SOS-Convex Lyapunov Functions for Switched Systems

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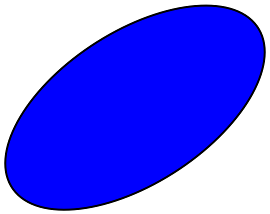
Convex Lyapunov functions

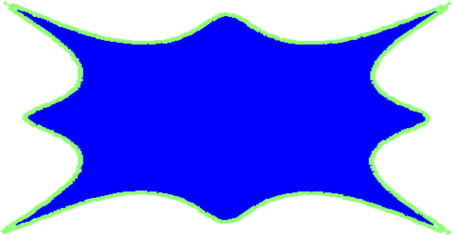
- **Q:** What do we gain/lose by requiring a Lyapunov function to be convex?

- Quadratic Lyapunov functions are automatically convex:

$$x^T P x \text{ convex} \Leftrightarrow \text{ psd}$$

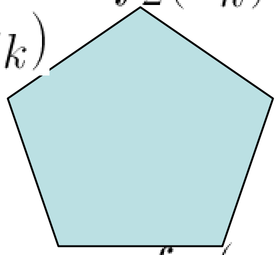


- Not necessarily true for more complicated Lyapunov functions, e.g., polynomials of higher degree:



Our setting: switched systems

- Uncertain and time-varying nonlinear system:



$$x_{k+1} = \tilde{f}(x_k)$$

$$\tilde{f}(x_k) \in conv\{f_1(x_k), \ldots, f_m(x_k)\}$$

$$f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}^n$$

- Interested in: *asymptotic stability under arbitrary switching*

- Special case: $x_{k+1} \in conv\{A_i x_k\}, \quad i = 1, \ldots, m$

- Stability equivalent to **"JSR<1"**

Joint spectral radius (JSR):

$$\rho(\mathcal{A}) = \lim_{k\to\infty} \max_{\sigma\in\{1,\ldots,m\}^k} \|A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

Agenda for today

1. Computational search for convex polynomial Lyapunov functions
    1. sos-convex Lyapunov functions (SDP based)
2. **Convexity in analysis of switched linear systems**
3. **Convexity in analysis of switched nonlinear systems**
4. Max-of-convex-polynomial Lyapunov functions and path-complete graphs

Algorithmic aspects

- How to search for convex polynomial Lyapunov functions?
- For quadratics is easy: CQLF is a straight LMI…
- Higher degree is a different story:

Thm: Deciding whether a degree-4 polynomial is convex is NP-hard in the strong sense.

[AAA, Olshevsky, Parrilo, Tsitsiklis – *Math Prog.* '13]

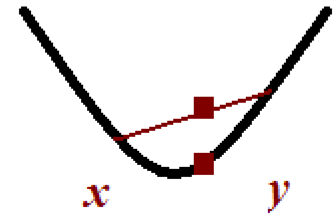
- If can't decide, certainly can't optimize
- Tractable relaxations?

Equivalent characterizations of convexity

- Basic definition:

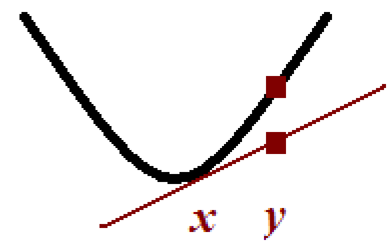
$$\lambda\, p(x) + (1-\lambda)\, p(y) \ge p(\lambda x + (1-\lambda) y)$$

$$\forall x, y\, \forall \lambda \in [0,1] \qquad (\lambda = \frac{1}{2}\, enough)$$



- First order condition:

$$p(y) \ge p(x) + \nabla^T p(x)(y - x) \;\; \forall x, y$$



- Second order condition:

$$y^T H(x) y \ge 0 \;\; \forall x, y$$

- Algebraic relaxations: **“simply replace ≥ with sos”**
- All become an SDP

Each condition can be SOS-ified

- Basic definition:

(A) $$g_{\frac{1}{2}}(x,y) = \frac{1}{2}p(x) + \frac{1}{2}p(y) - p(\frac{1}{2}x + \frac{1}{2}y) \;\; SOS$$

- First order condition:

(B) $$g_{\nabla}(x,y) = p(y) - p(x) - \nabla^T p(x)(y-x) \;\; SOS$$

- Second order condition:

(C) $$g_{\nabla^2}(x,y) = y^T H(x) y \;\; SOS$$

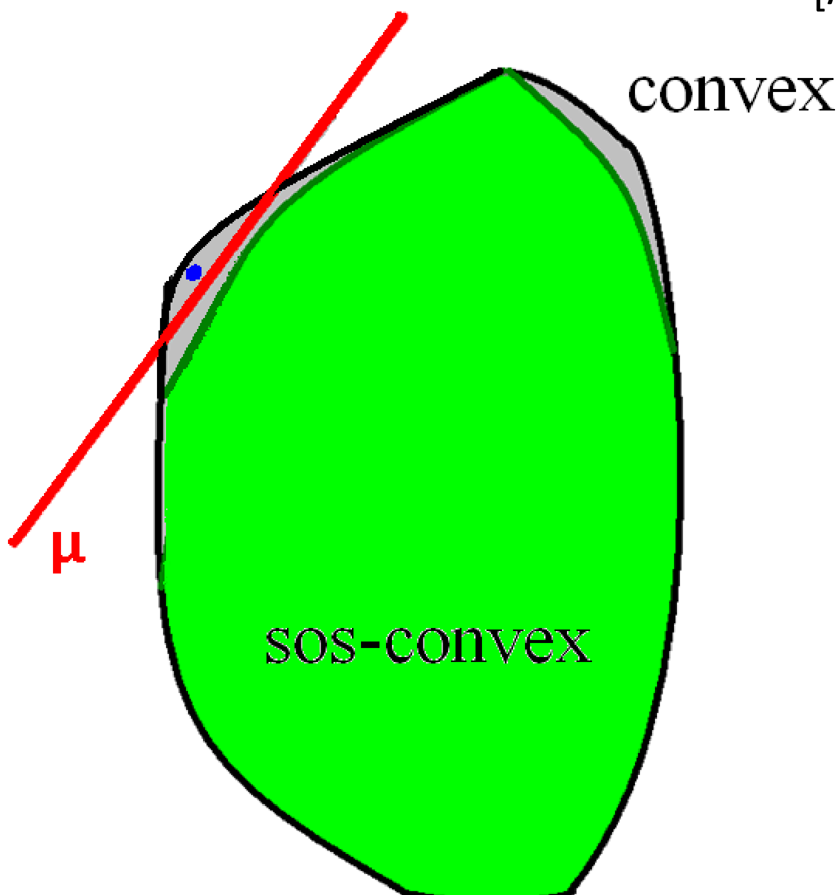
Thm: (A) $\Longleftrightarrow$ (B) $\Longleftrightarrow$ (C)

[AAA, Parrilo – *SIAM J. on Optimization*, '13]

A convex polynomial that is not sos-convex

$$p(x) = 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^4 - 43x_1^4x_2^2x_3^2 - 35x_1^4x_3^4 + 3x_1^2x_2^4x_3^2 - 16x_1^2x_2^2x_3^4 + 24x_1^2x_3^6 + 16x_2^8 + 44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8$$

[AAA, Parrilo – *Math Prog.*, ’11]



Convexity = SOS-Convexity?

- We have a complete characterization:

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms (homog. polynomials)

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

[AAA, Parrilo – *Math Prog.*, ’11]

[AAA, Parrilo – *SIAM J. on Opt.*, ‘13]

[AAA, Blekherman, Parrilo – upcoming]

SOS-convex Lyapunov functions

$$x_{k+1} \in conv\{A_i x_k\}, \quad i = 1, \ldots, m$$

Defn: An *sos-convex Lyapunov function* is a homogenous polynomial *V(x)* (of even degree) that satisfies:

$$\begin{array}{ll} V(x) & \text{sos-convex} \\ V(x) - V(A_i x) & \text{sos for } i = 1, \ldots, m. \end{array}$$

Notes:

- Search for *V(x)* is an SDP
- More tractable alternatives based on DSOS and SDSOS programming (LP and SOCP) [AAA, Majumdar'13]
- Convex forms are automatically nonnegative
- SOS-convex forms are sos

A converse Lyapunov theorem

$$x_{k+1} \in conv\{A_i x_k\}, \quad i = 1, \ldots, m$$

Thm: SOS-convex Lyapunov functions are *universal* (i.e., *necessary and sufficient*) for stability.

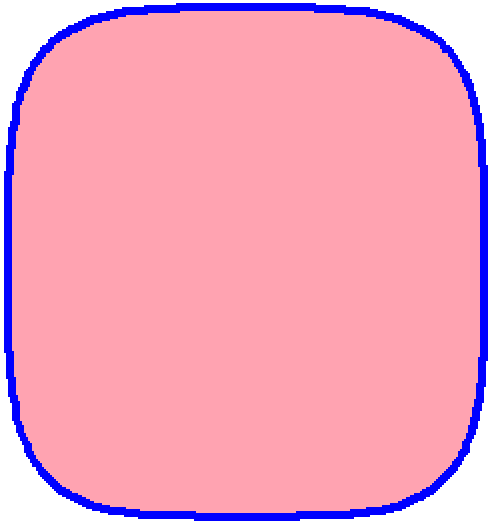
Notes:

- Previously known:
  - Convex (norm) Lyapunov functions universal (classical)
  - Polynomial (and sos) Lyapunov functions universal [Parrilo, Jadbabaie]

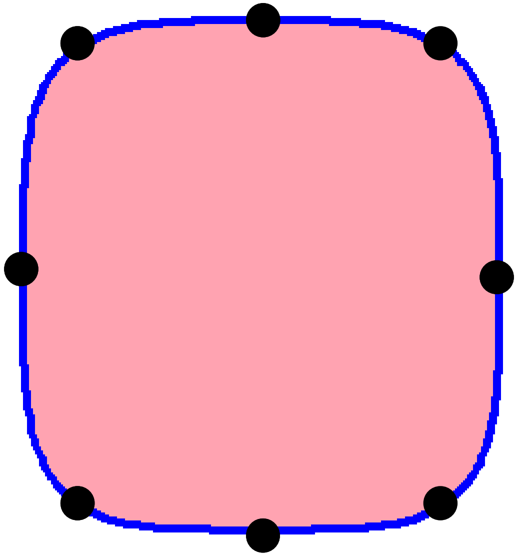
Proof idea:

- Approximate norms with convex polynomials
  - Similar construction by Mason et al. in continuous time
- In a second step, go from convex to sos-convex

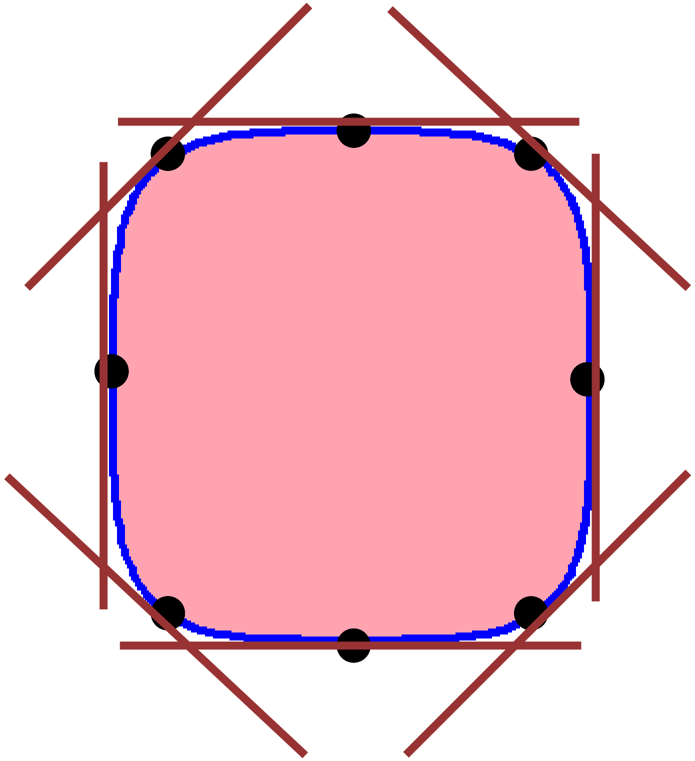
Proof idea



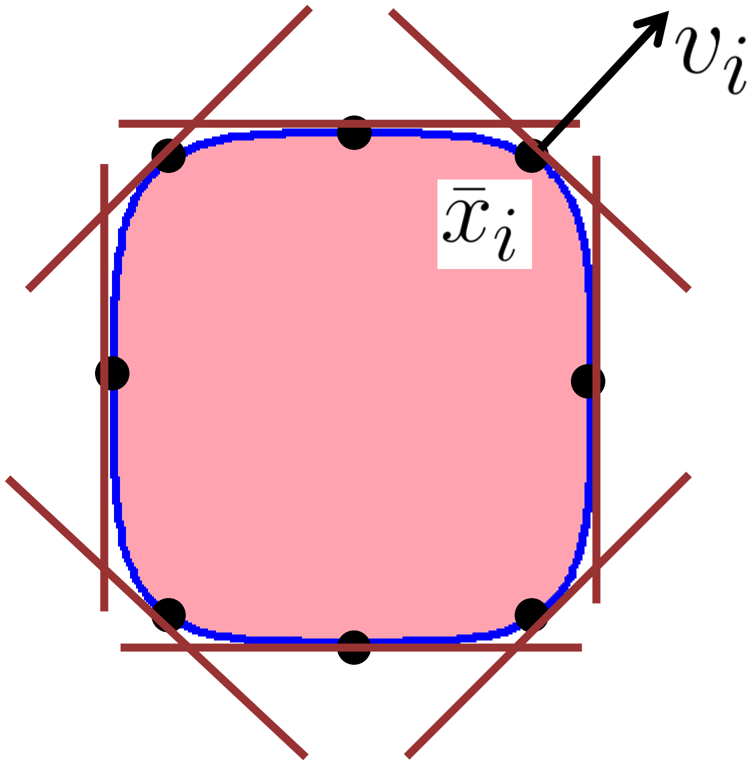
Proof idea



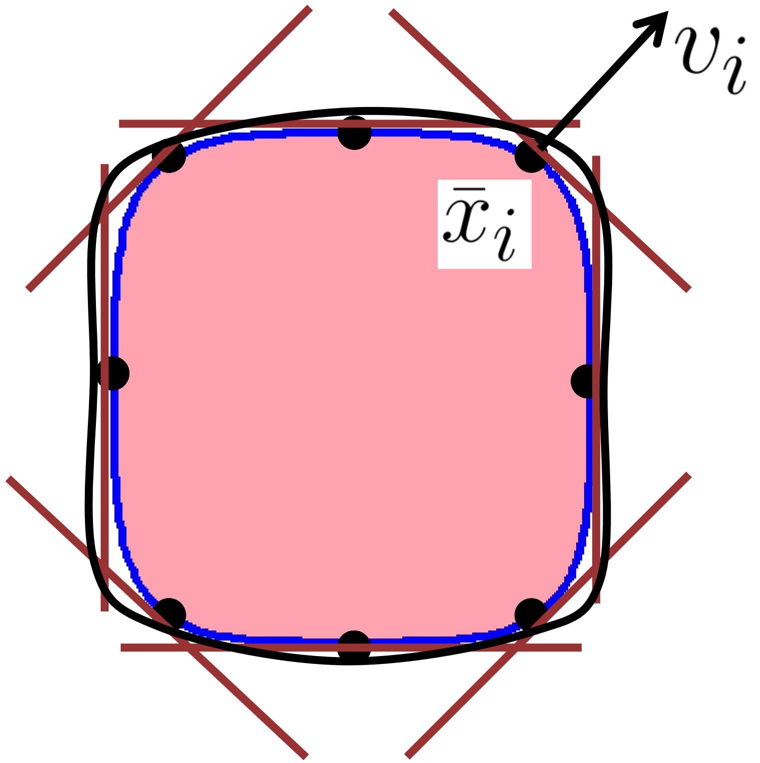
Proof idea



Proof idea



Proof idea



$$p(x) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{v_i^T x}{v_i^T \bar{x}_i}\right)^{2d}$$

(clearly convex)

From convex to sos-convex

Thm:
convex polynomial Lyapunov function ➔ **sos-convex** Lyapunov function

- Proof is algebraic.

$$p(x) = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{v_i^T x}{v_i^T \bar{x}_i} \right)^{2d}$$

  - This is already an sos-convex polynomial
  - But need to worry about:

$$p(x) - p(A_i x), \ i = 1, \ldots, m$$

not being sos.
This can happen; see [AAA, Parrilo, CDC'11]

From convex to sos-convex

- A recent Positivstellensatz result of Claus Scheiderer:

Thm: (Scheiderer'11) Given any two positive definite forms $\boldsymbol{g}$ and $\boldsymbol{h}$, there exists an integer $\boldsymbol{k}$ such that $\boldsymbol{g.h^k}$ is sos.

Our new Lyapunov funciton: $q(x) = p^k(x)$ (for large enough k)

$$q(x) - q(A_i x) = p^k(x) - p^k(A_i x)$$ ←will be sos

Nice identity: $$a^k - b^k = (a-b)\sum_{l=0}^{k-1} a^{k-1-l} b^l$$

Polynomial vs. convex polynomial

Thm: The minimum degree of a convex polynomial Lyapunov function can be **arbitrarily higher** than a non-convex one.

Proof:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\{\gamma A_1, \gamma A_2\}$$

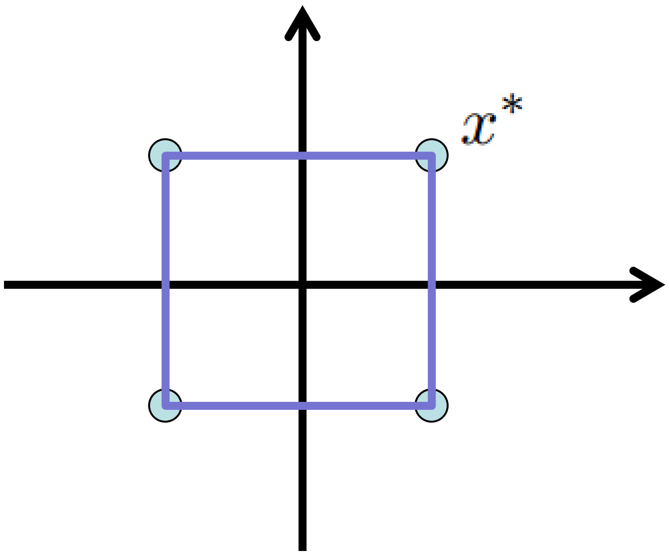
- Stable for $\gamma < 1$
- But degree of convex polynomial Lyapunov function→∞ as γ →1
- However, a polynomial Lyapunov function of degree 4 works for all $\gamma < 1$

Proof

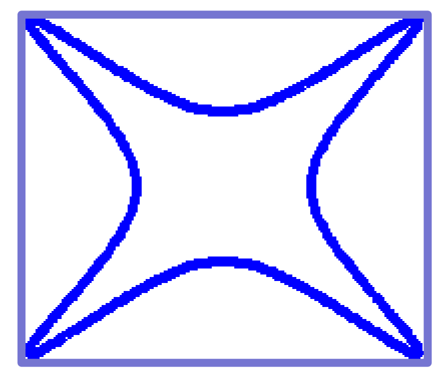
What can the 1-level set of a convex Lyapunov function look like?

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathcal{A}^*$$



How can a non-convex polynomial do this?



Lemma:

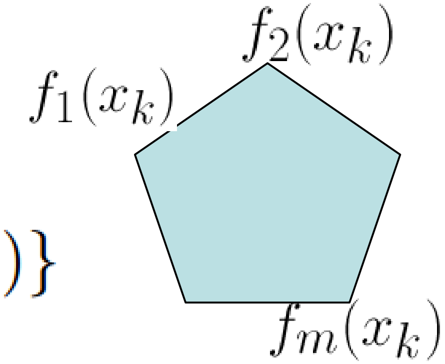
- If S is invariant, then *conv(S)* is also invariant.
- The **Minkowski norm** that *conv(S)* defines is a new convex (non-polynomial) Lyapunov function.

Nonlinear switched systems

$$x_{k+1} = \tilde{f}(x_k)$$

$$\tilde{f}(x_k) \in conv\{f_1(x_k), \dots, f_m(x_k)\}$$

$$f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}^n$$



Lemma:

- Unlike the linear case, stability of a switched system on the corners only **does not imply** stability of the convex hull.
- Unlike the linear case, a common Lyapunov function for the corners **does not imply** stability of the convex hull.

Ex.

$$\begin{aligned} f_1(x) &= (x_1 x_2, 0)^T \\ f_2(x) &= (0, x_1 x_2)^T \end{aligned}$$

Common Lyapunov function:

$$V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2)$$

$$V(f_i(x)) = x_1^2 x_2^2 < V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2)$$

But unstable:

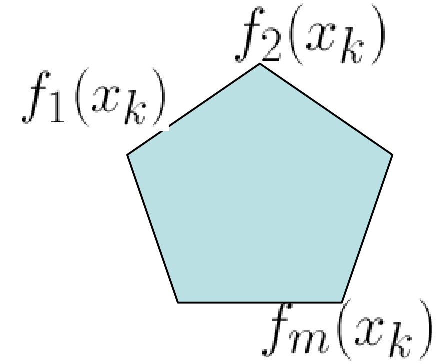
$$f(x) = \left(\frac{x_1 x_2}{2}, \frac{x_1 x_2}{2}\right) \in conv\{f_1(x_k), f_2(x_k)\}$$

But a *convex* Lyapunov function implies stability

$$x_{k+1} = \tilde{f}(x_k)$$

$$\tilde{f}(x_k) \in conv\{f_1(x_k), \ldots, f_m(x_k)\}$$

$$f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}^n$$



Suppose we can find a **convex** common Lyapunov function:

$$V(x) > 0, \quad V(f_i(x)) < V(x) \text{ for } i = 1, \ldots, m$$

Then, then we have stability of the convex hull.

Proof: $V(\tilde{f}(x)) = V(\sum_{i=1}^{m} \alpha_i f_i(x)) \leq \sum_{i=1}^{m} \alpha_i V(f_i(x)) < V(x)$

Example revisited

Ex.

$$f_1(x) = (x_1x_2, 0)^T$$
$$f_2(x) = (0, x_1x_2)^T$$

Common Lyapunov function:

$$V(x) = x_1^2x_2^2 + (x_1^2 + x_2^2)$$
$$V(f_i(x)) = x_1^2x_2^2 < V(x) = x_1^2x_2^2 + (x_1^2 + x_2^2)$$

But unstable:

$$f(x) = \left(\frac{x_1x_2}{2}, \frac{x_1x_2}{2}\right) \in conv\{f_1(x_k), f_2(x_k)\}$$

A different Lyapunov function: $W(x) = x_1^2 + x_2^2$

$$\begin{aligned} W(f_i(x)) &= x_1^2x_2^2 \\ &< x_1^2 + x_2^2 \\ &= W(x) \end{aligned}$$

$$\text{on } S = \{x : -1 \leq x_1, x_2 \leq 1\}$$

And S is invariant→ S is part of the ROA under switching

ROA Computation via SDP

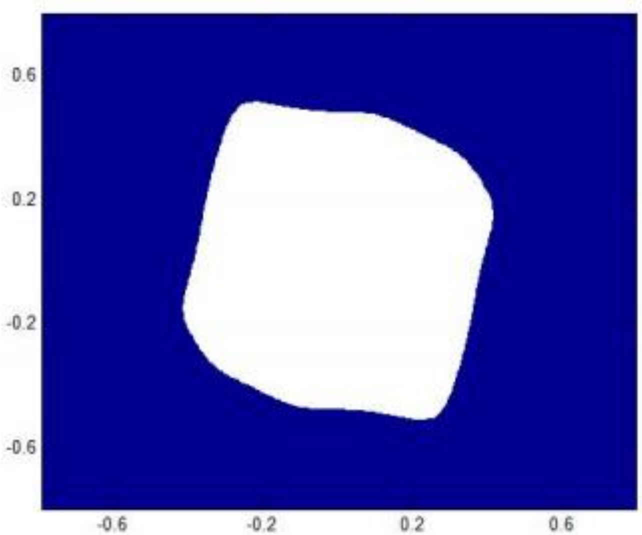
$$f_1(x) \quad = \quad \begin{pmatrix} 0.687x_1 + 0.558x_2 - .0001x_1x_2 \\ -0.292x_1 + 0.773x_2 \end{pmatrix}$$

$$f_2(x) \quad = \quad \begin{pmatrix} 0.369x_1 + 0.532x_2 - .0001x_1^2 \\ -1.27x_1 + 0.12x_2 - .0001x_1x_2 \end{pmatrix}$$

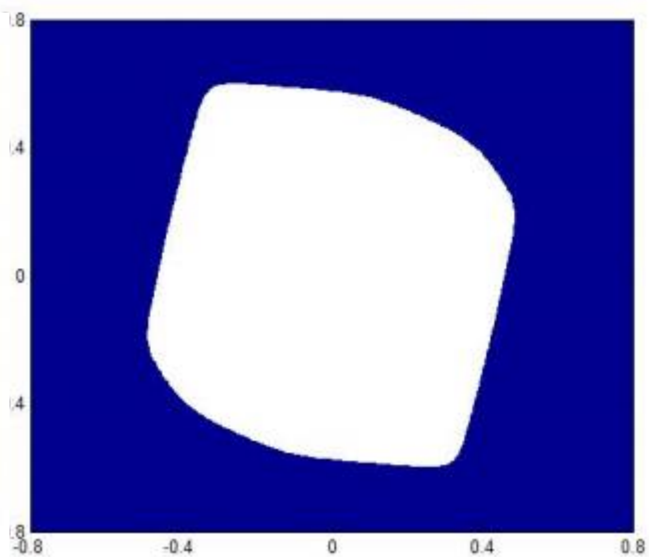
Linearization:

$$A_1 = \begin{pmatrix} 0.687 & 0.558 \\ -0.292 & 0.773 \end{pmatrix}, A_2 = \begin{pmatrix} 0.369 & 0.532 \\ -1.27 & 0.12 \end{pmatrix}$$

- **JSR<1**
- **No polynomial Lyapunov function of degree less than 12**
- **No convex polynomial Lyapunov function of degree less than 14**



non-convex, deg=12



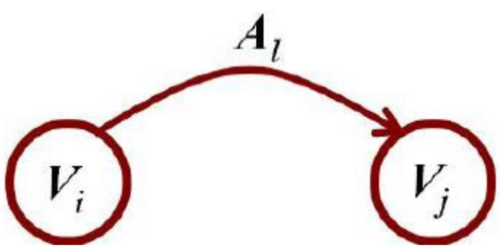
sos-convex, deg=14

- **Left:**
  Cannot make any statements about ROA
- **Right:**
  A second SOS problem verifies the largest level set included in the ROA

Convex, but beyond polynomials

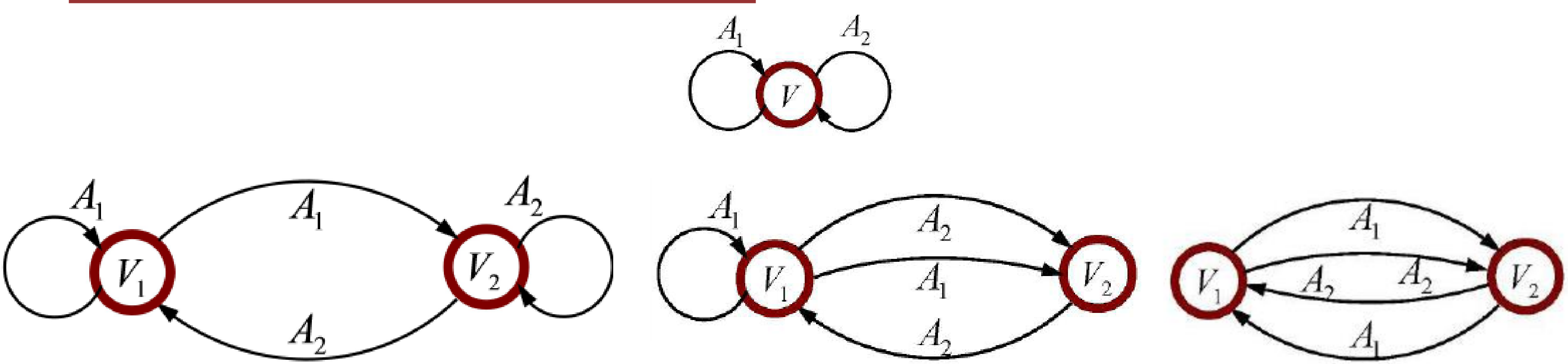
- Parameterization of *pointwise maximimum* of (sos-convex) Lyapunov functions

$$V_j(A_l x) \leq V_i(x) \quad \forall x \in \mathbb{R}^n$$



Nodes: Lyapunov functions
Edges: Lyapunov inequalities
Edge labels: transition maps

- Take ***any*** graph whose nodes have incoming edges with all labels



More general story:

- **Path-completeness**: a characterization of ***all*** stability-proving LMIs

[AAA, Jungers, Parrilo, Roozbehani – *SIAM J. on Control and Optimization*, to appear]

Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu/