Stability

$$\Downarrow ?$$

Polynomial Lyapunov Function

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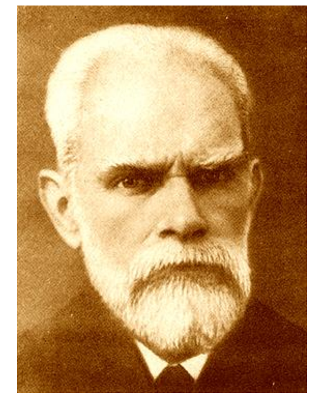
Lyapunov analysis

Consider a **polynomial** vector field:

$$\dot{x} = f(x) \quad (f : \mathbb{R}^n \to \mathbb{R}^n)$$

Goal: prove global asymptotic stability (GAS)

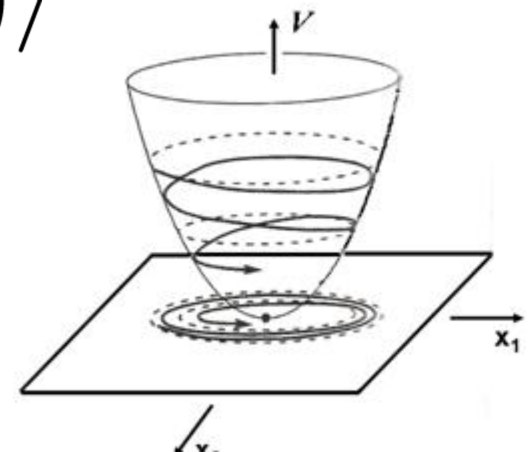


Radially unbounded Lyapunov function

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

with derivative $$\dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle$$

$$V(x) > 0$$
$$-\dot{V}(x) > 0$$
$\Rightarrow$GAS



[Chaos, Yorke]

Lyapunov analysis and computation

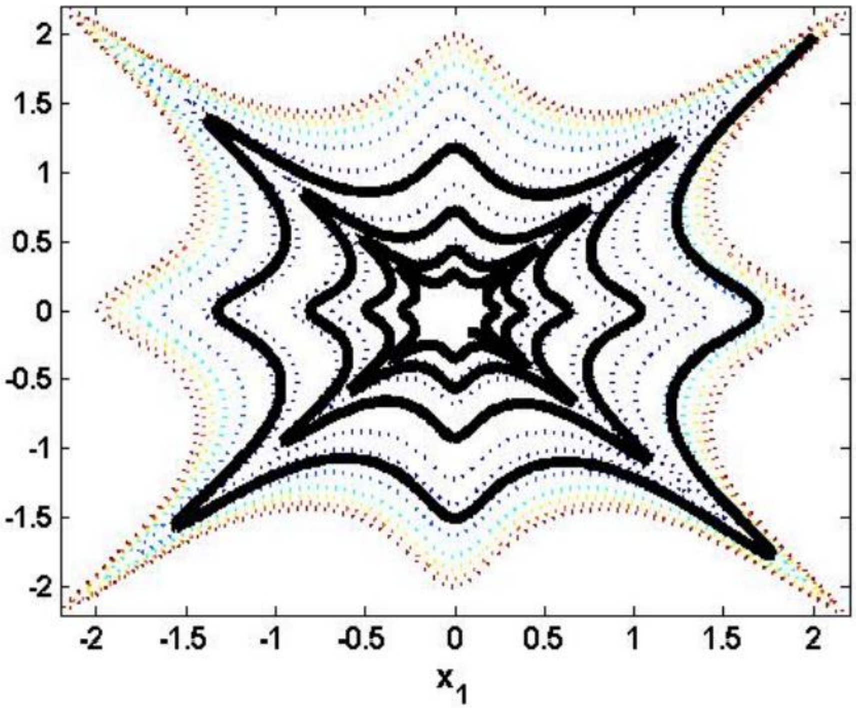
- Classical converse Lyapunov theorem:
  - GAS $\Rightarrow$ C^1 Lyapunov function
  - But how to find one?
- Most common (and quite natural) to search for **polynomial Lyapunov functions**
- Has become further prevalent over the last decade because of **SOS Lyapunov functions**
  - Fully algorithmic search for polynomial Lyapunov functions using semidefinite programming

$$\begin{array}{ll} V(x) & \text{sos} \\ -\dot{V}(x) & \text{sos} \end{array} \Rightarrow \begin{array}{l} V(x) > 0 \\ -\dot{V}(x) > 0 \end{array} \Rightarrow \text{GAS}$$

An example

$$
\begin{aligned}
\dot{x_1} &= -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 + 14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7 \\
\dot{x_2} &= -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 + 807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7
\end{aligned}
$$

- No polynomial Lyapunov function of degree 2, 4, 6.
- But SOS-programming finds one of degree 8.



Output of SDP solver:

$$
\begin{aligned}
V &= 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 - 3.517x_1^4x_2^4 \\
&\quad +0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 - 0.015x_1x_2^7 + 0.02x_2^8.
\end{aligned}
$$

Converse questions

$$\begin{matrix} V(x) \ \text{sos} \\ -\dot{V}(x) \ \text{sos} \end{matrix} \Rightarrow \begin{matrix} V(x) > 0 \\ -\dot{V}(x) > 0 \end{matrix} \Rightarrow \text{GAS}$$

$\overset{?}{\Leftarrow}$ **Talked about this morning**

$\overset{?}{\Leftarrow}$ ($V(x)$ polynomial)

Focus of this talk

Relation to decidability

- Conjecture of Arnold: testing stability is **undecidable**
  - For what n,d?
- Linear systems (d=1): decidable and polynomial time
  - Quadratic Lyapunov functions always exist
- What about d=2?

Fact: If for a class of polynomial vector fields one proves existence of **polynomial Lyapunov functions**, together with a **computable upper bound** on the degree,
then stability becomes **decidable** for that class
(quantifier elimination)

Agenda for the talk

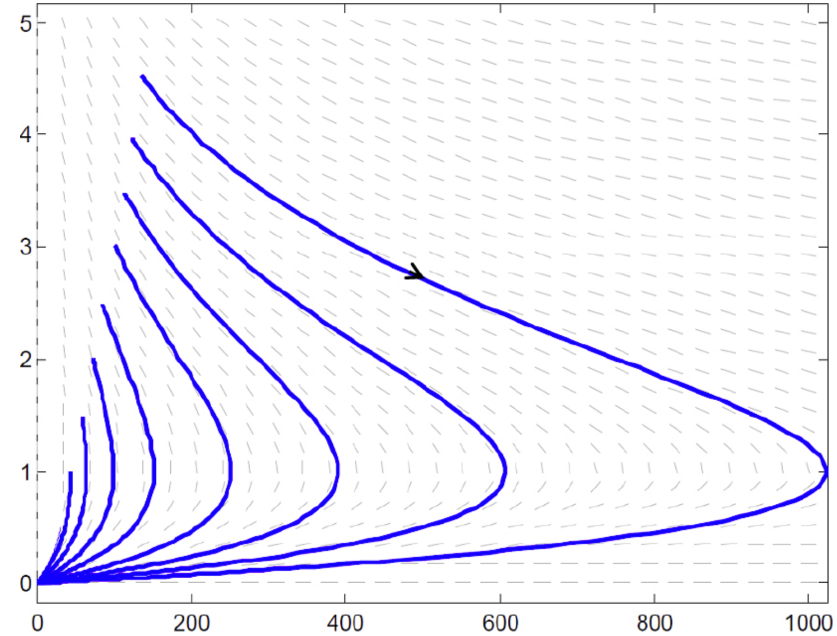
- A counterexample: GAS but no polynomial Lyapunov function
  - Focus of our 2-page CDC paper
- More misery (even for homogeneous cubic vector fields)
  - NP-hardness of testing stability
  - Lack of bounds on degree of Lyapunov functions
  - Non-monotonicity in degree of Lyapunov functions
- Some open problems

Nonexistence of polynomial Lyapunov functions

$$\dot{x} = -x + xy$$
$$\dot{y} = -y$$

Claim 1: System is GAS.

Claim 2: No polynomial Lyapunov function (of any degree) exists!



Proof:

$$\boxed{V(x,y) = \ln(1+x^2) + y^2}$$

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= -\frac{x^2+2y^2+x^2y^2+(x-xy)^2}{1+x^2}$$

Nonexistence of polynomial Lyapunov functions

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= -y \end{aligned}$$

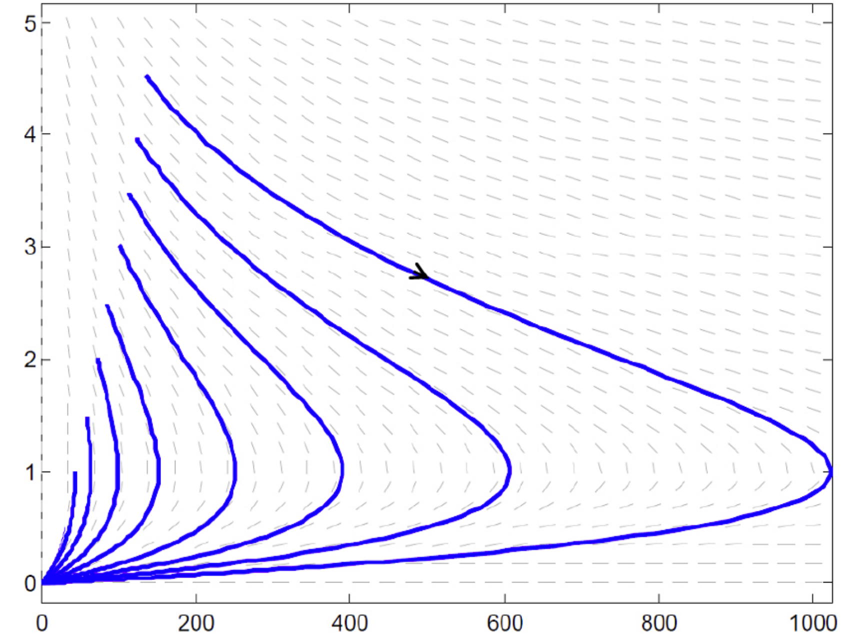
Claim 2: No polynomial Lyapunov function (of any degree) exists!

Proof:

$$\begin{aligned} x(t) &= x(0)e^{[y(0)-y(0)e^{-t}-t]} \\ y(t) &= y(0)e^{-t} \end{aligned}$$

$$t^* = \ln(k)$$

$$(k, \alpha k) \longrightarrow (e^{\alpha(k-1)}, \alpha)$$



$$V(e^{\alpha(k-1)}, \alpha) < V(k, \alpha k)$$

Impossible. ■

Another counterexample

- An earlier independent counterexample appears in a book by Bacciotti and Rosier
  - n=2, d=5
  - Relies crucially on use of **irrational coefficients**
  - Complementary to our example:
    - Problem occurs arbitrarily close to the origin (as opposed to arbitrarily far as in our example)
    - No polynomial Lyapunov function even locally

Homogeneous systems

$$\dot{x} = f(x)$$

$$f(\lambda x) = \lambda^d f(x)$$

- All monomials in f have the same degree
- Local Asymptotic Stability = Global Asymptotic Stability
- Can take Lyapunov function to be homogeneous

$$\begin{array}{c} V(x) \ \text{sos} \\ -\dot{V}(x) \ \text{sos} \end{array} \Rightarrow \begin{array}{c} V(x) > 0 \\ -\dot{V}(x) > 0 \end{array} \Rightarrow \text{GAS}$$

$\Leftarrow$ ✓ $\qquad\qquad\qquad$ $\Leftarrow$ Conjecture

Stability of homogeneous systems: complexity

- Linear systems (d=1): polynomial time
  - Quadratic Lyapunov functions always exist
- d=2 and homogeneous: never asymptotically stable
- d=3 and homogeneous: we show strongly NP-hard

NP-hardness of deciding asymptotic stability for cubics

Thm: Deciding asymptotic stability of cubic homogeneous vector fields is strongly NP-hard.

Implication: Unless P=NP, there cannot be *any* polynomial time (or even pseudo-polynomial time) algorithm.
(In particular suggests SOS Lyapunov functions of "small" degree shouldn't always exist.)

Reduction from: **ONE-IN-THREE 3SAT**

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_4)$$

NP-hardness of deciding asymptotic stability for cubics

ONE-IN-THREE 3SAT

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_4)$$

Positivity of quartic forms

$$\begin{aligned} p(x) \;=\; & \textstyle\sum_{i=1}^{5} x_i^2(1-x_i)^2 \\ & +(x_1+(1-x_2)+x_4-1)^2+((1-x_2)+(1-x_3)+x_5-1)^2 \\ & +((1-x_1)+x_3+(1-x_5)-1)^2+(x_1+x_3+x_4-1)^2 \end{aligned}$$

$$p_h(x,y) = y^4 p\left(\frac{x}{y}\right)$$

Asymptotic stability of cubic homogeneous vector fields

$$\dot{x} = -\nabla p_h$$

(suggests a method for proving positivity)

Lyapunov degree can be arbitrarily large

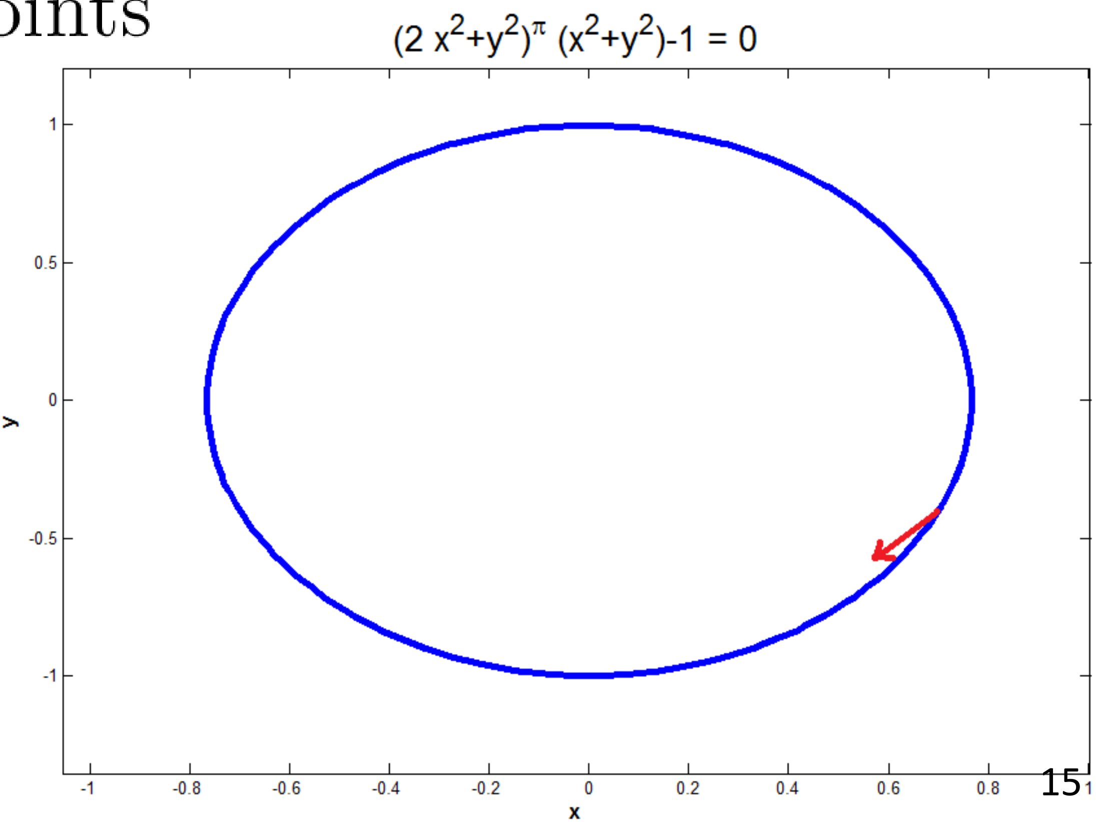
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -2\lambda y(x^2+y^2) - 2y(2x^2+y^2) \\ 4\lambda x(x^2+y^2) + 2x(2x^2+y^2) \end{pmatrix}$$

Thm: Given any degree d, there exists an integer k, such that the system above with

$$\lambda = \pi \text{ to } k \text{ decimal points}$$

$$\theta = \frac{1}{k}$$

is GAS but has no polynomial Lyapunov function of degree $\leq d$.

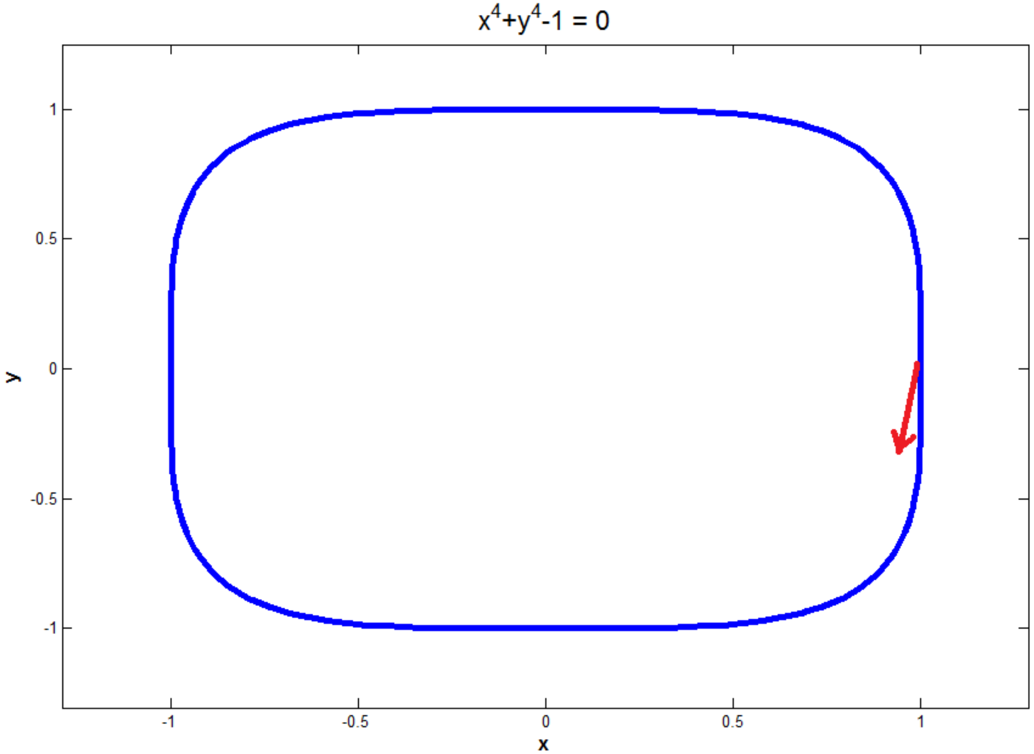


Lack of monotonicity in Lyapunov function degree

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \\ -\cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} x^3 \\ y^3 \end{pmatrix}$$

Thm: There exists a range of values for $\theta > 0$ such that the system is GAS, has **no homog. polynomial Lyapunov fn. of degree 6,**

but admits one of degree 4.



Messages to take home…

$$\begin{aligned} V(x) &> 0 \\ -\dot{V}(x) &> 0 \end{aligned} \quad \Rightarrow \text{ GAS}$$

$$\Leftarrow$$

?

- **No.** Even when n=2, d=2!
- Homogeneous cubic vector fields:
  - NP-hard to test asymptotic stability
  - Lack of bounds on degree of Lyapunov functions (even for fixed dimension, n=2)
  - Non-monotonicity in degree of Lyapunov functions
- Linear to nonlinear: very sharp transition in complexity!

Open questions

1. Is asymptotic stability decidable?

2. Local asymptotic stability with rational coefficients
   ?$\Rightarrow$? Polynomial Lyapunov function

3. Global *exponential* stability
   ?$\Rightarrow$? Polynomial Lyapunov function

Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu/