Approximation of the Joint Spectral Radius via Multiple Lyapunov Functions on Path-Complete Graphs

Amir Ali Ahmadi

Laboratory for Information and Decision Systems
Massachusetts Institute of Technology

Joint work with:

Raphaël Jungers, Pablo A. Parrilo, Mardavij Roozbehani

(UC Louvain) (MIT) (MIT)

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JSR and switched linear systems

Given a finite set of *nxn* matrices $\mathcal{A} := \{A_1, ..., A_m\}$

decide Absolute Asymptotic Stability (AAS) of:

$$x_{k+1} = A_{\sigma(k)} x_k \qquad x_{k+1} \in \text{co}\mathcal{A}\ x_k$$

$$\sigma : \mathbb{Z}_+ \to \{1, \ldots, m\}$$

If only have one matrix: $\mathcal{A} = \{A\}$

Spectral Radius

$$\rho(A) = \lim_{t \to \infty} ||A^t||^{1/t}$$

Asymptoticly stable iff $\rho(A) < 1$

Joint Spectral Radius (JSR):

$$\rho(\mathcal{A}) = \lim_{k \to \infty} \max_{\sigma \in \{1, \ldots, m\}^k} \|A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

The switched systems are ASS iff $\rho < 1$

Computation of JSR

Testing $\rho(\mathcal{A}) \leq 1$? undecidable [Blondel, Tsitsiklis]

Unless P=NP, no approximation $\hat{\rho}$ of ρ that satisfies

$$|\hat{\rho} - \rho| \leq \epsilon\rho$$

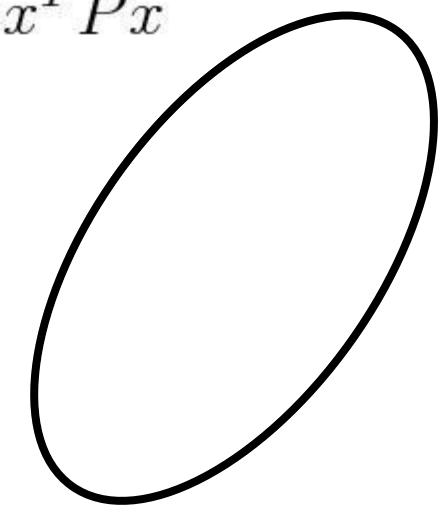
[Blondel, Tsitsiklis]

in a number steps polynomial in size of $\mathcal{A}$ and ϵ.

- **Our goal:** efficiently compute "good" upper bounds with guaranteed accuracy
- **Our approach:** use **semidefinite programming (SDP)** to search for **multiple Lyapunov functions** that together prove stability

Common Lyapunov function approach

- If AAS, a convex common Lyapunov function always exists
- Approximate this function with "simple" functions where search is feasible
- Common quadratic Lyapunov function $V(x) = x^T P x$



$$\begin{aligned} P &\succ 0 \\ \gamma^2 A_i^T P A_i &\preceq P \quad \forall i = 1, \ldots, m. \end{aligned}$$

$$\frac{1}{\sqrt{n}} \hat{\rho}_{CQ}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{CQ}(\mathcal{A})$$

[Ando, Shih]
[Blondel, Nesterov, Theys]

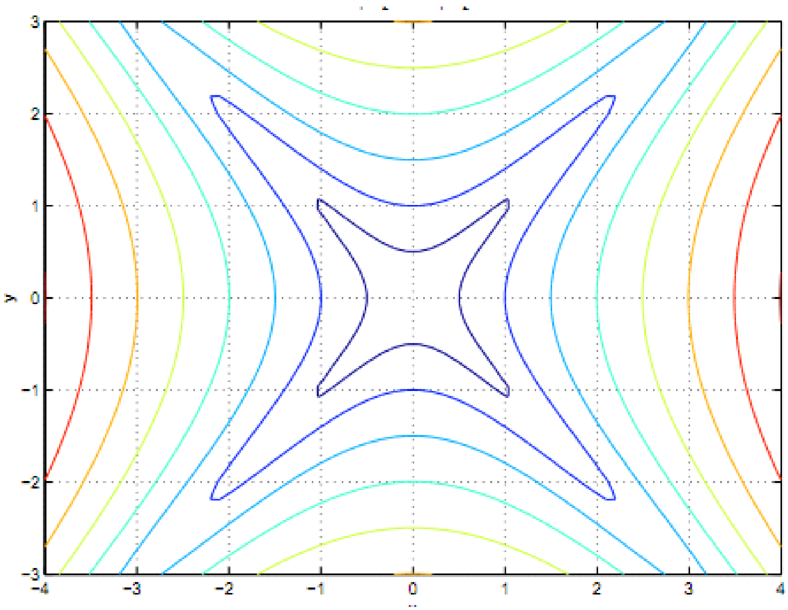
Common SOS Lyapunov functions

$$V(x) \text{ sos}$$

$$V(x) - V(A_i x) \text{ sos } i = 1, \ldots, m$$

$$\frac{1}{\sqrt[2d]{\eta}} \hat{\rho}_{SOS,2d}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{SOS,2d}(\mathcal{A})$$

$$\eta = \min\{m, \binom{n+d-1}{d}\}$$



[Parrilo, Jadbabaie]

Multiple Lyapunov functions

- Can we do better with more than one Lyapunov function?

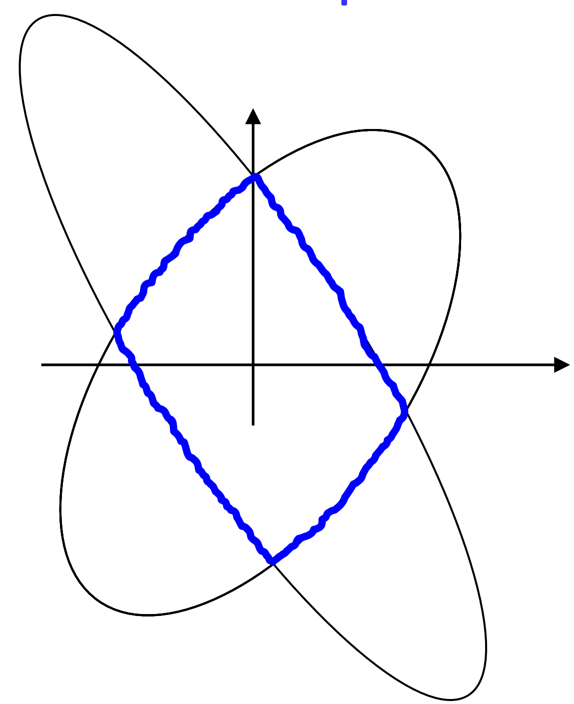
- How?

- Consider the SDP:

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

$\Rightarrow \quad \rho \leq r$

max-of-quadratics



A numerical example

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$ [Ando, Shih,'98]

$$\rho(\mathcal{A}) = 1$$

- Common quadratic does "as badly as possible":

$$\hat{\rho}_{CQ}(\mathcal{A}) = \sqrt{2}$$

- Even worse, CQ applied to products of higher length, never gets the JSR exactly:

$$\hat{\rho}_{CQ}^{\frac{1}{t}}(\mathcal{A}^t) = 2^{1/2t}$$

- But the LMIs on the last slide (max of 2 quadratics) get the JSR exactly

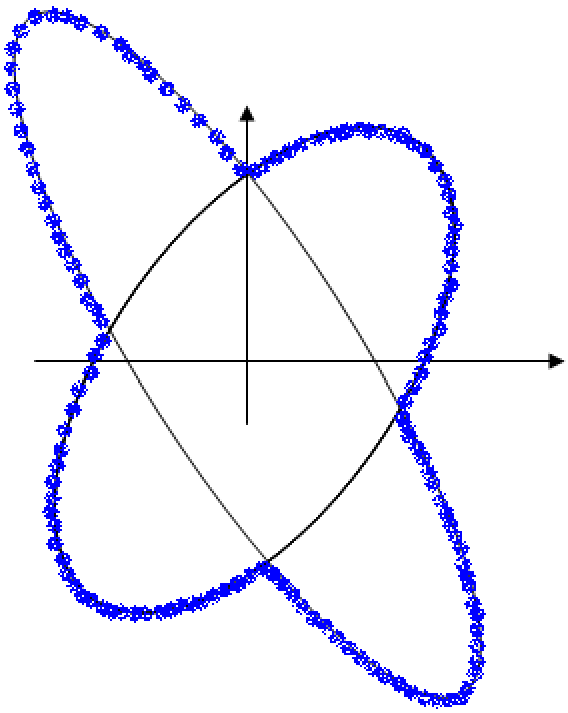
Multiple Lyapunov functions

- Consider another SDP:

$$
\begin{array}{lll}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_1, \\
A_1^T P_1 A_1 & \preceq & r^2 P_2, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

min-of-quadratics



Can bring in products of higher length

- e.g., a common quadratic for $A_1^2, A_1A_2, A_2A_1, A_2^2$ clearly implies AAS

- But so does the following strange SDP:

$$
\begin{array}{lcl}
\min_{r\in\mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P A_1 & \preceq & r^2 P, \\
(A_2A_1)^T P (A_2A_1) & \preceq & r^4 P, \\
(A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\
P & \succeq & 0.
\end{array}
$$

$\Rightarrow \quad \rho \leq r$

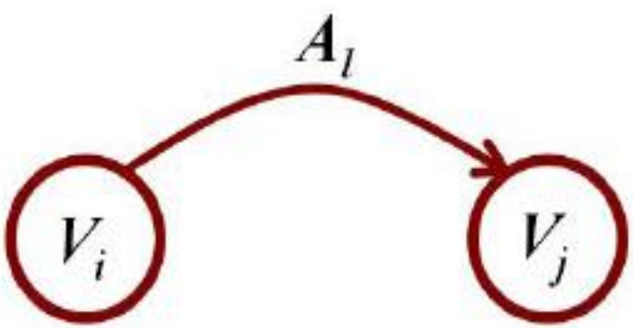
[AAA, Jungers, Parrilo, Roozbehani]

Natural questions

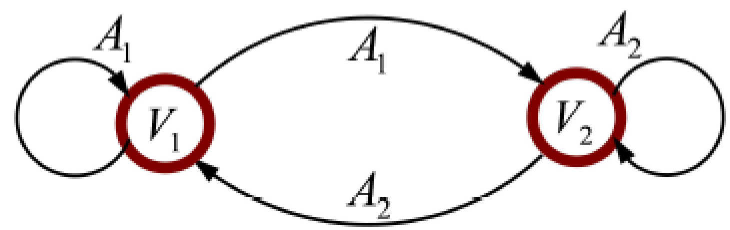
- Where do these conditions come from?
- Can we give a unifying framework?
- How do different methods compare in terms of conservatism?
- What about approximation guarantees for these methods?
(i.e., converse Lyapunov results)

Representation of Lyapunov inequalities via labeled graphs

$$V_j(A_l x) \leq V_i(x) \quad \forall x \in \mathbb{R}^n$$



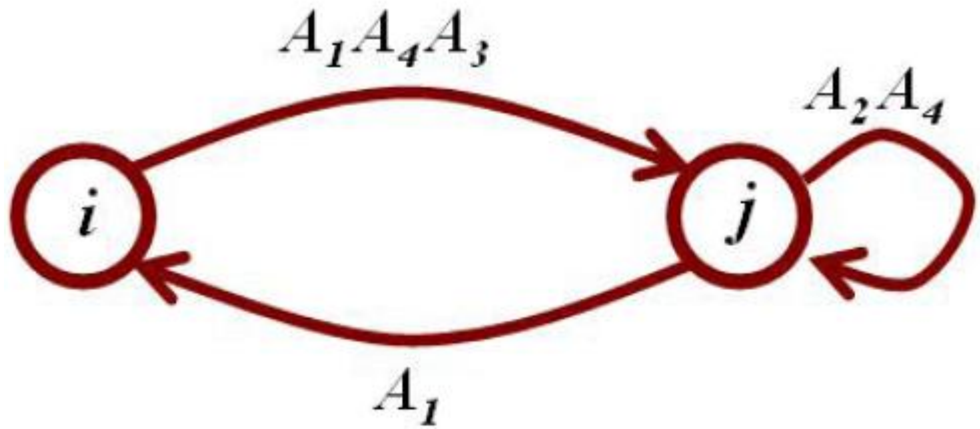
$$\begin{aligned} A_1^T P_1 A_1 &\preceq P_1 \\ A_2^T P_1 A_2 &\preceq P_2 \\ A_1^T P_2 A_1 &\preceq P_1 \\ A_2^T P_2 A_2 &\preceq P_2 \end{aligned}$$



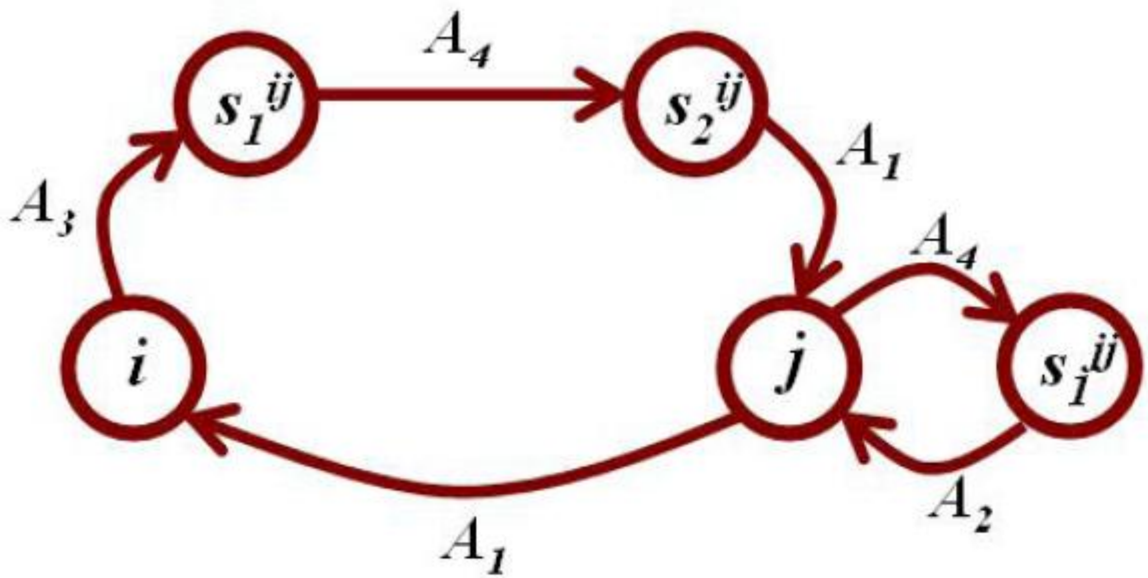
- What property of the graph implies stability?

Graph expansion

Graph $G(N,E)$



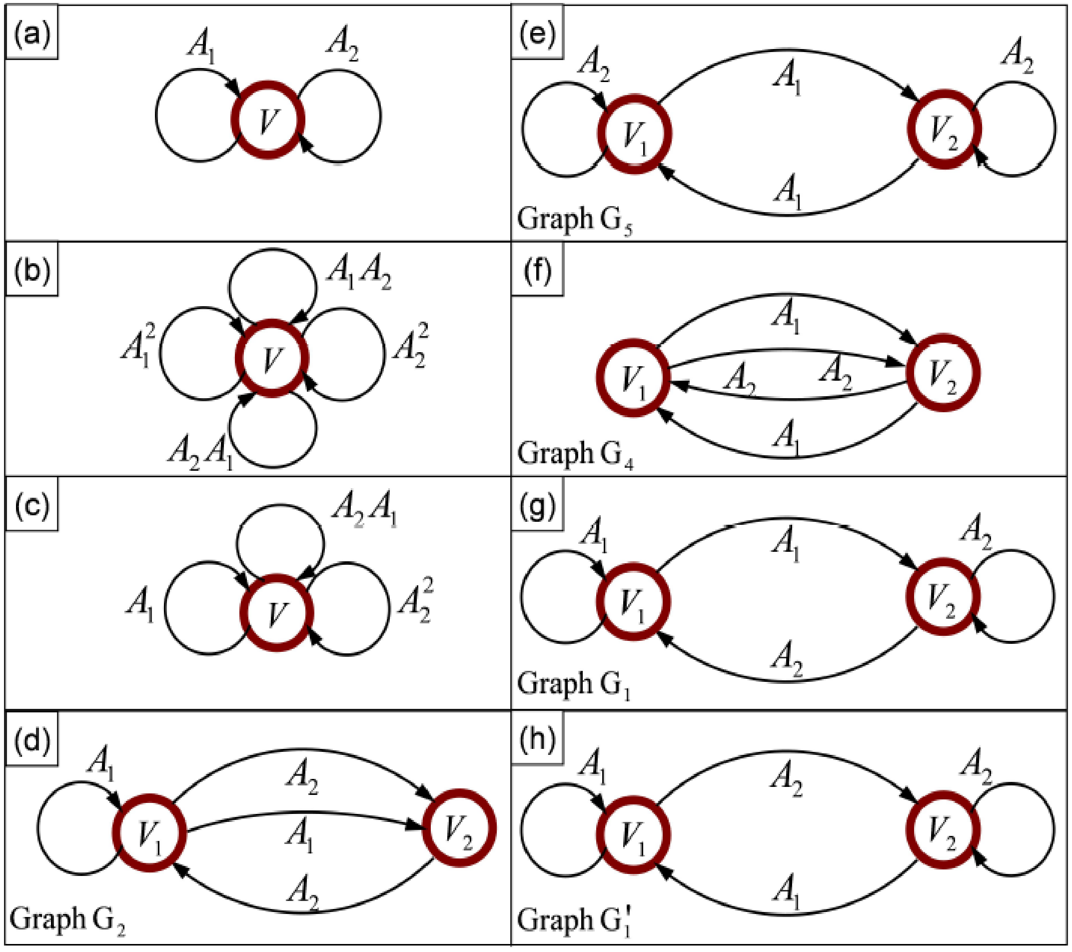
Expanded Graph $G^e(N^e,E^e)$



Path-complete graphs

Definition. A labeled directed graph $G(N,E)$ is path-complete if for every word of finite length there is an associated directed path in its expanded graph $G^e(N^e,E^e)$.

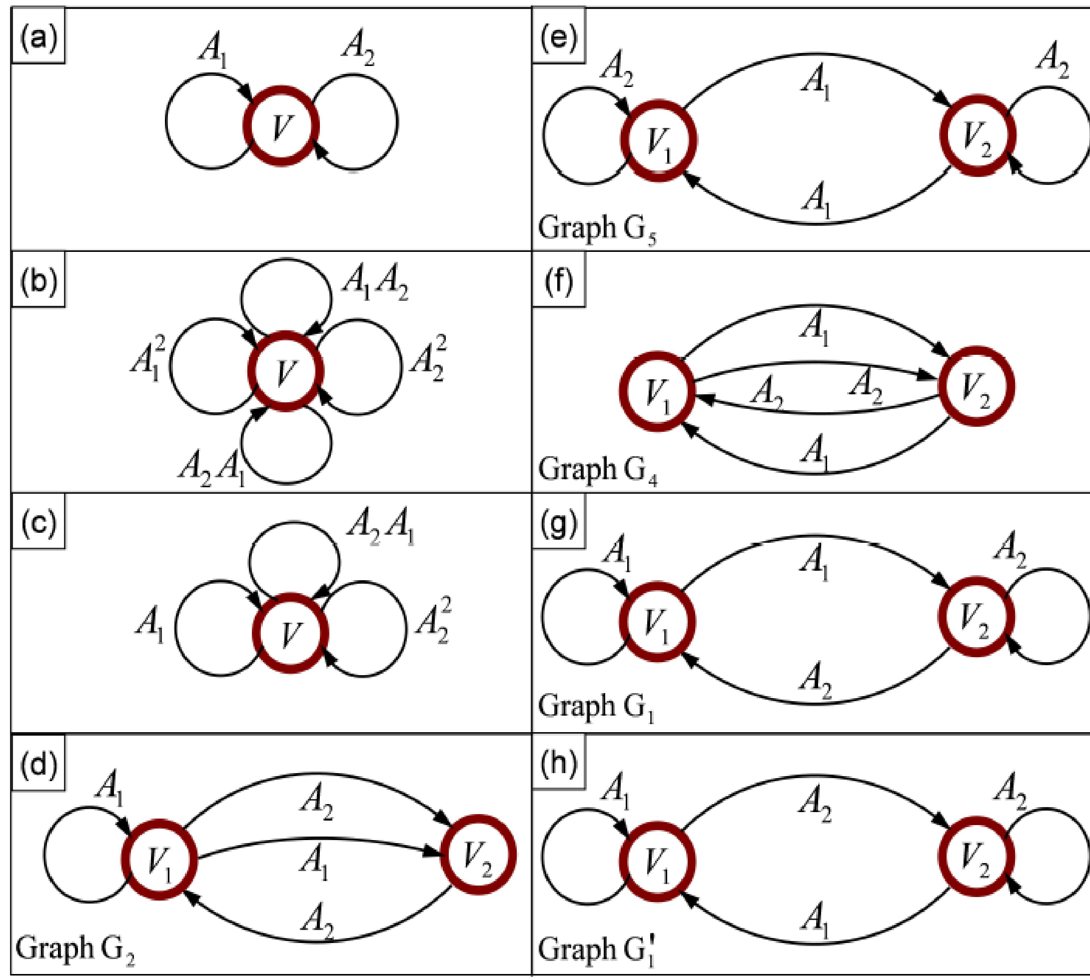
- Path-completeness can be checked with standard algorithms in **automata theory**



Path-complete graphs and stability

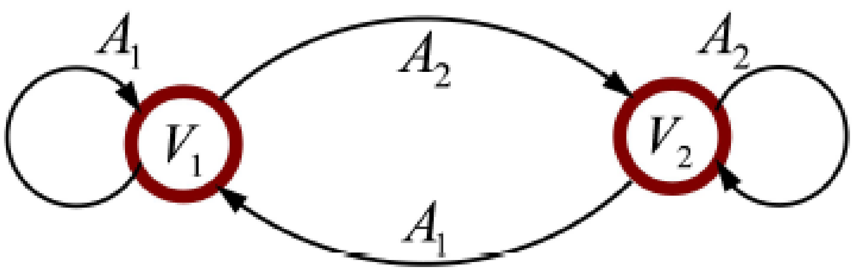
Theorem. If Lyapunov functions satisfying Lyapunov inequalities associated with **any path-complete graph** are found, then the switched system is absolutely asymptotically stable.

- Provides immediate proofs of stability for various techniques in the literature
- Introduces many new ones

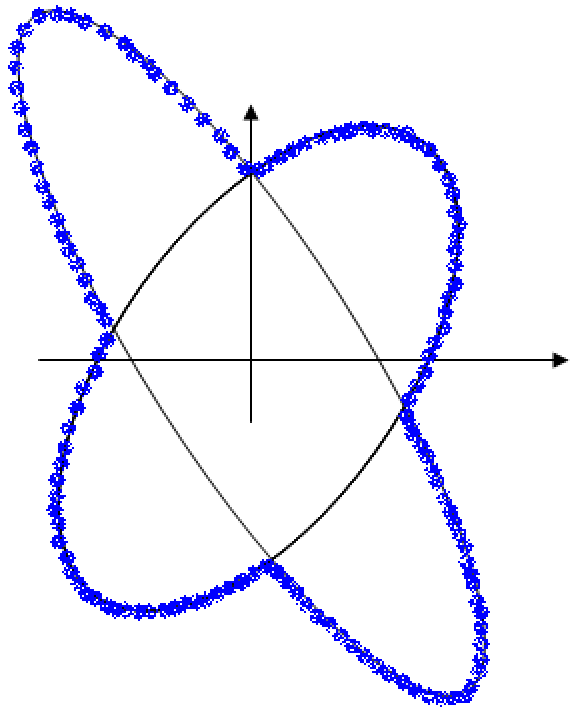


Quick proofs

$$
\begin{aligned}
A_1^T P_1 A_1 &\preceq P_1 \\
A_2^T P_2 A_2 &\preceq P_1 \\
A_1^T P_1 A_1 &\preceq P_2 \\
A_2^T P_2 A_2 &\preceq P_2
\end{aligned}
$$



min-of-quadratics

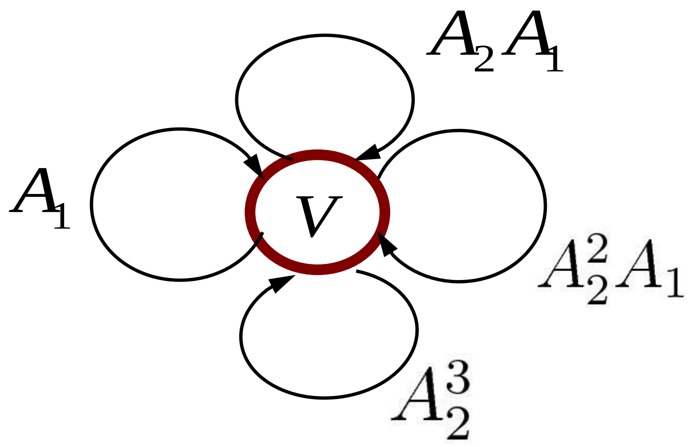


Quick proofs for previous methods

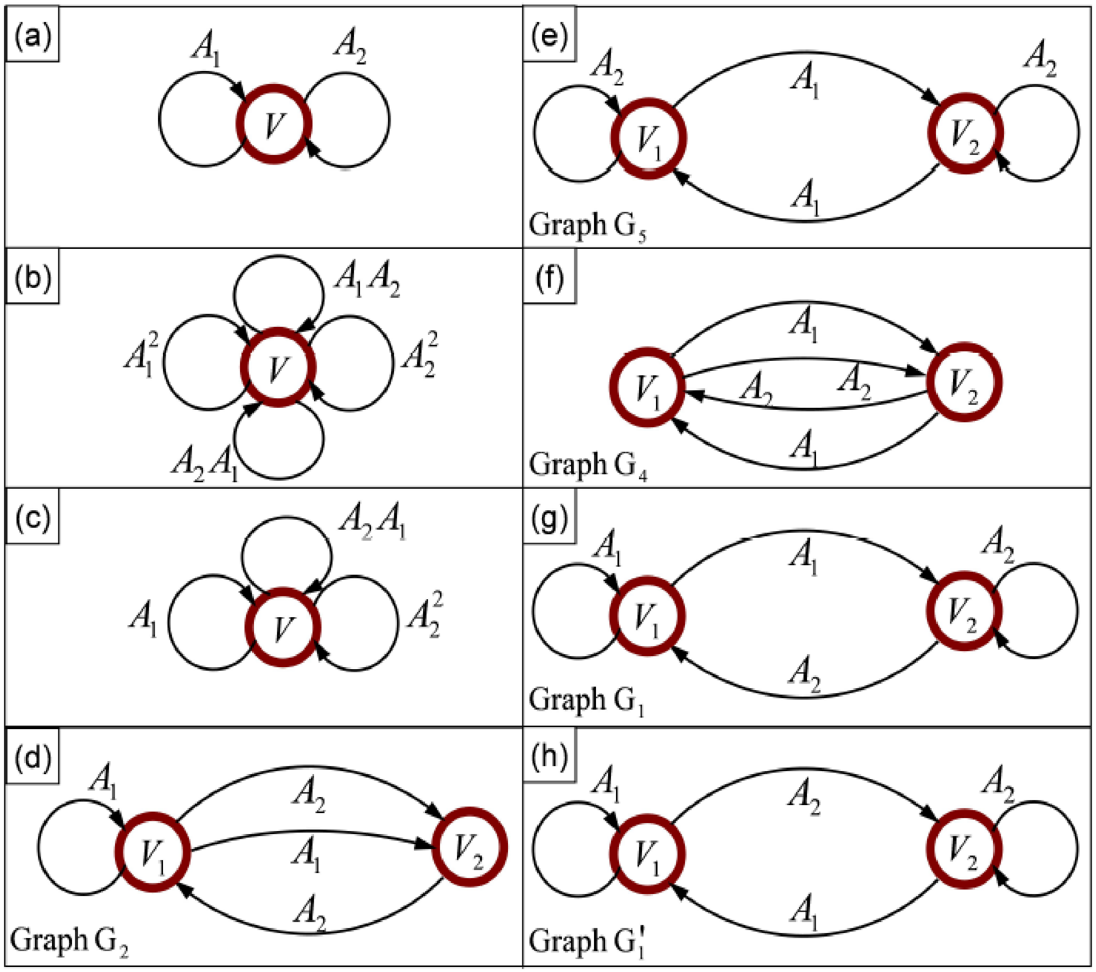
- Max/min-of-quadratics
- Composite quadratic Lyapunov functions [Hu, Lin] [Goebel, Hu, Teel]
- "Path dependent Lyapunov functions" [Lee, Dullerud]
  - De Bruijn graph
- "Parameter dependent Lyapunov functions" [Daafouz, Bernussou]

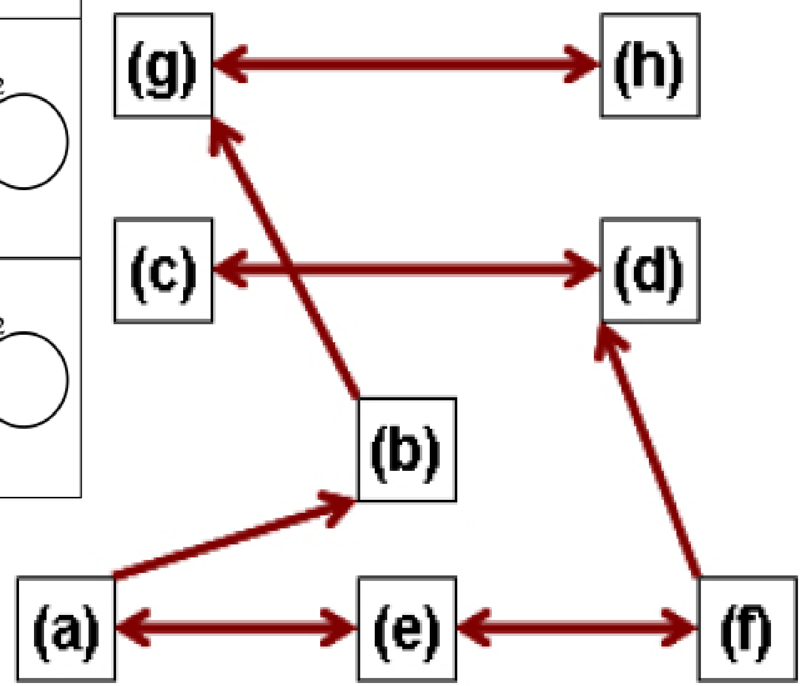
Example of a new method

$$
\begin{aligned}
A_1^T P A_1 &\preceq P \\
(A_2 A_1)^T P (A_2 A_1) &\preceq P \\
(A_2^2 A_1)^T P (A_2^2 A_1) &\preceq P \\
(A_2^3)^T P (A_2^3) &\preceq P
\end{aligned}
$$



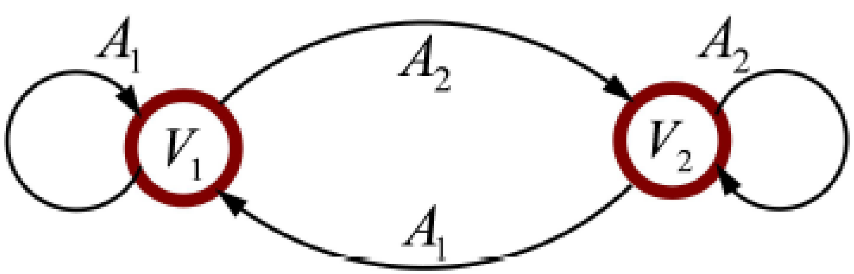
Comparison of different methods



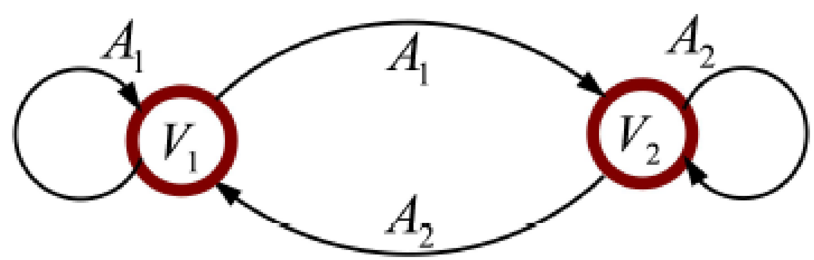


Comparison of different methods (ctnd.)

Theorem (Duality). If the bound obtained by graph G is invariant under transposing the matrices, then this bound equals the bound obtained by the graph G', which is obtained by reversing the edges of G.

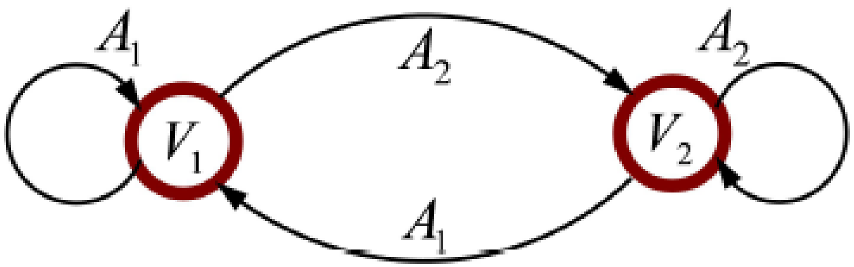


min-of-quadratics

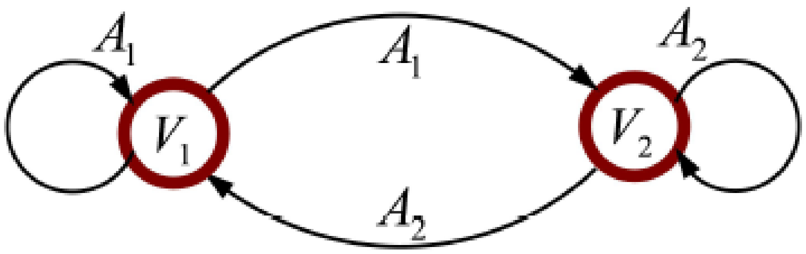


max-of-quadratics

Approximation guarantees



min-of-quadratics



max-of-quadratics

$$\frac{1}{\sqrt[4]{n}}\hat{\rho}_{\mathcal{V}^2,G_1}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2,G_1}(\mathcal{A})$$

- Independent of m
- Tighter than known SOS bounds

Converse Lyapunov theorem for max/min-of-quadratics

Theorem. Given any desired accuracy

$$\frac{1}{\sqrt[2l]{n}}\hat{\rho}_{\mathcal{V}^2,G}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2,G}(\mathcal{A})$$

We can explicitly construct a graph G (**with m^{l-1} nodes**) that results in a max/min-of-quadratics Lyapunov function and the corresponding **SDP** achieves the accuracy

Path-complete graphs and stability

Theorem. If Lyapunov functions satisfying Lyapunov inequalities associated with **any path-complete graph** are found, then the switched system is absolutely asymptotically stable.

Conjecture: converse is also true.

If true, it would essentially mean that any convex optimization approach for proving stability of arbitrary switched systems comes from this framework

Messages to take home…

- The joint spectral radius characterizes the maximum rate of growth of the trajectories of a switched linear system
- We gave a hierarchy of asymptotically tight SDP based approximation algorithms
- Unifying framework of path-complete graphs for use of multiple Lyapunov functions
- Comparison of a family of path-complete graphs
- Approximation guarantees for families of path-complete graphs

Open questions

- Analogues of path-complete graphs and JSR for continuous time?
- Given a special family of matrices, what is the “best” path-complete graph to use?
- Other families of powerful path-complete graphs that we haven’t discovered yet?
- If a set of LMIs imply stability, do they come from a path-complete graph?

Thank you for your attention!

Questions?

Want to know more?

Come to my poster! Or visit:

http://aaa.lids.mit.edu/