On Higher Order Derivatives of Lyapunov Functions

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Lyapunov stability analysis

\[ \dot{x} = f(x) \quad (f : \mathbb{R}^n \rightarrow \mathbb{R}^n) \]

**Goal:** prove local or global asymptotic stability

Asymptotic stability established if we find a Lyapunov function

\[ V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \]

with derivative

\[ \dot{V}(x) = \left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle \]

such that

\[ V(x) > 0 \]
\[ \dot{V}(x) < 0 \]
Algorithmic search for Lyapunov functions

- Advances in **convex optimization** and in particular **semidefinite programming (SDP)** have led to algorithmic techniques for Lyapunov functions.
- Can parameterize certain classes of Lyapunov functions and pose the search as a **convex feasibility problem**.
  - Quadratic Lyapunov functions for linear systems (SDP)
  - Piecewise quadratic Lyapunov functions (SDP)
  - Surface Lyapunov functions (SDP)
  - Polytopic Lyapunov functions (LP)
  - SOS polynomial Lyapunov functions (SDP)
  - Relaxations for pointwise maximum or minimum of quadratics (SDP)

**Key:** Lyapunov inequalities are **affine** in the parameters of $V$.

\[ V(x) > 0 \]
\[ \dot{V}(x) < 0 \]
This is great, but...

- We can only search for a restricted class of (low complexity) functions
- “Simple” dynamics may have “complicated” Lyapunov functions
  e.g. 
  \[
  \begin{align*}
  \dot{x} &= -x + xy \\
  \dot{y} &= -y 
  \end{align*}
  \]
  is GAS but has **no (global) polynomial Lyapunov function of any degree**!
- Existence of Lyapunov functions that we can efficiently search for are almost **never necessary** for stability
- Often Lyapunov functions that we find are too complicated, e.g., polynomial of high degree or piecewise quadratics with many pieces
- Recall: a polynomial in \( n \) variables and degree \( d \) has \( \binom{n+d}{d} \) coefficients! (≈460 for \( n=5, d=6 \))
  - Explore simpler parameterizations?
- Can we relax the conditions of Lyapunov’s theorem to prove stability with simpler functions?
Main motivation

**Q:** If all we need is $V \rightarrow 0$, why require a monotonic decrease?

![Diagram showing Complicated $V$ and Simpler $V$](image)
Main motivation

- Similarly in continuous time:

  Complicated $V$

  Simpler $V$
Questions of interest

- **Q1**: Conditions that allow the Lyapunov functions to increase locally but guarantee their convergence to zero in the limit?
- **Q2**: Can the search for non-monotonic Lyapunov functions satisfying the new conditions be cast as a convex program?
- **Q3**: Connections between non-monotonic Lyapunov functions and standard Lyapunov functions?

- **Discrete time (DT) – idea: use higher order differences**
  - **Q1, Q2, Q3**: [Ahmadi, Parrilo ’08]

- **Continuous time (CT) – idea: use higher order derivatives**
  - **Q1**: [Butz ’69], [Heinen, Vidyasagar ‘70], [Gunderson ’71], [Meigoli, Nikravesh ’09]

Focus of this (2-page) ACC paper:

Simple observation on **Q2** and **Q3**
CT: relaxing monotonicity via higher order order derivatives

\[
\dot{x} = f(x)
\]

- Allow \( \dot{V} > 0 \) at some points in space
- Limit the rate at which \( V \) can increase by imposing constraints on higher order derivatives

\[
\dot{V}(x) = \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle
\]

\[
\ddot{V}(x) = \left\langle \frac{\partial \dot{V}(x)}{\partial x}, f(x) \right\rangle
\]

\[
\dddot{V}(x) = \left\langle \frac{\partial \ddot{V}(x)}{\partial x}, f(x) \right\rangle
\]

...  
- Cheap to compute
- All linear in \( V \)!
First two derivatives alone don’t help for inferring stability

- A condition of type
  \[ \min\{\dot{V}(x), \ddot{V}(x)\} < 0 \quad \forall x \neq 0 \]
  is vacuous. In particular,
  \[ \tau \ddot{V}(x) + \dot{V}(x) < 0 \quad \forall x \neq 0, \]
  \[ \text{for some } \tau \geq 0 \]
  is never satisfied unless
  \[ \dot{V}(x) < 0 \text{ everywhere } [\text{Butz,'69}]. \]
But the first three derivatives help

**Thm** (Butz, 1969): existence of a positive Lyapunov function $V$ and nonnegative scalars $\tau_{1,2}$ satisfying

$$\tau_2 \dddot{V} + \tau_1 \dddot{V} + \dot{V} < 0 \quad (*)$$

implies (global) asymptotic stability.

- Proof by comparison lemma type arguments and basic facts about ODEs
- (*) imposed on complements of compact sets implies Lagrange stability (boundedness of trajectories) [Heinen, Vidyasagar ’70]
- (*) is nonconvex (bilinear in decision vars. $V$ and $\tau_i$)

We will get around this issue shortly
Condition is non-vacuous

\[ \tau_2 \dddot{V} + \tau_1 \ddot{V} + \dot{V} < 0 \quad (*) \]

- An example by Butz:

\[ x = Ax \]

\[ A = \begin{bmatrix} -4 & -5 \\ 1 & 0 \end{bmatrix} \quad \text{Eig: } -2 \pm j \]

\[ V(x) = \frac{1}{2} x^T P x, \quad \text{with } P = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \]

\[ \dot{V}(x) = \frac{1}{2} x^T Q x, \quad \text{with } Q = \begin{bmatrix} -7 & -6 \\ -6 & -5 \end{bmatrix} \]

But (*) is satisfied with:

\[ \tau_1 = 0 \]

\[ 0.0021 < \tau_2 < 0.0486 \]
A more interesting example

\[ \dot{x}(t) = \begin{bmatrix} \cos(20t) - 0.2 & 1 \\ -1 & \cos(20t) - 0.2 \end{bmatrix} x(t) \]

Claims:

- A time-independent standard Lyap fn. (if there is one) must have a complicated structure
- But

\[ V(x) = x_1^2 + x_2^2 \]

\[ \tau_1 = 0.0039 \quad \tau_2 = 0.0025 \]

satisfies

\[ \tau_2 \ddot{V}(x) + \tau_1 \dot{V}(x) + \dot{V}(x) < 0 \]
Generalization to derivatives of higher order

**THM** ([Meigoli, Nikravesh,’09-a]):

If you find $V > 0$ satisfying

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0$$

with scalars $\tau_i$ such that the characteristic polynomial

$$p(s) = s^m + \tau_{m-1} s^{m-1} + \cdots + \tau_1 s$$

has all roots negative and real, then the system is (locally/globally) asymptotically stable.

**Condition later relaxed to** ([Meigoli, Nikravesh,’09-b]):

- $p(s)$ being Hurwitz
- $p(s)$ having nonnegative coefficients
- (generalization to time-varying dynamics also done)
Links to standard Lyapunov functions?

We make the following simple observation:

**THM:** No matter what conditions are placed on the function $V$ and the scalars $\tau_i$, if $V(0)=0$,

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0$$

holds, and the system is (locally/globally) asymptotically stable, then,

$$W(x) = V^{(m-1)}(x) + \tau_{m-1} V^{(m-2)} + \cdots + \tau_2 \dot{V}(x) + \tau_1 V(x)$$

is a standard Lyapunov function.

**Proof:** easy.
Let’s revisit our example

\[
\dot{x}(t) = \begin{bmatrix} \cos(20t) - 0.2 & 1 \\ -1 & \cos(20t) - 0.2 \end{bmatrix} x(t)
\]

\[
V(x) = x_1^2 + x_2^2 \quad \text{satisfies}
\]

\[
\tau_2 \dddot{V} + \tau_1 + \dddot{V} + \dddot{\dot{V}} < 0
\]

\[
W(x, t) = \tau_2 \dddot{V}(x, t) + \tau_1 \dddot{V}(x, t) + V(x) = x^T x \left\{ \tau_2 \left[ -40 \sin(20t) + 4(\cos(20t) - 0.2)^2 \right] + \tau_1 [2(\cos(20t) - 0.2)] + 1 \right\}
\]

\[
\text{satisfies} \quad W > 0 \quad \dot{W} < 0
\]

but \( W \) is time-varying and more complicated
Implications of this observation (I)

→ Non-monotonic Lyapunov functions can be interpreted as standard Lyapunov functions of a very specific structure:

\[ W(x) = V^{(m-1)}(x) + \tau_{m-1} V^{(m-2)} + \cdots + \tau_2 \dot{V}(x) + \tau_1 V(x) \]

▪ This is a Lyapunov function that has the vector field \( f(x) \) and its derivatives embedded in its structure

▪ Reminiscent of Krasovskii’s method: use \( f(x) \) in the parametrization of the Lyapunov function

→ Our observation does not necessarily imply that higher order derivatives are not useful

▪ \( W \) is often more complicated than \( V \)
Implications of this observation (II)

→ Instead of requiring

\[ V(x) > 0 \]
\[ \tau_i \geq 0 \]

\[ V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0 \]

it is always less conservative to require

\[ V^{(m-1)}(x) + \tau_{m-1} V^{(m-2)}(x) + \cdots + \tau_2 \dot{V}(x) + \tau_1 V(x) > 0 \]
\[ V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_2 \ddot{V}(x) + \tau_1 \dot{V}(x) < 0 \]

(with no condition on \( V \) or \( \tau_i \))
Implications of this observation (III)

→ With this observation, we can **convexify** the previously nonconvex condition:

- Simply search for different functions \( V_1(x), \ldots, V_m(x) \)

  with no conditions on them individually, such that

\[
V_{m-1}^{(m-1)}(x) + V_{m-1}^{(m-2)}(x) + \cdots + \dot{V}_2(x) + V_1(x) > 0
\]

\[
V_{m-1}^{(m)}(x) + V_{m-1}^{(m-1)}(x) + \cdots + \ddot{V}_2(x) + \dot{V}_1(x) < 0
\]

- Guaranteed to have a solution if any of the previous conditions had a feasible solution
- Specific parametrization, depends on vector field
- Can be cast as a convex program
Example

\[
\begin{align*}
\dot{x}_1 &= -0.8x_1^3 - 1.5x_1x_2^2 - 0.4x_1x_2 - 0.4x_1x_3^2 - 1.1x_1 \\
\dot{x}_2 &= x_1^4 + x_3^6 + x_1^2x_3^4 \\
\dot{x}_3 &= -0.2x_1^2x_3 - 0.7x_2^2x_3 - 0.3x_2x_3 - 0.5x_3^3 - 0.5x_3.
\end{align*}
\]

- No quadratic standard Lyapunov function exists
- But
  \[
  \begin{align*}
  V_1(x) &= 0.47x_1^2 + 0.89x_2^2 + 0.91x_3^2 \\
  V_2(x) &= 0.36x_2
  \end{align*}
  \]

satisfy

\[
\dot{V}_2(x) + V_1(x) \quad \text{SOS}
\]

\[
-(\ddot{V}_2(x) + \dot{V}_1(x)) \quad \text{SOS}
\]

- Proves GAS. If desired, can construct a standard (sextic) Lyapunov function from it:

\[
W(x) = \dot{V}_2(x) + V_1(x) = 0.36x_1^4 + 0.36x_1^2x_3^4 + 0.47x_1^2 + 0.89x_2^2 + 0.36x_3^6 + 0.91x_3^2.
\]

- Number of decision variables saved as compared to a search for a standard sextic polynomial Lyapunov function: 68
Messages to take home...

- Monotonicity requirement of Lyapunov’s theorem can be relaxed by using higher order differences/derivatives.

- When the higher order differential inclusions are satisfied, one can always construct a (more complicated) standard Lyapunov function.

- This observation allows us to write conditions that are
  - Always less conservative
  - Checkable with convex programs

- Employing higher order derivatives in the structure of the Lyapunov function may lead to proofs of stability with simpler functions.

  - Computationally, this translates to fewer decision variables.

<table>
<thead>
<tr>
<th>$V(x)$ simple</th>
<th>$W(x)$ complicated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial of low degree</td>
<td>Polynomial of high degree</td>
</tr>
<tr>
<td>Smooth</td>
<td>Piecewise with many pieces</td>
</tr>
<tr>
<td>Time independent</td>
<td>Time dependent</td>
</tr>
</tbody>
</table>
Thank you for your attention!

Want to know more?

Amir Ali’s homepage:

http://aaa.lids.mit.edu