Complexity of Deciding Convexity in Polynomial Optimization

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Convexity

Rockafellar, ’93:

“In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.”

But how easy is it to distinguish between convexity and nonconvexity?

This talk:

-- Given a multivariate polynomial, can we efficiently **decide** if it is **convex**?

-- Given a basic semialgebraic set, can we efficiently **decide** if it is a **convex set**?

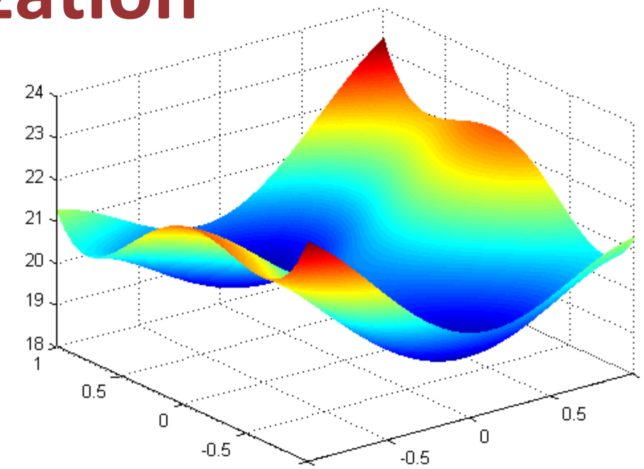
Convexity in optimization

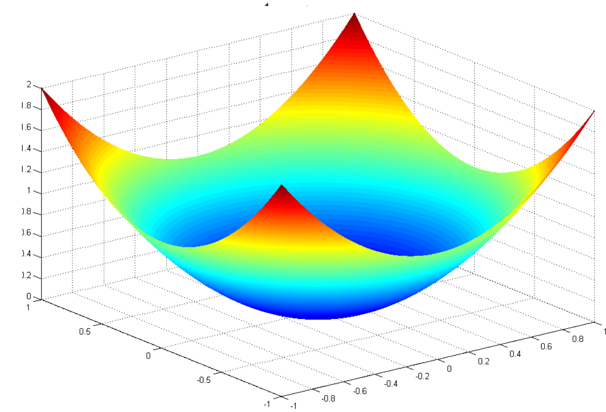
- Global optimization

  -- Minimizing polynomials is **NP-hard for degree ≥ 4**

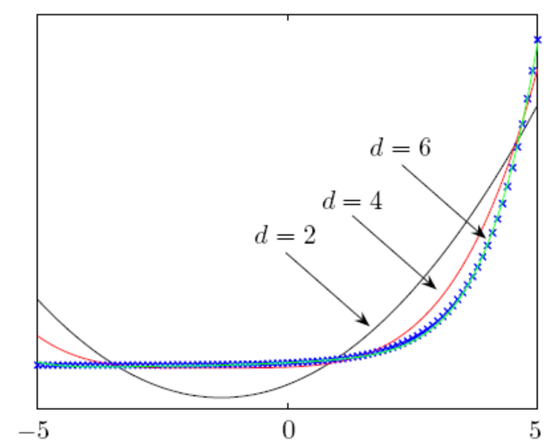
  -- But if polynomial is known to be convex, even simple gradient descent methods can find a global min

(often we check convexity based on “simple rules” from calculus of convex functions)





Applications: convex envelopes, convex data fitting, defining norms



[Magnani, Lall, Boyd]

Complexity of deciding convexity

- Input to the problem: an ordered list of the coefficients (all rational)
- Degree d odd: trivial
- d=2, i.e., $p(x)=x^TQx+q^Tx+c$: check if Q is PSD
- d=4, first interesting case
    - Question of N. Z. Shor:

"What is the complexity of deciding convexity of a multivariate polynomial of degree four?"

(appeared on a list of seven open problems in complexity of numerical optimization in 1992, [Pardalos, Vavasis])

Our main result: problem is **strongly NP-hard**

Agenda for the rest of the talk

1. Idea of the proof

2. Complexity of deciding variants of convexity

-- (strong, strict, pseudo,quasi)-convexity

NP-hardness of deciding convexity of quartics

Thm: Deciding convexity of quartic forms is strongly NP-hard.

- Reduction from problem of deciding **"nonnegativity of biquadratic forms"**

- **Biquadratic form:** $$b(x;y) = \sum_{i \leq j,\, k \leq l} \alpha_{ijkl} x_i x_j y_k y_l$$

- Can write any biquadratic form as $y^T A(x) y,$

where $A(x)$ is a matrix whose entries are quadratic forms

- Example: $y^T A(x) y,$ with $$A(x) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_3^2 + 2x_1^2 \end{bmatrix}$$

Sequence of reductions

STABLE SET

[Motzkin, Straus] $\frac{1}{\alpha(G)} = \min_{\substack{\sum x_i = 1 \\ x_i \geq 0}} x^T(A+I)x$

Minimizing an indefinite quadratic form over the simplex

[Gurvits], [Ling, Nie, Qi, Ye]

Nonnegativity of biquadratic forms

Our work

Convexity of quartic forms

The Hessian structure

- **Biquadratic form:** a form of the type $y^T A(x) y,$

where $A(x)$ is a matrix whose entries are quadratic forms

- Example: $y^T A(x) y,$ with $A(x) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_3^2 + 2x_1^2 \end{bmatrix}$

- **Biquadratic Hessian form:**

Special biquadratic form where $A(x)$ is a **valid Hessian**

- $A(x)$ above is ***not*** a valid Hessian:

$$\frac{\partial A_{1,1}(x)}{\partial x_3} = 0 \neq -x_3 = \frac{\partial A_{1,3}(x)}{\partial x_1}$$

From biquadratic forms to biquadratic Hessian forms

- We give a constructive procedure to go from *any* biquadratic form $y^T A(x) y$ to a biquadratic Hessian form $z^T H(x,y) z$ by **doubling the number of variables**, such that:

$$y^T A(x) y \;\; \text{psd} \Leftrightarrow z^T H(x,y) z \;\; \text{psd}$$

- In fact, we construct the polynomial ***f(x,y)*** that has ***H(x,y)*** as its Hessian directly

- Let's see this construction…

The main reduction

Thm: Given any biquadratic form $b(x;y)$,

Let $[C(x,y)]_{ij} := \frac{\partial b(x;y)}{\partial x_i \partial y_j}$ Let $\gamma := \max |\mathrm{coeff}(C(x,y))|$

Let

$$f(x,y) := b(x;y) + \underbrace{\frac{n^2\gamma}{2}\Big(\sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} y_i^4 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} x_i^2 x_j^2 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} y_i^2 y_j^2\Big)}_{g(x,y)}$$

Then

$$b(x;y)\text{psd} \iff f(x,y)\text{convex}$$

$$H(x,y) = H_b(x,y) + H_g(x,y)$$

$$b(x;y) \ \text{psd} \Leftrightarrow z^T H(x,y) z \ \ \text{psd}$$

Observation on the Hessian of a biquadratic form

$$b(x;y) = \sum_{i \leq j,\, k \leq l} \alpha_{ijkl} x_i x_j y_k y_l$$

$$[A(x)]_{ij} := \frac{\partial b(x;y)}{\partial y_i \partial y_j} \qquad [B(y)]_{ij} := \frac{\partial b(x;y)}{\partial x_i \partial x_j}$$

$$\frac{1}{2} y^T A(x) y = b(x;y) \qquad \frac{1}{2} x^T B(y) x = b(x;y)$$

$$[C(x,y)]_{ij} := \frac{\partial b(x;y)}{\partial x_i \partial y_j}$$

$$H_b(x,y) = \begin{bmatrix} B(y) & C(x,y) \\ C^T(x,y) & A(x) \end{bmatrix}$$

Proof of correctness of the reduction

Start with $b(x;y)$,

Let $[C(x,y)]_{ij} := \dfrac{\partial b(x;y)}{\partial x_i \partial y_j}$ Let $\gamma := \max |\text{coeff}(C(x,y))|$

$$f(x,y) := b(x;y) + \underbrace{\frac{n^2\gamma}{2}\Big(\sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} y_i^4 + \sum_{\substack{i,j=1,\ldots,n\\ i<j}} x_i^2 x_j^2 + \sum_{\substack{i,j=1,\ldots,n\\ i<j}} y_i^2 y_j^2\Big)}_{g(x,y)}$$

$$H(x,y) = H_b(x,y) + H_g(x,y)$$

Claim: $b(x;y)$ psd $\Leftrightarrow z^T H(x,y) z$ psd

$$H(x,y) = \begin{bmatrix} B(y) & C(x,y) \\ C^T(x,y) & A(x) \end{bmatrix} + \frac{n^2\gamma}{2}\begin{bmatrix} H_g^{11}(x) & 0 \\ 0 & H_g^{22}(y) \end{bmatrix}$$

Reduction on an instance

$$A(x) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_3^2 + 2x_1^2 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{y_1^2+2y_3^2}+24x_1^2+4x_2^2+4x_3^2 & \boxed{-y_1y_2}+8x_1x_2 & \boxed{-y_1y_3}+8x_1x_3 & 2x_1y_1-x_2y_2-x_3y_3 & -x_2y_1 & 4x_1y_3-x_3y_1 \\ \boxed{-y_1y_2}+8x_1x_2 & \boxed{2y_1^2+y_2^2}+24x_2^2+4x_1^2+4x_3^2 & \boxed{-y_2y_3}+8x_2x_3 & -x_1y_2+4x_2y_1 & -x_1y_1+2x_2y_2-x_3y_3 & -x_3y_2 \\ \boxed{-y_1y_3}+8x_1x_3 & \boxed{-y_2y_3}+8x_2x_3 & \boxed{2y_2^2+y_3^2}+24x_3^2+4x_1^2+4x_2^2 & -x_1y_3 & -x_2y_3+4x_3y_2 & -x_1y_1-x_2y_2+2x_3y_3 \\ 2x_1y_1-x_2y_2-x_3y_3 & -x_1y_2+4x_2y_1 & -x_1y_3 & \boxed{x_1^2+2x_2^2}+24y_1^2+4y_2^2+4y_3^2 & \boxed{-x_1x_2}+8y_1y_2 & \boxed{-x_1x_3}+8y_1y_3 \\ -x_2y_1 & -x_1y_1+2x_2y_2-x_3y_3 & -x_2y_3+4x_3y_2 & \boxed{-x_1x_2}+8y_1y_2 & \boxed{x_2^2+2x_3^2}+24y_2^2+4y_1^2+4y_3^2 & \boxed{-x_2x_3}+8y_2y_3 \\ 4x_1y_3-x_3y_1 & -x_3y_2 & -x_1y_1-x_2y_2+2x_3y_3 & \boxed{-x_1x_3}+8y_1y_3 & \boxed{-x_2x_3}+8y_2y_3 & \boxed{2x_1^2+x_3^2}+24y_3^2+4y_1^2+4y_2^2 \end{bmatrix}$$

A 6x6 **Hessian** with quadratic form entries

Other notions of interest in optimization

- Strong convexity
  - "Hessian uniformly bounded away from zero"
  - Appears e.g. in convergence analysis of Newton-type methods

$\Downarrow$

- Strict convexity
  - "curve strictly below the line"
  - Guarantees uniqueness of optimal solution

$\Downarrow$

- Convexity

$\Downarrow$

- Pseudoconvexity
  - "Relaxation of first order characterization of convexity"
  - Any point where gradient vanishes is a global min

$\Downarrow$

- Quasiconvexity
  - "Convexity of sublevel sets"
  - Deciding convexity of basic semialgebraic sets

Summary of complexity results

property vs. degree	1	2	odd ≥ 3	even ≥ 4
strong convexity	no	P	no	strongly NP-hard
strict convexity	no	P	no	strongly NP-hard
convexity	yes	P	no	strongly NP-hard
pseudoconvexity	yes	P	P	strongly NP-hard
quasiconvexity	yes	P	P	strongly NP-hard

Quasiconvexity

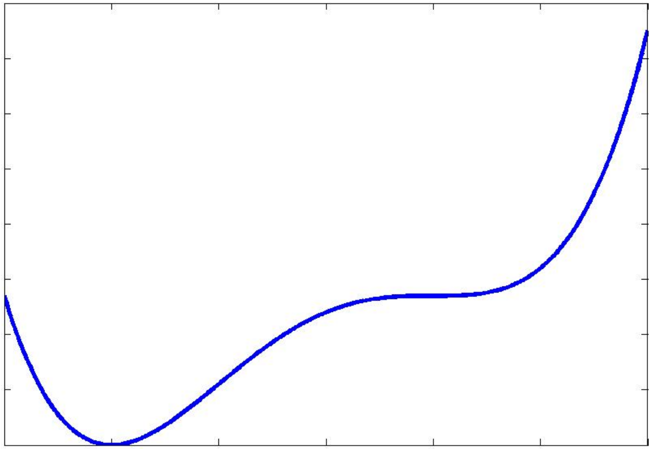
A multivariate polynomial $p(x)=p(x_1,\ldots,x_n)$ is *quasiconvex* if all its sublevel sets

$$\mathcal{S}_\alpha := \{x \in \mathbb{R}^n \mid p(x) \leq \alpha\}$$

are convex.

Convexity $\Rightarrow$ Quasiconvexity

(converse fails)



- Deciding quasiconvexity of polynomials of **even degree 4 or larger** is **strongly NP-hard**
- Quasiconvexity of **odd degree** polynomials can be decided in **polynomial time**

Quasiconvexity of even degree forms

Lemma: A homogeneous polynomial $p(x)$ of even degree d is quasiconvex if and only if it is convex.

Proof:

- A homogeneous quasiconvex polynomial is nonnegative
- The unit sublevel sets of $p(x)$ and $p(x)^{1/d}$ are the same convex set
- $p(x)^{1/d}$ is the Minkowski norm defined by this convex set and hence a convex function
- A convex nonnegative function raised to a power d larger than one remains convex

Corollaries:

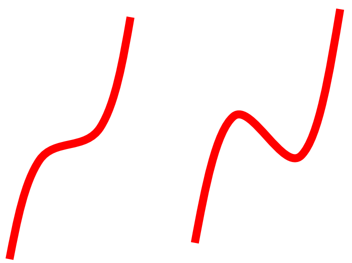
-- Deciding quasiconvexity is NP-hard

-- Deciding convexity of basic semialgebraic sets is NP-hard

Quasiconvexity of odd degree polynomials

Thm: The sublevel sets of a quasiconvex polynomial *p(x)* of odd degree are halfspaces.

Proof:



- Show super level sets must also be convex sets
- Only convex set whose complement is also convex is a halfspace

Thm: A polynomial *p(x)* of odd degree d is quasiconvex iff it can be written as

$$p(x) = h(\xi^T x)$$

$\xi \in \mathbb{R}^n, h(t)$ monotonic univariate polynomial of degree d

This representation can be checked in polynomial time

What can we do?

One possibility: natural relaxation based on sum of squares

Defn. ([Helton, Nie]): A polynomial $p(x) := p(x_1, \ldots, x_n)$ is ***sos-convex*** if its Hessian factors as

$$H(x) = M^T(x)M(x)$$

for a possibly nonsquare polynomial matrix $M(x)$.

- $p(x)$ sos-convex $\Rightarrow p(x)$ convex (obvious)
- Deciding sos-convexity: a **semidefinite program (SDP)**

Gap between convexity and sos-convexity

A convex form that is not sos-convex:

$$
\begin{aligned}
p(x) = {} & x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 \\
& + 2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_4^2x_5^2 + x_4^2x_6^2 + x_5^2x_6^2) \\
& + \frac{1}{2}(x_1^2x_4^2 + x_2^2x_5^2 + x_3^2x_6^2) \\
& + x_1^2x_6^2 + x_2^2x_4^2 + x_3^2x_5^2 \\
& - (x_1x_2x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_5x_6)
\end{aligned}
$$

[Ahmadi, Parrilo, ‘10]

convex=sos-convex?

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

[Ahmadi, Parrilo, ‘10]

[Ahmadi, Blekherman, Parrilo, ‘10]

Messages to take home…

property vs. degree	1	2	odd ≥ 3	even ≥ 4
strong convexity	no	P	no	strongly NP-hard
strict convexity	no	P	no	strongly NP-hard
convexity	yes	P	no	strongly NP-hard
pseudoconvexity	yes	P	P	strongly NP-hard
quasiconvexity	yes	P	P	strongly NP-hard

- SOS-Convexity: a powerful SDP relaxation for convexity

convex=sos-convex?

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu/