Complexity of 10 Decision Problems in Continuous Time Dynamical Systems

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Given a **polynomial** vector field:

$$\dot{x} = f(x) \quad (f : \mathbb{R}^n \to \mathbb{R}^n)$$

Decide if the origin is locally (or globally) asymptotically stable.

Example.

$$\begin{aligned} \dot{x} &= -y + \frac{3}{2}x^2 - \frac{1}{2}x^3 \\ \dot{y} &= 3x - y \end{aligned}$$

- **How difficult is this problem?**
- We study the complexity of testing 10 of the most basic qualitative properties of ODEs in control, such as the one above

Linear systems: pretty much the only well-understood case

$$\dot{x} = Ax$$

- d=1 (linear systems): decidable, and polynomial time
  - **Iff A is Hurwitz** (i.e., eigenvalues of A have negative real part)
  - **Quadratic Lyapunov functions** always exist
  - A polynomial time algorithm is the following:
    - Solve $A'P + PA = -I$
    - Check if P is positive definite (e.g. by checking positivity of all *n* leading principal minors)

What about higher degree?

- Classical converse Lyapunov theorem:
  - a.s. $\Rightarrow$ C^1 Lyapunov function, but how to find one?
- Numerous computational techniques available:
  - Lyapunov's linearization test, polytopic Lyapunov functions (LP), piecewise quadratic Lyapunov functions (SDP), polynomial Lyapunov functions (SOS), …
- None known to be exact – ugly things can happen to innocent vector fields:

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= -y \end{aligned}$$

GAS, but no polynomial Lyapunov function (of any degree) exists!

[AAA, Krstic, Parrilo- CDC'12]

Can there be ***any*** easily checkable necessary and sufficient condition?

Thm: Deciding (local or global) asymptotic stability of cubic vector fields is strongly NP-hard.

[AAA - ACC’12]

Implication: Unless P=NP, there cannot be *any* polynomial time (or even pseudo-polynomial time) algorithm.

(In particular suggests that the tests based on polynomially-sized convex programs can never be exact.)

Our NP-hardness results in this paper

1. Inclusion of the unit ball in region of attraction (d=3)
2. Invariance of the unit ball (d=3)
3. Invariance of a quartic semialgebraic set (d=1)
4. Boundedness of trajectories (d=3)
5. Stability in the sense of Lyapunov (d=4)
6. Local attractivity (d=3)
7. Local collision avoidance (d=4)
8. Existence of a quadratic Lyapunov function (d=3)
9. Existence of a stabilizing control law (d=3)
10. Local asymptotic stability for trigonometric vector fields (d=4)

We only prove #10 here...

> **Thm:** Deciding asymptotic stability of quartic trigonometric vector fields is strongly NP-hard.

Reduction from: **ONE-IN-THREE 3SAT**

$$\overset{1}{(x_1} \vee \overset{0}{\bar{x}_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3})$$

(satisfiable)

$$x_1 = 1, x_2 = 1, x_3 = 0$$

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$

(unsatisfiable)

Goal: Design a quartic trig differential equation which is a.s. iff ONE-IN-THREE 3SAT has no solution

Sketch of the reduction

$$(b_1 \vee b_2 \vee b_3) \wedge (b_1 \vee \bar{b}_4 \vee b_5)$$

$$t(x) = \Sigma_{i=1}^{n} \sin^2(x_i)(1 - \sin(x_i))^2$$
$$+ [\sin(x_1) + \sin(x_2) + \sin(x_3) - 1]^2$$
$$+ [\sin(x_1) + (1 - \sin(x_4)) + \sin(x_5) - 1]^2$$

$$x_i = \sin^{-1}(b_i^*)$$

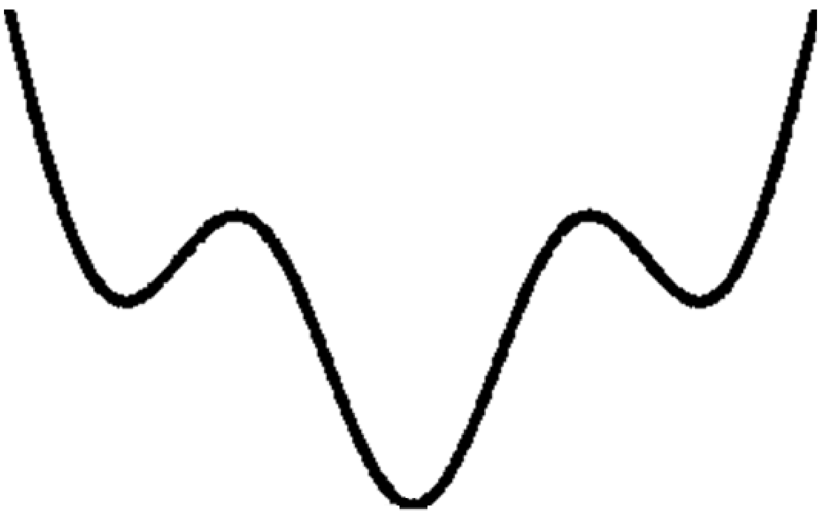
$$t_h(z) = \Sigma_{i=1}^{n} \sin^2(x_i)(\sin(y) - \sin(x_i))^2$$
$$+ [\sin(x_1)\sin(y) + \sin(x_2)\sin(y) \ldots$$
$$+ \sin(x_3)\sin(y) - \sin^2(y)]^2$$
$$+ [\sin(x_1)\sin(y) + (\sin^2(y) - \sin(x_4)\sin(y)) \ldots$$
$$+ \sin(x_5)\sin(y) - \sin^2(y)]^2$$

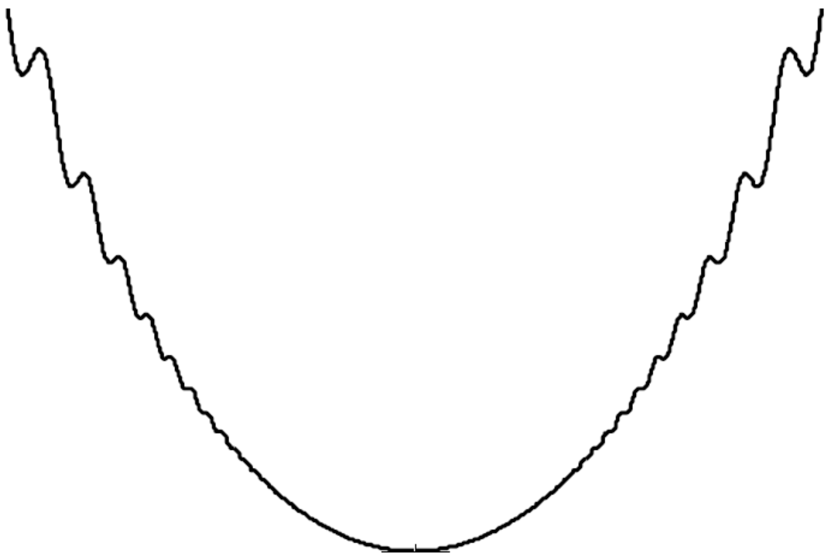
$$x_i(\alpha) = \sin^{-1}(\alpha b_i^*)$$
$$y(\alpha) = \sin^{-1}(\alpha)$$

Lemma: $t_h(z)$ is locally positive definite if and only if the ONE-IN-THREE 3SAT instance is unsatisfiable.

Lemma we will use

Lemma: If the ONE-IN-THREE 3SAT instance is unsatisfiable, there exists a neighborhood around the origin in which $\nabla t_h(z)$ does not vanish.





cannot happen

Sketch of the reduction (cont'd)

Thm: Let $t_h(z)$ be as before. Then,

$$\dot{z} = -\nabla t_h(z)$$

is locally asymptotically stable if and only if $t_h(z)$ is locally positive definite.

Proof: $\Leftarrow$

$$\dot{t}_h(z) = \langle \nabla t_h(z), -\nabla t_h(z) \rangle = -\|\nabla t_h(z)\|^2 \leq 0$$

Apply Lyapunov's theorem…

$\Rightarrow$

- $\nabla t_h(z)$ does not vanish locally because…
- $t_h(z)$ must be locally positive semidefinite because…
- If $t_h(z)$ were to vanish, its gradient would vanish also…

Proof summary

ONE-IN-THREE 3SAT

$$(b_1 \vee b_2 \vee b_3) \wedge (b_1 \vee \bar{b}_4 \vee b_5)$$

↓

Local positivity of a quartic trigonometric function

$$\begin{aligned} t_h(z) = \ & \Sigma_{i=1}^{n} \sin^2(x_i)(\sin(y) - \sin(x_i))^2 \\ & + [\sin(x_1)\sin(y) + \sin(x_2)\sin(y) \dots \\ & + \sin(x_3)\sin(y) - \sin^2(y)]^2 \\ & + [\sin(x_1)\sin(y) + (\sin^2(y) - \sin(x_4)\sin(y)) \dots \\ & + \sin(x_5)\sin(y) - \sin^2(y)]^2 \end{aligned}$$

↓

Asymptotic stability of a quartic trig vector field

$$z := (x, y)$$

$$\dot{z} = -\nabla t_h(z)$$

Summary of NP-hardness results

1. Inclusion of the unit ball in region of attraction (d=3)
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Note: decidability is not known

- Conjecture of Arnold (1976):
asymptotic stability of polynomial vector fields is **undecidable.**

Obviously, there is a connection between complexity of the problem and the type of Lyapunov functions we can expect

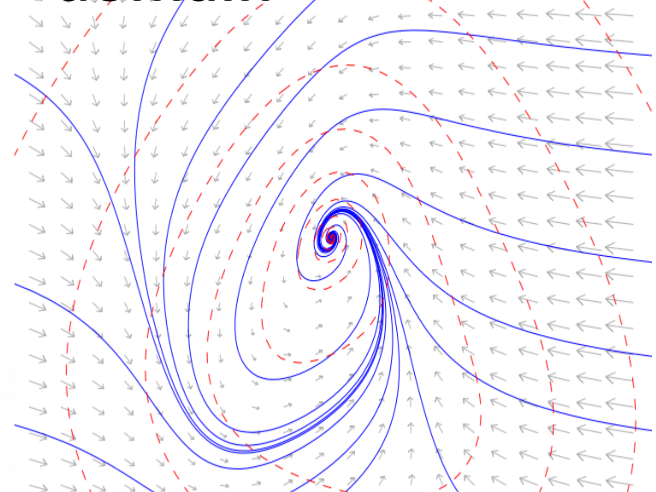
Fact: Existence of **polynomial Lyapunov functions**, together with a **computable upper bound** on their degree, implies decidability (quantifier elimination)

- Similarly, negative "time complexity" results rule out existence of polynomial Lyapunov functions that we can efficiently search for

A two-way bridge

Discrete and combinatorial domain

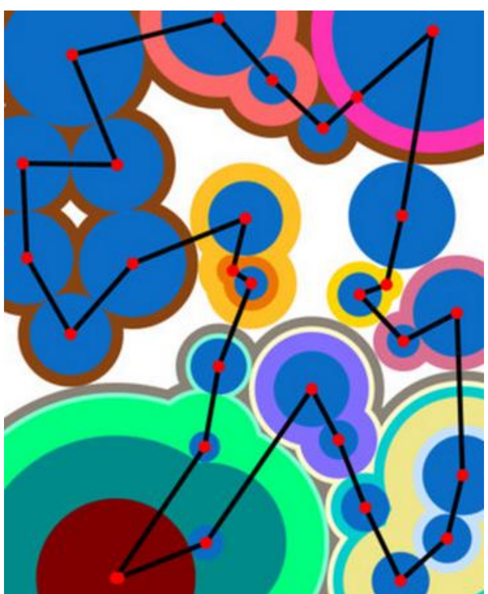
Continuous dynamics domain

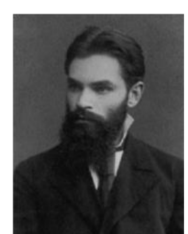


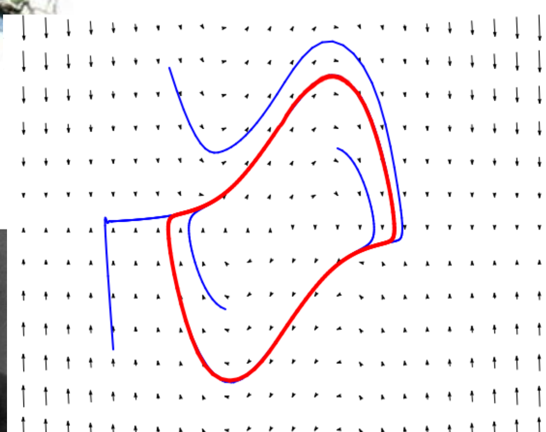
3SAT

$$(\overset{1}{x_1} \vee \overset{0}{\bar{x}_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3})$$

TSP







Thank you for your attention!

Questions?

Want to know more?

http://aaa.lids.mit.edu/