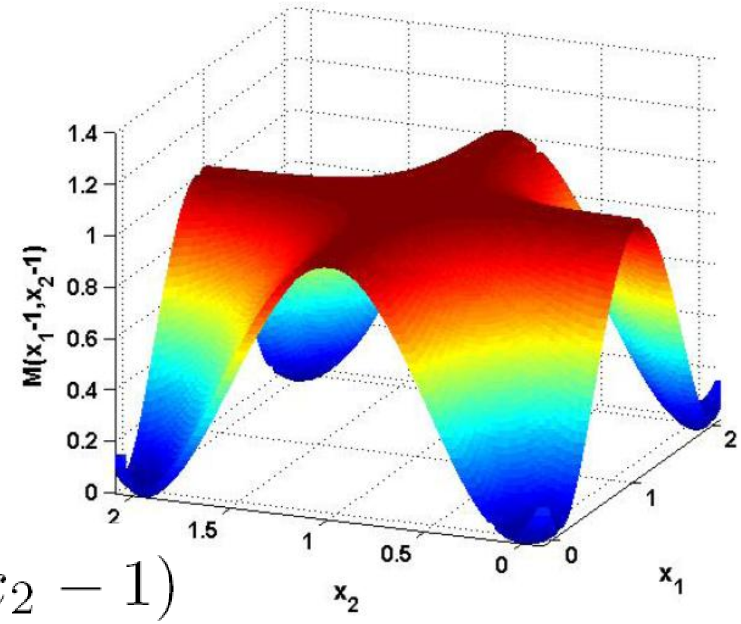
Proof: a certificate of infeasibility from dual SDP.

Counterexample (ctnd.)

$$\dot{x}_1 = -x_1^3x_2^2 + 2x_1^3x_2 - x_1^3 + 4x_1^2x_2^2 - 8x_1^2x_2 + 4x_1^2 - x_1x_2^4 + 4x_1x_2^3 - 4x_1 + 10x_2^2$$
$$\dot{x}_2 = -9x_1^2x_2 + 10x_1^2 + 2x_1x_2^3 - 8x_1x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2$$

Claim 2:

$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ proves GAS.



$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -M(x_1 - 1, x_2 - 1)$$

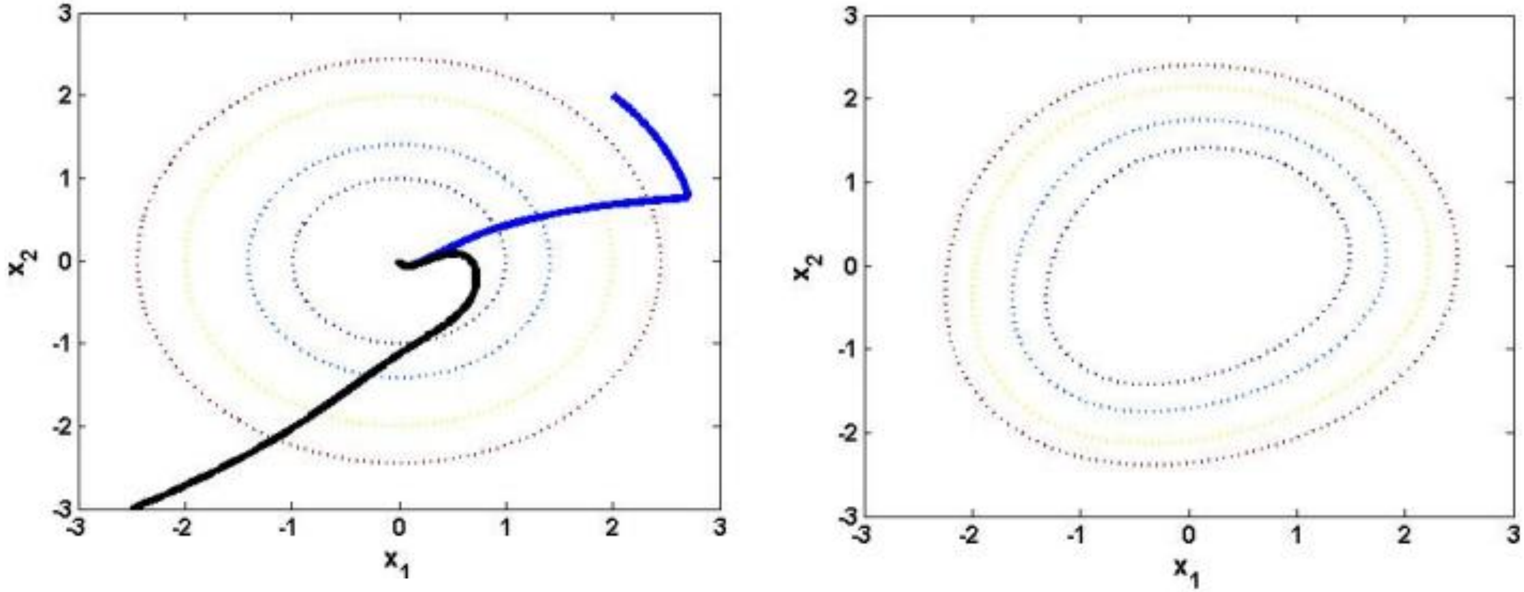
$$M(x_1, x_2) = x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2 + 1$$

- M is psd but not sos
- Use LaSalle's invariance principle

Counterexample (ctnd.)

$$\dot{x}_1 = -x_1^3 x_2^2 + 2x_1^3 x_2 - x_1^3 + 4x_1^2 x_2^2 - 8x_1^2 x_2 + 4x_1^2 - x_1 x_2^4 + 4x_1 x_2^3 - 4x_1 + 10x_2^2$$

$$\dot{x}_2 = -9x_1^2 x_2 + 10x_1^2 + 2x_1 x_2^3 - 8x_1 x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2$$



SOS-program succeeds in finding a quartic Lyapunov function:

$$W = 0.08x_1^4 - 0.04x_1^3 + 0.13x_1^2 x_2^2 + 0.03x_1^2 x_2 + 0.13x_1^2 + 0.04x_1 x_2^2 - 0.15x_1 x_2 + 0.07x_2^4 - 0.01x_2^3 + 0.12x_2^2$$

Is this always the case?

Converse SOS Lyapunov Theorem: Previous Work

[Peet, Papachristodoulou,'10]
A Converse Sum-of-Squares Lyapunov Result: An Existence Proof Based on Picard Iteration

If there is a polynomial Lyapunov function $V(x)$ such that

$$V(x) > 0 \text{ and } -\dot{V}(x) > 0$$

then there is a polynomial Lyapunov function $W(x)$ such that

$$W(x) \text{ sos and } -\dot{W}(x) > 0$$

- No guarantee that the sos-program will find $W(x)$ (recall our counterexample)
- To get this result, it's enough to take $W(x) = V^2(x)$

Converse SOS Lyapunov Theorems

Thm: Given a homogeneous polynomial vector field, if there is a polynomial Lyapunov function $V(x)$ such that

$$V(x) > 0 \text{ and } -\dot{V}(x) > 0$$

then there is a polynomial Lyapunov function $W(x)$ such that

$$W(x) \text{ sos and } -\dot{W}(x) \text{ sos}$$

Thm: Given a polynomial vector field in 2 variables, if there is a polynomial Lyapunov function $V(x)$ such that

$$V(x) > 0 \text{ and } -\dot{V}(x) > 0$$

and the h.o.t. of $V(x)$ is positive definite,
then there is a polynomial Lyapunov function $W(x)$ such that

$$W(x) \text{ sos and } -\dot{W}(x) \text{ sos}$$

A Word on the Proof

- Proofs are quite short and easy
- They use two powerful Positivstellensatz results of Scheiderer:

Thm: Given any two positive definite forms p and q, there exists k such that pq^k is sos.

Thm: Given any two forms p and q in 3 variables, with q positive definite and p positive semidefinite, there exists k such that pq^k is sos.

- Removing the homogeneity and planarity assumptions from our converse sos Lyapunov theorems is open.

Generalization to Switched Systems

Thm: Consider a switched system:

$$\dot{x} = f_i(x), \quad i \in \{1, \ldots, m\}$$

$f_i(x)$ homogeneous polynomial of degree d_i

if there is a common polynomial Lyapunov function $V(x)$ such that

$$V(x) > 0$$

$$-\dot{V}_i(x) = -\langle \nabla V(x), f_i(x) \rangle > 0$$

then there is a common polynomial Lyapunov function $W(x)$ such that

$$W(x) \text{ sos}$$

$$-\dot{W}_i = -\langle \nabla W(x), f_i(x) \rangle \text{ sos}$$

SOS is universal for stability of switched linear systems

Cor: A switched linear system

$$\dot{x} = A_i x, \quad i \in \{1, \ldots, m\}$$

is (globally) asymptotically stable under arbitrary switching

if and only if there exists a common polynomial Lyapunov function $W(x)$ such that

$$W \quad \text{sos}$$

$$-\dot{W}_i = -\langle \nabla W(x), A_i x \rangle \quad \text{sos}$$

- Combines previous theorem with a result of [Mason, Boscain] on existence of polynomial Lyapunov functions

SOS Lyapunov result for switched linear systems

Thm: Consider the arbitrary switched linear system

$$x_{k+1} = A_i x_k$$
$$A_i \in \{A_1, \ldots, A_m\}$$

If a common polynomial Lyapunov function $V(x)$ satisfies

$$V(x) > 0 \text{ and } V(x) - V(A_i x) \text{ sos}$$

then $V(x)$ is sos.

- Same result holds in continuous time.

One Last Observation

- Lyapunov's theorem:

$$\begin{aligned} V(x) &> 0 \\ -\dot{V}(x) &> 0 \end{aligned} \quad \Rightarrow \text{GAS}$$

- In fact, the following is also true:

$$\begin{aligned} h.o.t.\ V(x) &> 0 \\ -\dot{V}(x) &> 0 \end{aligned} \quad \Rightarrow \text{GAS}$$

(can even decouple the two inequalities)

Implications:

- Cheaper inequality to impose: $\binom{n+d-1}{d}$ coeffs instead of $\binom{n+d}{d}$

  (e.g. when n=6, d=4, 126 coeffs instead of 210)

- For planar systems $V(x)$ sos is never conservative.

Messages to take home...

$$\begin{array}{c} V(x) \ \text{sos} \\ -\dot{V}(x) \ \text{sos} \end{array} \Rightarrow \begin{array}{c} V(x) > 0 \\ -\dot{V}(x) > 0 \end{array} \Rightarrow \text{GAS}$$

$\Leftarrow$?

No, if the same degree
Yes, if higher degree
& homogeneous or planar

$\Leftarrow$?

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- Switched linear systems always admit sos Lyapunov functions
- $V(x)$ sos is never conservative for switched linear systems

Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu/