Joint Spectral Radius of Rank One Matrices and the Maximum Cycle Mean Problem

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The Joint Spectral Radius

Given a finite set of $n \times n$ matrices

$$\mathcal{A} := \{A_1, \ldots, A_m\}$$

Joint spectral radius (JSR):

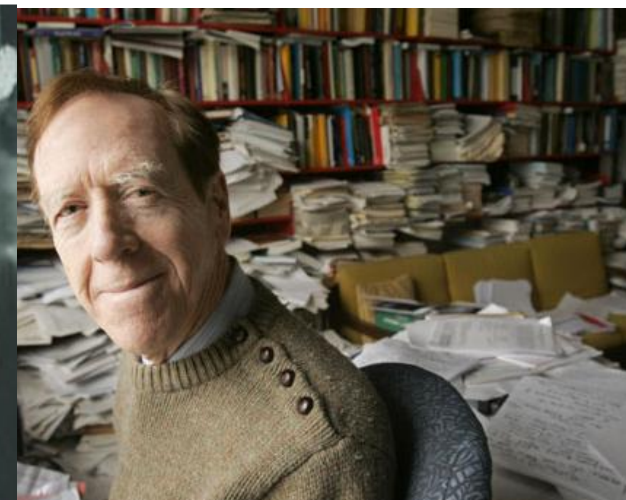
$$\rho(\mathcal{A}) = \lim_{k\to\infty} \max_{\sigma \in \{1,\ldots,m\}^k} \|A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

If only one matrix:

$$\mathcal{A} = \{A\}$$

Spectral Radius

$$\rho(A) = \lim_{k\to\infty} ||A^k||^{\frac{1}{k}}$$



G. C. Rota and W. G. Strang
A note on the joint spectral radius
Indag. Math., 22:379–381, 1960.

JSR and switched linear systems

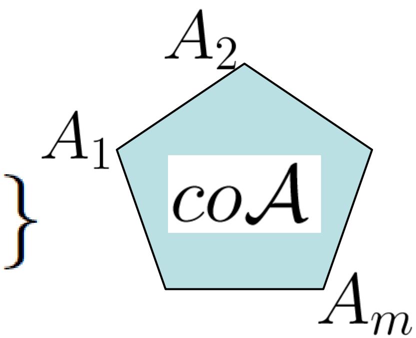
Linear dynamics: $x_{k+1} = Ax_k$

Spectral radius: $\rho(A) = \lim_{k \to \infty} ||A^k||^{\frac{1}{k}}$

"Stable" iff $\rho(A) < 1$

Switched linear dynamics: $x_{k+1} = A_i x_k$

$$\mathcal{A} := \{A_1, \ldots, A_m\}$$



Joint spectral radius (JSR):

$$\rho(\mathcal{A}) = \lim_{k \to \infty} \max_{\sigma \in \{1, \ldots, m\}^k} ||A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}||^{1/k}$$

"Stable under arbitrary switching" iff $\rho(\mathcal{A}) < 1$

Undecidability and the rational finiteness conjecture

If only one matrix: $\mathcal{A} = \{A\}$

$$\text{Testing } \text{“}\rho(A) < 1?\text{”} \text{ decidable in polynomial time}$$

For more than one matrix:

$$\text{Testing } \text{“}\rho(\mathcal{A}) \leq 1?\text{”} \text{ undecidable}$$

[Blondel, Tsitsiklis]

(even for 2 matrices of size 47x47 !!)

[Blondel, Canterini]

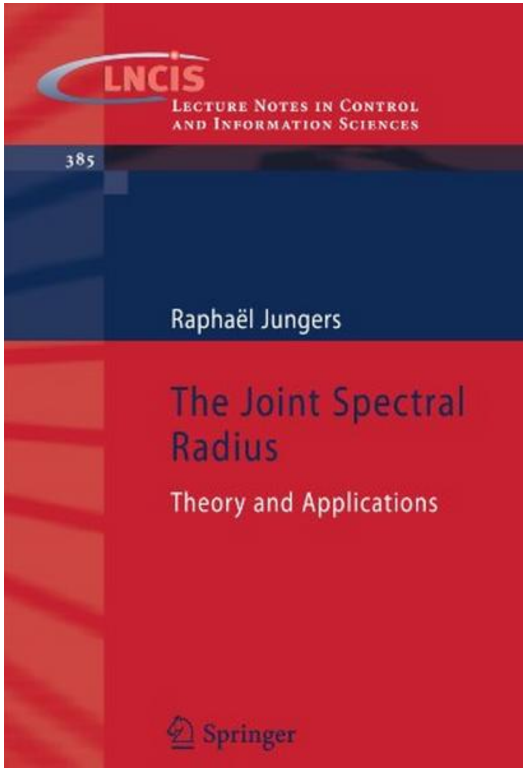
- **Open problem:** decidability of testing $\text{“}\rho(\mathcal{A}) < 1?\text{”}$
- Would become decidable if *rational finiteness conjecture* is true

- **Fact:** (spectral radius of any matrix product of length k)$^{1/k}$ **≤JSR**

Finiteness property: equality achieved at finite k

Computation of the JSR

- **Applications**
  Stability of switched systems, capacity of codes, asymptotics of overlap-free words, convergence of consensus algorithms, continuity of wavelet functions, trackability of graphs, …



- **Methods of approximation**

  CQLF [Ando, Shih], Kronecker liftings [Blondel, Nesterov], SOS [Parrilo, Jadbabaie], Path dependent Lyapunov functions [Lee, Dullerud], Polytopic norms [Guglielmi, Protasov, Zennaro],
  Path-complete graph Lyapunov functions [AAA, Jungers, Parrilo, Roozbehani], …

Focus of this 2-page paper: JSR of rank one matrices

- Motivated by recent work of [Liu, Xiao `11]
  - Correction to a claim in their paper

But our main contribution is:

- An **$O(m^3+m^2n)$** algorithm for computing the **exact JSR**

  Based on a **connection to a problem in graph theory**

- As a corollary: **finiteness property** & some structural property of the optimal word

Basic facts about rank one matrices

A rank one iff $A = xy^T$

spectral radius: $|y^T x|$

Products of rank one matrices have rank at most one:

$$\begin{aligned} A_i A_j &= x_i y_i^T x_j y_j^T \\ &= (y_i^T x_j) x_i y_j^T \end{aligned}$$

Cycles and cycle gains

Easy definitions:

- Cycle
- Simple cycle
- Cycle gain

$$g(c) = \rho_c^{1/k}$$

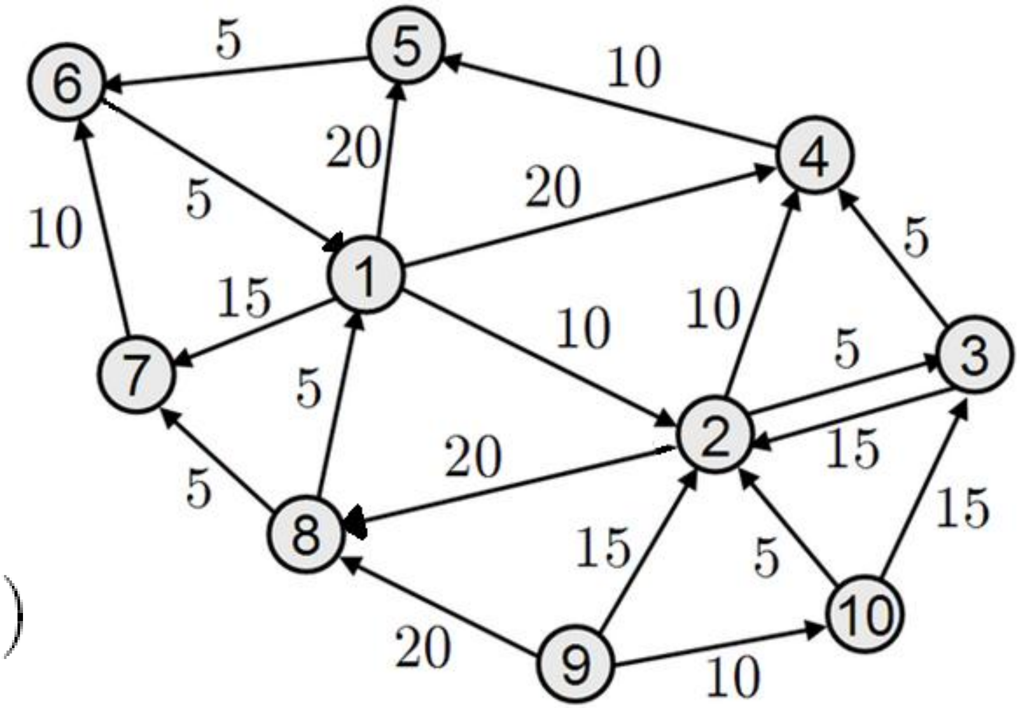
$$\rho_c = \prod_{i=1}^{k} w(e_i)$$

- Maximum cycle gain

$$\max_c g(c)$$

- Gain-maximizing cycle

$$c_{max}$$



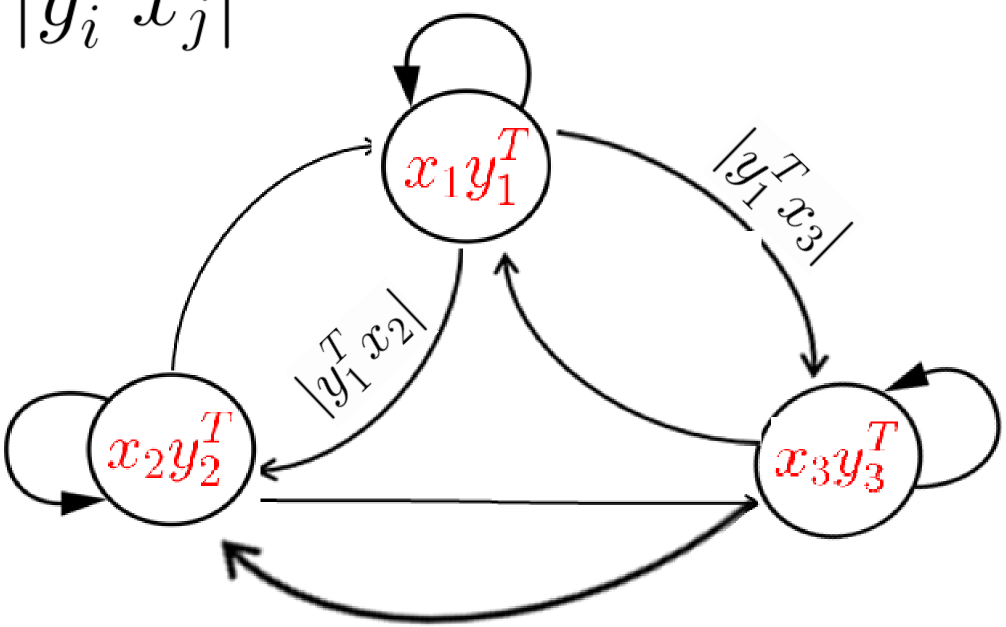
From matrix products to cycles in graphs

$$\mathcal{A} = \{A_1, \ldots, A_m\} \quad A_i = x_i y_i^T$$

$G_{\mathcal{A}}$ Complete directed graph on *m* nodes:

Nodes: matrices A_i

Edge weights: $w(e_{ij}) = |y_i^T x_j|$



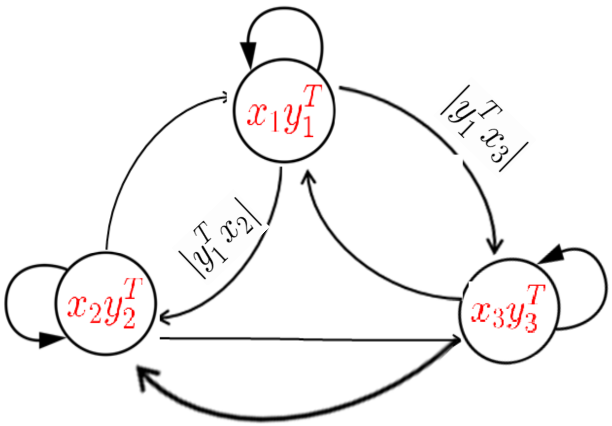
Maximum cycle gain gives the JSR

Thm: Let c_{max} be a gain-maximizing cycle, with l_{max} and ρ_{cmax} denoting its length and the product of the weights on its edges, respectively.
Then, the joint spectral radius is given by:

$$\rho(\mathcal{A}) = \rho_{c_{max}}^{1/l_{max}}$$

Proof sketch:

$$\rho(A_{\sigma_k} \cdots A_{\sigma_1}) = \rho_c$$



$$\begin{aligned} \rho(A_{\sigma_1} \cdots A_{\sigma_k})^{1/k} &= \left(\prod_{i=1}^{s} \rho_{c_i}^{m_i}\right)^{1/k} \\ &= \prod_{i=1}^{s} (\rho_{c_i}^{1/l_i})^{m_i l_i / k} \\ &\leq \rho_{c_{max}}^{1/l_{max}}, \end{aligned}$$

$$\rho(\mathcal{A}) = \limsup_{k\to\infty} \max_{A\in\mathcal{A}^k} \rho^{\frac{1}{k}}(A)$$

■

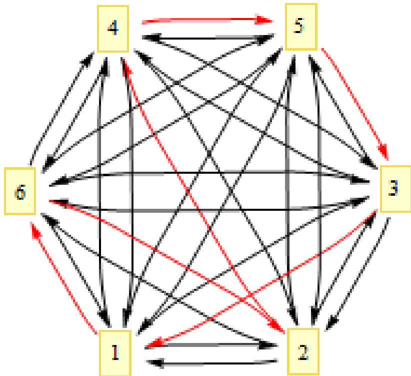
Finiteness property and the optimal product

Corollary:

- The JSR is achieved by the spectral radius of a finite matrix product, of length at most m. (In particular, the finiteness property holds.)
- There always exists an optimal product where no matrix appears more than once.

Proof: A simple cycle does not visit a node twice. ■

Not fun to enumerate all simple cycles…



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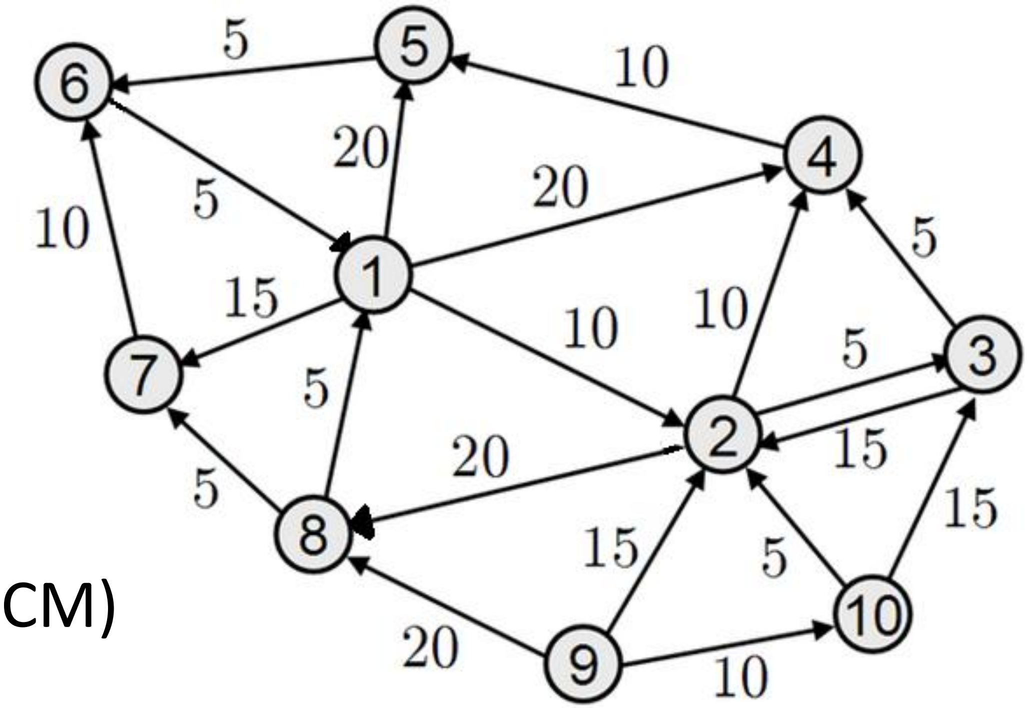
Maximum Cycle Mean Problem (MCMP)

- Cycle mean

$$m(c) = \sum_{i=1}^{k} \frac{w(e_i)}{k}$$

- Maximum cycle mean (MCM)

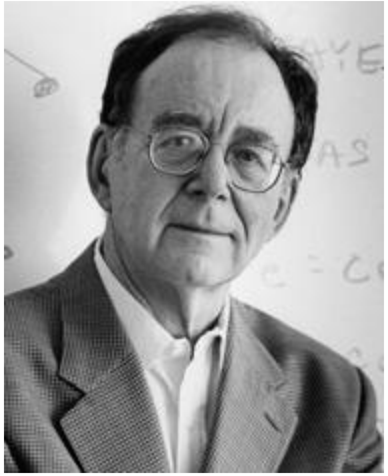
$$\lambda^* = \max_c m(c)$$



Karp's algorithm for MCMP

- Proposed by Karp in 1978, based on *dynamic programming*
- Let *s* be an arbitrary vertex
- For every vertex *v* and integer *k*, define $F_k(v)$ as the minimum weight of an edge progression of length *k* from *s* to *v*
- $F_k(v)$ can be computed via a simple DP recursion
- From this, the MCM can be computed as:

$$\min_{v} \max_{0 \le k \le n-1} \left[\frac{F_n(v) - F_k(v)}{n-k} \right]$$



- Running time O(|N||E|)

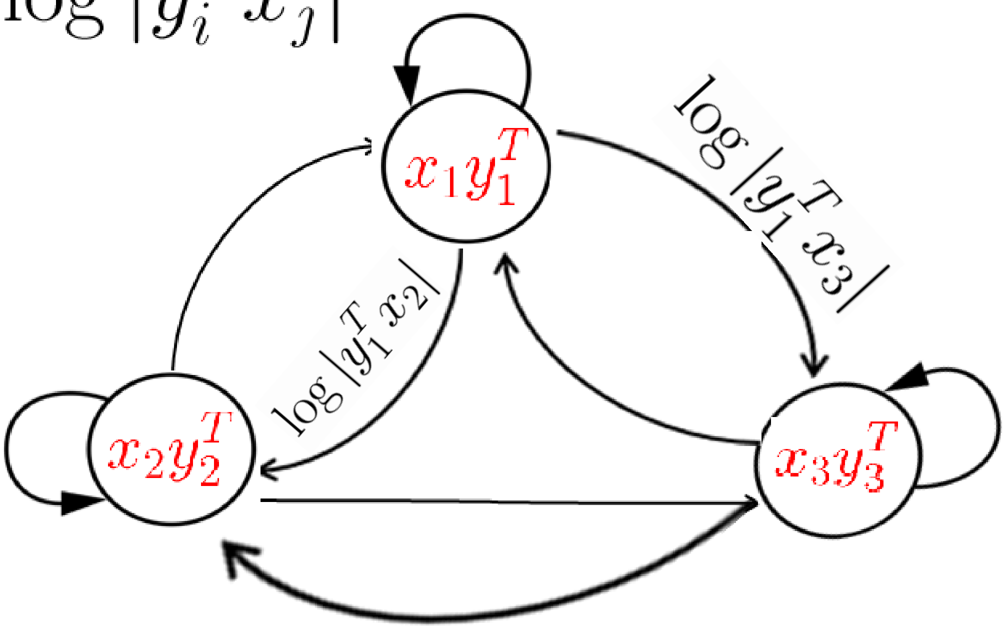
Take logs and apply Karp

$$\mathcal{A} = \{A_1, \ldots, A_m\} \quad A_i = x_i y_i^T$$

$\tilde{\mathbf{G}}_{\mathcal{A}}$ Complete directed graph on *m* nodes:

Nodes: matrices A_i

Edge weights: $w(e_{ij}) = \log |y_i^T x_j|$



$$\rho(\mathcal{A}) = e^{\lambda^* / k^*}$$

2 remarks, 2 examples, and that's it...

(1)

There is a different and independent characterization of the JSR of rank one matrices due to Gurvits and Samorodnitsky (2005), brought to our attention by Vincent Blondel

- Leads to solving a geometric program

(2)

Our approach based on MCMP can directly be used for deciding stability of rank one matrices under constrained switching

Length *m* may indeed be needed

- Counterexample to the claim by [Liu, Xiao '11]
  (claim was that only products of length at most 2 are needed)

$$\mathcal{A} = \{A_1, \ldots, A_m\}$$

$$A_1 = e_1 e_2^T, \quad A_2 = e_2 e_3^T, \quad \ldots \quad A_m = e_m e_1^T$$

- Spectral radius of all products is zero except for cyclic repetitions of

$$A_1 A_2 \cdots A_m$$

Hence $\rho(\mathcal{A}) = 1$

Common quadratic Lyapunov function can fail

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

- $\rho(\mathcal{A}) = 1$ (can be proven e.g. using our algorithm)
- An LMI searching for a common quadratic Lyapunov function can only prove $\rho(\mathcal{A}) \leq \sqrt{2}$
- In fact, CQLF does "as badly as possible" on this example

Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu/