Convexity and SOS-Convexity

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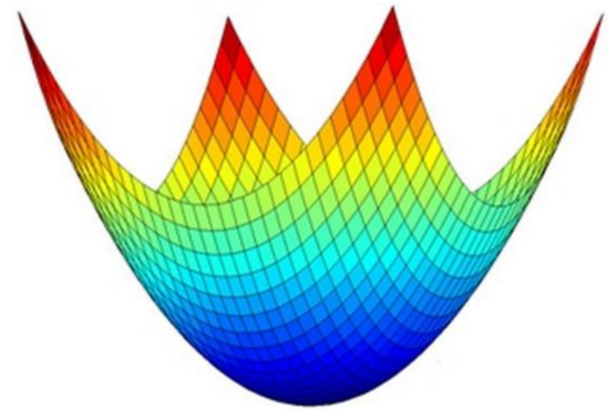
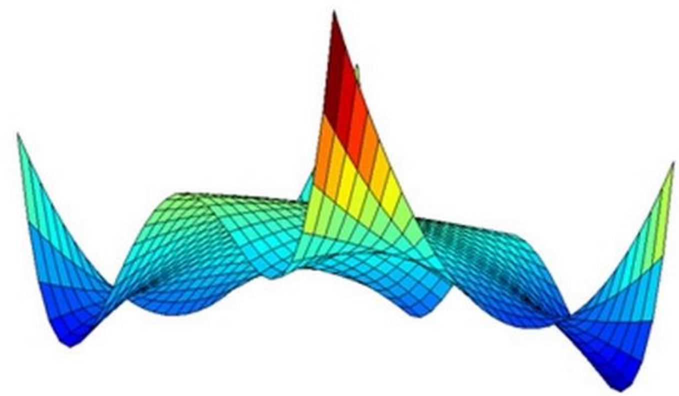
Deciding Convexity

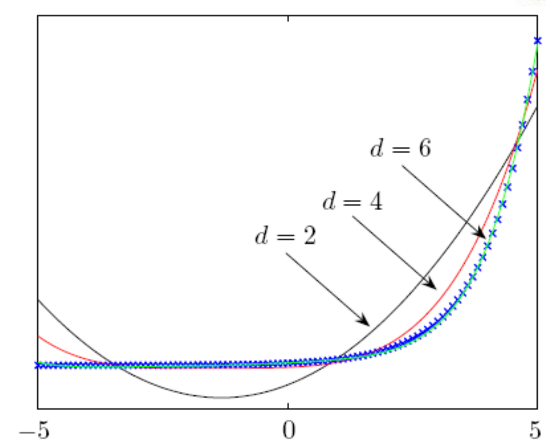
- Given a multivariate polynomial, how to ***decide*** if it is **convex**?
- How to ***search and optimize*** over a family of convex polynomials?





Applications:

- Global optimization
- Convex data fitting, convex envelopes, Lyapunov analysis, defining norms



[Magnani, Lall, Boyd]

Application in Lyapunov analysis

Suppose we are given *m* discrete dynamical systems:

$$\begin{aligned} x_{k+1} &= f_1(x) \\ &\vdots \\ x_{k+1} &= f_m(x) \end{aligned}$$

Suppose we can find a **convex** common Lyapunov function:

$$V(x) > 0, \quad V(f_i(x)) < V(x) \text{ for } i = 1, \ldots, m$$

Then, any other dynamical system

$$x_{k+1} = \tilde{f}(x), \text{ with } \tilde{f}(x) \in \text{conv}\{f_1(x), \ldots, f_m(x)\}$$

is also globally asymptotically stable.

Proof: $V(\tilde{f}(x)) = V(\sum_{i=1}^{m} \alpha_i f_i(x)) \leq \sum_{i=1}^{m} \alpha_i V(f_i(x)) < V(x)$

Outline

- (A word on) complexity of deciding convexity
- **SOS-Convexity**: an algebraic SDP based relaxation for convexity
  - Equivalent characterizations
  - Convex but not sos-convex polynomials
  - Complete characterization of the gap between convexity and sos-convexity
- An open problem!

Complexity of deciding convexity

- Input to the problem: an ordered list of the coefficients (all rational)
- Degree d odd: trivial
- d=2, i.e., $p(x)=x^TQx+q^Tx+c$: check if Q is PSD
- d=4, first interesting case
  - Question of N. Z. Shor:

"What is the complexity of deciding convexity of a multivariate polynomial of degree four?"

(appeared on a list of seven open problems in complexity of numerical optimization in 1992, [Pardalos, Vavasis])

- Our recent result: problem is **strongly NP-hard**
- Same result for (strong, strict, quasi, pseudo)-convexity

[Ahmadi, Olshevsky, Parrilo, Tsitsiklis,'10]

Nonnegativity and Sum of Squares

- Deciding if a polynomial is nonnegative (psd): NP-hard when $d \geq 4$
- Deciding if a polynomial is a sum of squares (sos): an SDP
- sos $\Rightarrow$ psd (obvious), but what about the converse?

Hilbert's 1888 Paper

psd=sos?

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

From Logicomix

Motzkin (1967):

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6$$

Robinson (1973):

$$\begin{aligned} R(x_1, x_2, x_3, x_4) \quad = \quad & x_1^2(x_1 - x_4)^2 + x_2^2(x_2 - x_4)^2 + x_3^2(x_3 - x_4)^2 \\ & +2x_1 x_2 x_3(x_1 + x_2 + x_3 - 2x_4) \end{aligned}$$

SOS-Convexity

Defn. ([Helton, Nie]): A polynomial $p(x) := p(x_1, \ldots, x_n)$ is ***sos-convex*** if its Hessian factors as

$$H(x) = M^T(x)M(x)$$

for a possibly nonsquare polynomial matrix $M(x)$.

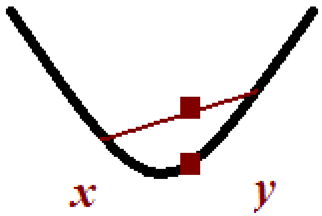
(We call such matrices an "sos-matrix".)

Equivalent Defn. $y^T H(x) y \quad sos$

- $p(x)$ sos-convex $\Rightarrow p(x)$ convex (obvious)
- Deciding sos-convexity: an SDP!

Equivalent characterizations of convexity

- Basic definition:

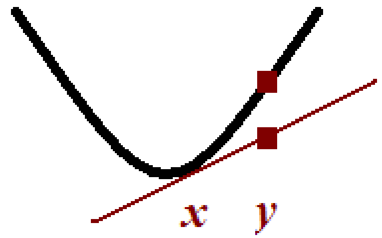


$$\lambda\, p(x) + (1-\lambda)\, p(y) \geq p(\lambda x + (1-\lambda) y)$$

$$\forall x, y\ \forall \lambda \in [0,1] \qquad (\lambda = \frac{1}{2}\ enough)$$

- First order condition:

$$p(y) \geq p(x) + \nabla^T p(x)(y-x) \quad \forall x, y$$



- Second order condition:

$$y^T H(x) y \geq 0 \quad \forall x, y$$

- Algebraic sos-based relaxations: **"simply replace ≥ with sos"**

Each condition can be SOS-ified

- Basic definition:

(A) $g_\lambda(x,y) = \lambda\, p(x) + (1-\lambda)\, p(y) - p(\lambda x + (1-\lambda) y)$ SOS $\forall \lambda \in [0,1]$

(A') $g_{\frac{1}{2}}(x,y) = \frac{1}{2} p(x) + \frac{1}{2} p(y) - p(\frac{1}{2} x + \frac{1}{2} y)$ SOS

- First order condition:

(B) $g_\nabla(x,y) = p(y) - p(x) - \nabla^T p(x)(y-x)$ SOS

- Second order condition:

(C) $g_{\nabla^2}(x,y) = y^T H(x) y$ SOS

(equivalent to SOS-Convexity: $H(x)$ an SOS-matrix $H(x) = M^T(x) M(x)$)

Thm: (A) $\Longleftrightarrow$ (A') $\Longleftrightarrow$ (B) $\Longleftrightarrow$ (C)

[Ahmadi, Parrilo, '10]

Convex polynomials that are not sos-convex?

A convex but not sos-convex poly would be a poly ***p(x)*** such that:

A $$g_\lambda(x,y) = \lambda\, p(x) + (1-\lambda)\, p(y) - p(\lambda x + (1-\lambda) y) \quad \forall \lambda \in [0,1]$$

B $$g_\nabla(x,y) = p(y) - p(x) - \nabla^T p(x)(y-x)$$

C $$g_{\nabla^2}(x,y) = y^T H(x) y$$

are **psd but not sos**.

The first counterexample

$$p(x) = 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^4 - 43x_1^4x_2^2x_3^2 - 35x_1^4x_3^4 + 3x_1^2x_2^4x_3^2 - 16x_1^2x_2^2x_3^4 + 24x_1^2x_3^6 + 16x_2^8 + 44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8$$

[Ahmadi, Parrilo, '09]

A homogeneous polynomial in 3 variables, of degree 8.

Claim:

- ***p(x)*** is **convex**: ***H(x)*** is PSD
- ***p(x)*** is **not** sos-convex: $H(x) \neq M^T(x)M(x)$

Convex but not SOS-Convex, lower degree?

$$
\begin{aligned}
p(x) = \; & x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 \\
& + 2(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_4^2 x_5^2 + x_4^2 x_6^2 + x_5^2 x_6^2) \\
& + \frac{1}{2}(x_1^2 x_4^2 + x_2^2 x_5^2 + x_3^2 x_6^2) \\
& + x_1^2 x_6^2 + x_2^2 x_4^2 + x_3^2 x_5^2 \\
& - (x_1 x_2 x_4 x_5 + x_1 x_3 x_4 x_6 + x_2 x_3 x_5 x_6)
\end{aligned}
$$

[Ahmadi, Parrilo, '10]

To explain where this comes from, need to talk first about

"**biquadratic forms**"

Biquadratic forms and biquadratic Hessian forms

- **Biquadratic form:** $y^T A(x) y,$

where $A(x)$ is a matrix whose entries are quadratic forms

- Example: the celebrated **Choi biquadratic form** $y^T C(x) y$ *with*

$$C(x) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_2^2 + 2x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_3^2 + 2x_1^2 \end{bmatrix}$$

$$y^T C(x) y \;\; \text{psd but not sos}$$

- **Biquadratic Hessian form:**

Special biquadratic form where $A(x)$ is a **valid Hessian**

- Choi matrix is *not* a valid Hessian: $\dfrac{\partial C_{1,1}(x)}{\partial x_3} = 0 \neq -x_3 = \dfrac{\partial C_{1,3}(x)}{\partial x_1}.$

From biquadratic forms to biquadratic Hessian forms

- We give a constructive procedure to go from *any* biquadratic form $y^T A(x) y$ to a biquadratic Hessian form $z^T H(x,y) z$ by **doubling the number of variables**, such that:

$$y^T A(x) y \;\; \text{psd} \Leftrightarrow z^T H(x,y) z \;\; \text{psd}$$

$$y^T A(x) y \;\; \text{sos} \Leftrightarrow z^T H(x,y) z \;\; \text{sos}$$

  - Proves NP-hardness of checking convexity of quartic forms!
    (nonnegativity of biquadratic forms known to be NP-hard; [Gurvits, '03], [Ling et al., '09], reduction from CLIQUE via Motzkin-Straus)

  - Gives an explicit method to construct **quartic convex but not sos-convex** polynomials!

Reduction on an instance

$$A(x) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_3^2 + 2x_1^2 \end{bmatrix}$$

$$\begin{bmatrix} y_1^2+2y_3^2+24x_1^2+4x_2^2+4x_3^2 & -y_1y_2+8x_1x_2 & -y_1y_3+8x_1x_3 & 2x_1y_1-x_2y_2-x_3y_3 & -x_2y_1 & 4x_1y_3-x_3y_1 \\ -y_1y_2+8x_1x_2 & 2y_1^2+y_2^2+24x_2^2+4x_1^2+4x_3^2 & -y_2y_3+8x_2x_3 & -x_1y_2+4x_2y_1 & -x_1y_1+2x_2y_2-x_3y_3 & -x_3y_2 \\ -y_1y_3+8x_1x_3 & -y_2y_3+8x_2x_3 & 2y_2^2+y_3^2+24x_3^2+4x_1^2+4x_2^2 & -x_1y_3 & -x_2y_3+4x_3y_2 & -x_1y_1-x_2y_2+2x_3y_3 \\ 2x_1y_1-x_2y_2-x_3y_3 & -x_1y_2+4x_2y_1 & -x_1y_3 & \boxed{x_1^2+2x_2^2}+24y_1^2+4y_2^2+4y_3^2 & \boxed{-x_1x_2}+8y_1y_2 & \boxed{-x_1x_3}+8y_1y_3 \\ -x_2y_1 & -x_1y_1+2x_2y_2-x_3y_3 & -x_2y_3+4x_3y_2 & \boxed{-x_1x_2}+8y_1y_2 & \boxed{x_2^2+2x_3^2}+24y_2^2+4y_1^2+4y_3^2 & \boxed{-x_2x_3}+8y_2y_3 \\ 4x_1y_3-x_3y_1 & -x_3y_2 & -x_1y_1-x_2y_2+2x_3y_3 & \boxed{-x_1x_3}+8y_1y_3 & \boxed{-x_2x_3}+8y_2y_3 & \boxed{2x_1^2+x_3^2}+24y_3^2+4y_1^2+4y_2^2 \end{bmatrix}$$

A 6x6 **Hessian** with quadratic form entries

Convex but not SOS-Convex Quartic

$$
\begin{aligned}
p(x) = {} & x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 \\
& + 2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_4^2x_5^2 + x_4^2x_6^2 + x_5^2x_6^2) \\
& + \frac{1}{2}(x_1^2x_4^2 + x_2^2x_5^2 + x_3^2x_6^2) \\
& + x_1^2x_6^2 + x_2^2x_4^2 + x_3^2x_5^2 \\
& - (x_1x_2x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_5x_6)
\end{aligned}
$$

[Ahmadi, Parrilo, '10]

Has the Hessian we just presented

Convexity = SOS-Convexity?

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

[Ahmadi, Parrilo, '10]

Forms

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

Thm: Every convex ternary quartic is SOS-convex.

[Ahmadi, Blekherman, Parrilo, '10]

Convex=SOS-convex for bivariate quartics

Thm ("the biform theorem"):

Let $f := f(u_1, u_2, v_1, \ldots, v_m)$

be a form that is quadratic in v for fixed u and a form (of however large degree) in u for fixed v.

Then, f is psd iff it is sos.

$$y^T H(x_1, x_2) y$$

Convex=SOS-convex for ternary quartic forms

- $p(x) = p(x_1,x_2,x_3)$ form of degree 4
- Let $y'H(x)y$ be its Hessian (biquadratic) form
- Biquadratic from in 6 variables can be psd but not sos (e.g., Choi)
- But $y'H(x)y$ has a special symmetry: $y'H(x)y=x'H(y)x$
- Even with this symmetry, can have a psd but not sos biquadratic form:

$$
\begin{aligned}
y'A(x)y = {} & 4y_1y_2x_1^2+5x_1x_2y_3^2-10x_1x_3y_1y_3+5x_3^2y_1y_2+9x_1x_3y_1^2 \\
& +13x_1x_3y_2^2+3x_1x_3y_3^2+9y_1y_3x_1^2-10y_2y_3x_1^2-11x_1x_3y_2y_3+23x_2^2y_1y_2 \\
& +5y_2x_1x_2y_3-10x_2x_3y_1^2+13x_2^2y_1y_3+5x_2x_3y_1y_2-11x_2x_3y_1y_3 \\
& +13x_2^2y_2y_3+6x_3^2y_1^2+3x_1x_3y_1y_2-5x_2x_3y_2y_3+13x_2x_3y_2^2+12x_2^2y_2^2 \\
& +12x_2^2y_1^2+12x_3^2y_2^2+12x_3^2y_3^2+7x_2x_3y_3^2+3x_3^2y_1y_3+7x_3^2y_2y_3 \\
& +12x_2^2y_3^2+12y_1^2x_1^2+12y_2^2x_1^2+6y_3^2x_1^2+31y_1y_2x_1x_2 \\
& +3y_1y_3x_1x_2+4y_1^2x_1x_2+23y_2^2x_1x_2
\end{aligned}
$$

[Ahmadi, Blekherman, Parrilo, '10]

- But $y'H(x)y$ has yet a little more structure…

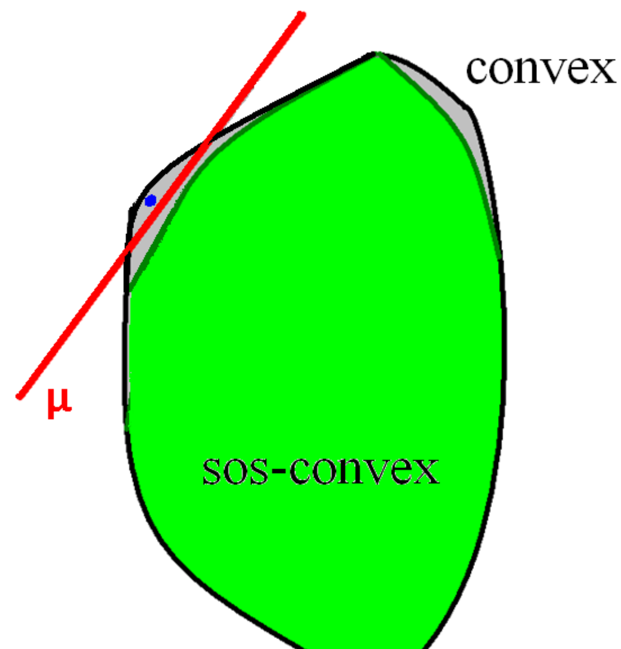
Minimal counterexample: n=3, d=6

$$
\begin{aligned}
f(x_1,x_2,x_3) \;=\;& 77x_1^6 - 155x_1^5x_2 + 445x_1^4x_2^2 + 76x_1^3x_2^3 + 556x_1^2x_2^4 + 68x_1x_2^5 \\
&+240x_2^6 - 9x_1^5x_3 - 1129x_1^3x_2^2x_3 + 62x_1^2x_2^3x_3 + 1206x_1x_2^4x_3 \\
&-343x_2^5x_3 + 363x_1^4x_3^2 + 773x_1^3x_2x_3^2 + 891x_1^2x_2^2x_3^2 - 869x_1x_2^3x_3^2 \\
&+1043x_2^4x_3^2 - 14x_1^3x_3^3 - 1108x_1^2x_2x_3^3 - 216x_1x_2^2x_3^3 - 839x_2^3x_3^3 \\
&+721x_1^2x_3^4 + 436x_1x_2x_3^4 + 378x_2^2x_3^4 + 48x_1x_3^5 - 97x_2x_3^5 + 89x_3^6
\end{aligned}
$$

Not sos-convex: $y^T H(x) y$ not sos

Convex: $y^T H(x) y$ psd

$(x_1^2 + x_2^2) \cdot y^T H_f(x) y$ sos



$$(x_1^2 + x_2^2) \cdot y^T H_f(x) y = \frac{1}{84} z^T Q z$$

$$z = [x_2 x_3^2 y_3,\ x_2 x_3^2 y_2,\ x_2 x_3^2 y_1,\ x_2^2 x_3 y_3,\ x_2^2 x_3 y_2,\ x_2^2 x_3 y_1,\ x_2^3 y_3,\ x_2^3 y_2,\ x_2^3 y_1,$$
$$x_1 x_3^2 y_3,\ x_1 x_3^2 y_2,\ x_1 x_3^2 y_1,\ x_1 x_2 x_3 y_3,\ x_1 x_2 x_3 y_2,\ x_1 x_2 x_3 y_1,\ x_1 x_2^2 y_3,\ x_1 x_2^2 y_2,\ x_1 x_2^2 y_1,$$
$$x_1^2 x_3 y_3,\ x_1^2 x_3 y_2,\ x_1^2 x_3 y_1,\ x_1^2 x_2 y_3,\ x_1^2 x_2 y_2,\ x_1^2 x_2 y_1,\ x_1^3 y_3,\ x_1^3 y_2,\ x_1^3 y_1]^T$$

$$Q_1 = \begin{bmatrix}
224280 & -40740 & 20160 & -81480 & 139692 & 93576 & -50540 & -27804 & -48384 & 0 & -29400 & -32172 & 21252 & 103404 \\
-40740 & 63504 & 36624 & 114324 & -211428 & -8316 & -67704 & 15372 & -47376 & 29400 & 0 & 16632 & 75936 & 15540 \\
20160 & 36624 & 121128 & 52920 & -27972 & -93072 & -42252 & -77196 & -58380 & 32172 & -16632 & 0 & 214284 & -57960 \\
-81480 & 114324 & 52920 & 482104 & -538776 & 36204 & -211428 & 362880 & -70644 & 19068 & 13524 & -588 & 179564 & 20258 \\
139692 & -211428 & -27972 & -538776 & 1020600 & -94416 & 338016 & -288120 & 188748 & -46368 & -33684 & -46620 & -113638 & -119112 \\
93576 & -8316 & -93072 & 36204 & -94416 & 266448 & -75348 & 216468 & 5208 & 3360 & -33432 & -31080 & -221606 & 254534 \\
-50540 & -67704 & -42252 & -211428 & 338016 & -75348 & 175224 & -144060 & 101304 & -20692 & 7826 & -4298 & -77280 & -108192 \\
-27804 & 15372 & -77196 & 362880 & -288120 & 216468 & -144060 & 604800 & 28560 & -35350 & -840 & -35434 & -132804 & 134736 \\
-48384 & -47376 & -58380 & -70644 & 188748 & 5208 & 101304 & 28560 & 93408 & -21098 & 2786 & -11088 & -104496 & -22680 \\
0 & 29400 & 32172 & 19068 & -46368 & 3360 & -20692 & -35350 & -21098 & 224280 & -40740 & 20160 & 35028 & 89964 \\
-29400 & 0 & -16632 & 13524 & -33684 & -33432 & 7826 & -840 & 2786 & -40740 & 63504 & 36624 & 51828 & -196476 \\
-32172 & 16632 & 0 & -588 & -46620 & -31080 & -4298 & -35434 & -11088 & 20160 & 36624 & 121128 & 29148 & -9408 \\
21252 & 75936 & 214284 & 179564 & -113638 & -221606 & -77280 & -132804 & -104496 & 35028 & 51828 & 29148 & 782976 & -463344 \\
103404 & 15540 & -57960 & 20258 & -119112 & 254534 & -108192 & 134736 & -22680 & 89964 & -196476 & -9408 & -463344 & 1167624 \\
267456 & -48132 & 27552 & -49742 & 78470 & 124236 & -100464 & 61404 & -90384 & 55524 & -50064 & -145908 & 41016 & -15456 \\
60872 & -25690 & -142478 & 22848 & -113820 & 259980 & -72996 & 237972 & 20412 & -95580 & -47964 & -27780 & -438732 & 514500 \\
37730 & -99036 & -10150 & -83160 & 473088 & 34188 & 167244 & 57120 & 159264 & 10752 & -93048 & -183540 & 230832 & -49980 \\
-119210 & 9170 & 81648 & 244356 & -41664 & -194124 & -9996 & 214368 & 19152 & -89184 & 2940 & -48480 & 204708 & -85344 \\
-116508 & 81564 & 26124 & 155832 & -308280 & -78180 & -74088 & 14616 & -49644 & 40320 & 87108 & 225456 & 135744 & 8568 \\
30660 & -14952 & 11844 & -21420 & 62604 & 14364 & 13608 & 1176 & 5124 & 59388 & -18144 & -99624 & 31332 & -178248 \\
35700 & 11340 & 52836 & -70788 & 86184 & 9396 & 12264 & -108024 & -11256 & 259056 & -86520 & -3528 & -19334 & 142128 \\
102156 & 2856 & 64536 & 22176 & -4200 & 77532 & -70896 & 54348 & -49616 & 72744 & -78876 & -144998 & 29316 & 23856 \\
-86100 & 47148 & 71820 & 230916 & -223692 & -131628 & -72156 & 59640 & -31416 & -75096 & -39396 & -44520 & 158508 & 308196 \\
95364 & -504 & -8412 & -23100 & 28140 & 81648 & -26768 & -25200 & -13944 & -51002 & -39228 & 71232 & 130298 & 298956 \\
11256 & 5208 & 32158 & -33264 & 45444 & 3122 & 6888 & -34440 & -5628 & 61320 & -19152 & 8988 & 18060 & -19467 \\
0 & -1344 & -3696 & -34692 & 33768 & 5964 & 9492 & -20244 & 5208 & -30072 & -9912 & 58884 & -50883 & 151956 \\
-51422 & 49056 & 32592 & 160370 & -229068 & -36792 & -68796 & 57708 & -39564 & 55944 & 31164 & -8008 & 141876 & -126483
\end{bmatrix}$$

$$Q_2 = \begin{bmatrix}
267456 & 60872 & 37730 & -119210 & -116508 & 30660 & 35700 & 102156 & -86100 & 95364 & 11256 & 0 & -51422 \\
-48132 & -25690 & -99036 & 9170 & 81564 & -14952 & 11340 & 2856 & 47148 & -504 & 5208 & -1344 & 49056 \\
27552 & -142478 & -10150 & 81648 & 26124 & 11844 & 52836 & 64536 & 71820 & -8412 & 32158 & -3696 & 32592 \\
-49742 & 22848 & -83160 & 244356 & 155832 & -21420 & -70788 & 22176 & 230916 & -23100 & -33264 & -34692 & 160370 \\
78470 & -113820 & 473088 & -41664 & -308280 & 62604 & 86184 & -4200 & -223692 & 28140 & 45444 & 33768 & -229068 \\
124236 & 259980 & 34188 & -194124 & -78180 & 14364 & 9396 & 77532 & -131628 & 81648 & 3122 & 5964 & -36792 \\
-100464 & -72996 & 167244 & -9996 & -74088 & 13608 & 12264 & -70896 & -72156 & -26768 & 6888 & 9492 & -68796 \\
61404 & 237972 & 57120 & 214368 & 14616 & 1176 & -108024 & 54348 & 59640 & -25200 & -34440 & -20244 & 57708 \\
-90384 & 20412 & 159264 & 19152 & -49644 & 5124 & -11256 & -49616 & -31416 & -13944 & -5628 & 5208 & -39564 \\
55524 & -95580 & 10752 & -89184 & 40320 & 59388 & 259056 & 72744 & -75096 & -51002 & 61320 & -30072 & 55944 \\
-50064 & -47964 & -93048 & 2940 & 87108 & -18144 & -86520 & -78876 & -39396 & -39228 & -19152 & -9912 & 31164 \\
-145908 & -27780 & -183540 & -48480 & 225456 & -99624 & -3528 & -144998 & -44520 & 71232 & 8988 & 58884 & -8008 \\
41016 & -438732 & 230832 & 204708 & 135744 & 31332 & -19334 & 29316 & 158508 & 130298 & 18060 & -50883 & 141876 \\
-15456 & 514500 & -49980 & -85344 & 8568 & -178248 & 142128 & 23856 & 308196 & 298956 & -19467 & 151956 & -126483 \\
610584 & 21840 & 127932 & -65184 & -323834 & 195636 & 90972 & 339794 & -100716 & -96012 & 24864 & -114219 & 36876 \\
21840 & 466704 & -110628 & -106820 & -54012 & -90636 & -111790 & -14952 & 63672 & 107856 & -67788 & 61404 & -88284 \\
127932 & -110628 & 1045968 & 142632 & -410592 & 171024 & 86268 & 176820 & 96516 & 199752 & 13524 & -70784 & -42756 \\
-65184 & -106820 & 142632 & 569856 & 21518 & -30156 & -159264 & -23016 & 410004 & -71484 & -62076 & -13860 & 74032 \\
-323834 & -54012 & -410592 & 21518 & 604128 & -229992 & -75516 & -297276 & 182385 & 75684 & -3528 & 94500 & 138432 \\
195636 & -90636 & 171024 & -30156 & -229992 & 169512 & 104748 & 187341 & -136332 & -145719 & 35364 & -94836 & 24612 \\
90972 & -111790 & 86268 & -159264 & -75516 & 104748 & 381920 & 147168 & -182595 & -36876 & 105504 & -24612 & -7560 \\
339794 & -14952 & 176820 & -23016 & -297276 & 187341 & 147168 & 346248 & -59304 & -137928 & 64932 & -90888 & 28392 \\
-100716 & 63672 & 96516 & 410004 & 182385 & -136332 & -182595 & -59304 & 776776 & 48972 & -98784 & 19152 & 180852 \\
-96012 & 107856 & 199752 & -71484 & 75684 & -145719 & -36876 & -137928 & 48972 & 494536 & -28392 & 118188 & -130200 \\
24864 & -67788 & 13524 & -62076 & -3528 & 35364 & 105504 & 64932 & -98784 & -28392 & 60984 & 0 & -3780 \\
-114219 & 61404 & -70784 & -13860 & 94500 & -94836 & -24612 & -90888 & 19152 & 118188 & 0 & 74760 & -65100 \\
36876 & -88284 & -42756 & 74032 & 138432 & 24612 & -7560 & 28392 & 180852 & -130200 & -3780 & -65100 & 194040
\end{bmatrix}.$$

Long story short…

psd=sos?

convex=sos-convex?

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

–A mere coincidence?

Nonnegativity and convexity: some similarities

- Both never hold for odd degree polynomials
- Nonnegativity=convexity for quadratic forms
- Both NP-hard for degree 4 or larger
- Nonnegativity = SOS if and only if convexity=SOS-convexity
- Perhaps there are more (and deeper) connections?

Open problem

Find a *convex* form that is not SOS.

- Blekherman (2009) has shown that they exist!
- Has implications for algebraic methods in polynomial optimization
- Convex forms are nonnegative
- sos-convex $\Rightarrow$ sos
  - So, such a polynomial must necessarily be convex but not sos-convex
- Our counterexamples pass this necessary condition but the polynomials themselves are sos

Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu