SOS-Convex Lyapunov Functions for Switched Systems

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Convex Lyapunov functions

- **Q**: What do we gain/lose by requiring a Lyapunov function to be convex?

- Quadratic Lyapunov functions are automatically convex:
  \[ x^T P x \text{ convex } \Leftrightarrow \text{ psd} \]

- Not necessarily true for more complicated Lyapunov functions, e.g., polynomials of higher degree:
Our setting: switched systems

- Uncertain and time-varying nonlinear system:

  \[ x_{k+1} = \tilde{f}(x_k) \]

  \[ \tilde{f}(x_k) \in \text{conv}\{f_1(x_k), \ldots, f_m(x_k)\} \]

  \[ f_1, \ldots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

- Interested in: *asymptotic stability under arbitrary switching*

- Special case: \( x_{k+1} \in \text{conv}\{A_i x_k\}, \quad i = 1, \ldots, m \)

- Stability equivalent to “JSR<1”

Joint spectral radius (JSR):

\[
\rho(A) = \lim_{k \to \infty} \max_{\sigma \in \{1, \ldots, m\}^k} \|A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}\|^{1/k}
\]
Agenda for today

1. Computational search for convex polynomial Lyapunov functions
   1. sos-convex Lyapunov functions (SDP based)
2. Convexity in analysis of switched linear systems
3. Convexity in analysis of switched nonlinear systems
4. Max-of-convex-polynomial Lyapunov functions and path-complete graphs
Algorithmic aspects

- How to search for convex polynomial Lyapunov functions?
- For quadratics is easy: CQLF is a straight LMI...
- Higher degree is a different story:

**Thm:** Deciding whether a degree-4 polynomial is convex is NP-hard in the strong sense.

[AAA, Olshevsky, Parrilo, Tsitsiklis – *Math Prog.* ‘13]

- If can’t decide, certainly can’t optimize
- Tractable relaxations?
Equivalent characterizations of convexity

- **Basic definition:**
  \[ \lambda p(x) + (1 - \lambda) p(y) \geq p(\lambda x + (1 - \lambda) y) \]
  \[ \forall x, y \forall \lambda \in [0, 1] \quad (\lambda = \frac{1}{2} \text{ enough}) \]

- **First order condition:**
  \[ p(y) \geq p(x) + \nabla^T p(x)(y - x) \quad \forall x, y \]

- **Second order condition:**
  \[ y^T H(x) y \geq 0 \quad \forall x, y \]

- **Algebraic relaxations:** “simply replace ≥ with sos”

- **All become an SDP**
Each condition can be SOS-ified

- **Basic definition:**
  \[ g_{\frac{1}{2}}(x, y) = \frac{1}{2} p(x) + \frac{1}{2} p(y) - p\left(\frac{1}{2} x + \frac{1}{2} y\right) \quad \text{SOS} \]

- **First order condition:**
  \[ g_{\nabla}(x, y) = p(y) - p(x) - \nabla^T p(x)(y - x) \quad \text{SOS} \]

- **Second order condition:**
  \[ g_{\nabla^2}(x, y) = y^T H(x) y \quad \text{SOS} \]

Thm: \( \text{A} \leftrightarrow \text{B} \leftrightarrow \text{C} \)

[AAA, Parrilo – SIAM J. on Optimization, ‘13]
A convex polynomial that is not sos-convex

\[ p(x) = 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^4 - 43x_1^4x_2^2x_3^2 - 35x_1^4x_3^4 + 3x_1^2x_2^4x_3^2 - 16x_1^2x_2^2x_3^4 + 24x_1^2x_3^6 + 16x_2^8 + 44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8 \]

[AAA, Parrilo – Math Prog., ’11]
Convexity = SOS-Convexity?

- We have a complete characterization:

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[AAA, Parrilo – *Math Prog.*, ’11]
[AAA, Parrilo – *SIAM J. on Opt.*, ‘13]
[AAA, Blekherman, Parrilo – upcoming]
SOS-convex Lyapunov functions

\[ x_{k+1} \in \text{conv}\{A_i x_k\}, \quad i = 1, \ldots, m \]

**Defn:** An *sos-convex Lyapunov function* is a homogenous polynomial \( V(x) \) (of even degree) that satisfies:

\[
V(x) - V(A_i x) \quad \text{sos for } i = 1, \ldots, m.
\]

**Notes:**
- Search for \( V(x) \) is an SDP
- More tractable alternatives based on DSOS and SDSOS programming (LP and SOCP) [AAA, Majumdar’13]
- Convex forms are automatically nonnegative
- SOS-convex forms are sos
A converse Lyapunov theorem

\[ x_{k+1} \in \text{conv}\{A_i x_k\}, \quad i = 1, \ldots, m \]

**Thm:** SOS-convex Lyapunov functions are *universal* (i.e., necessary and sufficient) for stability.

**Notes:**
- Previously known:
  - Convex (norm) Lyapunov functions universal (classical)
  - Polynomial (and sos) Lyapunov functions universal [Parrilo, Jadbabaie]

**Proof idea:**
- Approximate norms with convex polynomials
  - Similar construction by Mason et al. in continuous time
- In a second step, go from convex to sos-convex
Proof idea
Proof idea
Proof idea
Proof idea

\[ p(x) = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{v_i^T x}{v_i^T \bar{x}_i} \right)^{2d} \] (clearly convex)
From convex to sos-convex

Thm: convex polynomial Lyapunov function $\Rightarrow$ sos-convex Lyapunov function

- Proof is algebraic.

$$p(x) = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{v_i^T x}{v_i^T x_i} \right)^{2d}$$

- This is already an sos-convex polynomial

- But need to worry about:

$$p(x) - p(A_i x), \quad i = 1, \ldots, m$$

not being sos.

This can happen; see [AAA, Parrilo, CDC’11]
From convex to sos-convex

- A recent Positivstellensatz result of Claus Scheiderer:

**Thm:** (Scheiderer’11) Given any two positive definite forms $g$ and $h$, there exists an integer $k$ such that $g \cdot h^k$ is sos.

Our new Lyapunov function: $q(x) = p^k(x)$ (for large enough $k$)

$$q(x) - q(A_ix) = p^k(x) - p^k(A_ix) \quad \Leftarrow \text{will be sos}$$

Nice identity: $a^k - b^k = (a - b) \sum_{l=0}^{k-1} a^{k-1-l} b^l$
Polynomial vs. convex polynomial

**Thm:** The minimum degree of a convex polynomial Lyapunov function can be *arbitrarily higher* than a non-convex one.

**Proof:**

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \]

\[ \{\gamma A_1, \gamma A_2\} \]

- Stable for \( \gamma < 1 \)
- But degree of convex polynomial Lyapunov function \( \rightarrow \infty \) as \( \gamma \rightarrow 1 \)
- However, a polynomial Lyapunov function of degree 4 works for all \( \gamma < 1 \)
Proof

What can the 1-level set of a convex Lyapunov function look like?

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \]

\[ \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\} \subset A^* \]

How can a non-convex polynomial do this?

Lemma:

- If \( S \) is invariant, then \( \text{conv}(S) \) is also invariant.
- The \textbf{Minkowski norm} that \( \text{conv}(S) \) defines is a new convex (non-polynomial) Lyapunov function.
Nonlinear switched systems

\[ x_{k+1} = \tilde{f}(x_k) \]
\[ \tilde{f}(x_k) \in \text{conv}\{f_1(x_k), \ldots, f_m(x_k)\} \]
\[ f_1, \ldots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

**Lemma:**
- Unlike the linear case, stability of a switched system on the corners only **does not imply** stability of the convex hull.
- Unlike the linear case, a common Lyapunov function for the corners **does not imply** stability of the convex hull.

**Ex.**

**Common Lyapunov function:**

\[ V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2) \]
\[ V(f_i(x)) = x_1^2 x_2^2 < V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2) \]

But unstable:

\[ f(x) = \left( \frac{x_1 x_2}{2}, \frac{x_1 x_2}{2} \right) \in \text{conv}\{f_1(x_k), f_2(x_k)\} \]
But a **convex** Lyapunov function implies stability

\[ x_{k+1} = \tilde{f}(x_k) \]

\[ \tilde{f}(x_k) \in \text{conv}\{f_1(x_k), \ldots, f_m(x_k)\} \]

\[ f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}^n \]

Suppose we can find a **convex** common Lyapunov function:

\[ V(x) > 0, \quad V(f_i(x)) < V(x) \text{ for } i = 1, \ldots, m \]

Then, then we have stability of the convex hull.

**Proof:**

\[ V(\tilde{f}(x)) = V(\sum_{i=1}^{m} \alpha_i f_i(x)) \leq \sum_{i=1}^{m} \alpha_i V(f_i(x)) < V(x) \]
Example revisited

Ex. \[ f_1(x) = (x_1 x_2, 0)^T \]
\[ f_2(x) = (0, x_1 x_2)^T \]

Common Lyapunov function:
\[ V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2) \]
\[ V(f_i(x)) = x_1^2 x_2^2 < V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2) \]

But unstable:
\[ f(x) = \left( \frac{x_1 x_2}{2}, \frac{x_1 x_2}{2} \right) \in \text{conv}\{f_1(x_k), f_2(x_k)\} \]

A different Lyapunov function:
\[ W(x) = x_1^2 + x_2^2 \]
\[ W(f_i(x)) = x_1^2 x_2^2 < x_1^2 + x_2^2 \]
\[ = W(x) \]

on \( S = \{ x : -1 \leq x_1, x_2 \leq 1 \} \)

And \( S \) is invariant \( \rightarrow \) \( S \) is part of the ROA under switching
ROA Computation via SDP

Linearization:

- $JSR < 1$
- No polynomial Lyapunov function of degree less than 12
- No convex polynomial Lyapunov function of degree less than 14

\[
\begin{align*}
  f_1(x) &= \begin{pmatrix}
  0.687x_1 + 0.558x_2 - 0.0001x_1x_2 \\
  -0.292x_1 + 0.773x_2 
\end{pmatrix} \\
  f_2(x) &= \begin{pmatrix}
  0.369x_1 + 0.532x_2 - 0.0001x_1^2 \\
  -1.27x_1 + 0.12x_2 - 0.0001x_1x_2 
\end{pmatrix} \\
  A_1 &= \begin{pmatrix}
  0.687 & 0.558 \\
  -0.292 & 0.773 
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
  0.369 & 0.532 \\
  -1.27 & 0.12 
\end{pmatrix}
\]

\[
\text{non-convex, deg=12} \quad \text{sos-convex, deg=14}
\]

- **Left:** Cannot make any statements about ROA
- **Right:** A second SOS problem verifies the largest level set included in the ROA
Convex, but beyond polynomials

- Parameterization of **pointwise maximum** of (sos-convex) Lyapunov functions

\[ V_j(A_l x) \leq V_i(x) \quad \forall x \in \mathbb{R}^n \]

**Nodes:** Lyapunov functions
**Edges:** Lyapunov inequalities
**Edge labels:** transition maps

- Take any graph whose nodes have incoming edges with all labels

**More general story:**
- **Path-completeness:** a characterization of all stability-proving LMIs

Thank you for your attention!
Questions?

http://aaa.lids.mit.edu/