On Higher Order Derivatives of Lyapunov Functions

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Lyapunov stability analysis

$$\dot{x} = f(x) \quad (f : \mathbb{R}^n \to \mathbb{R}^n)$$

Goal: prove local or global asymptotic stability

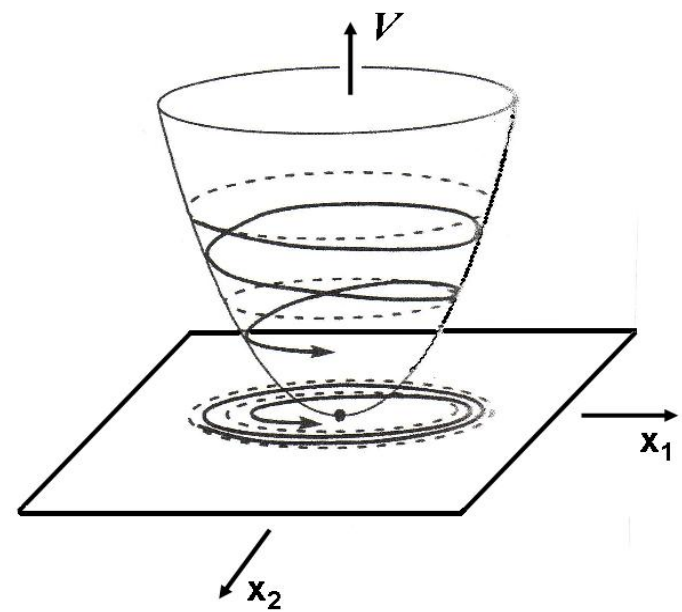
Asymptotic stability established if we find a Lyapunov function

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

with derivative $\dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle$

such that $V(x) > 0$

$$\dot{V}(x) < 0$$



Algorithmic search for Lyapunov functions

- Advances in **convex optimization** and in particular **semidefinite programming (SDP)** have led to algorithmic techniques for Lyapunov functions
- Can parameterize certain classes of Lyapunov functions and pose the search as a **convex feasibility problem**
  - Quadratic Lyapunov functions for linear systems (SDP)
  - Piecewise quadratic Lyapunov functions (SDP)
  - Surface Lyapunov functions (SDP)
  - Polytopic Lyapunov functions (LP)
  - SOS polynomial Lyapunov functions (SDP)
  - Relaxations for pointwise maximum or minimum of quadratics (SDP)

  …

- **Key:** Lyapunov inequalities are **affine** in the parameters of V

$$V(x) > 0$$
$$\dot{V}(x) < 0$$

This is great, but…

- We can only search for a restricted class of (low complexity) functions

- "Simple" dynamics may have "complicated" Lyapunov functions

e.g.

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= -y \end{aligned}$$

[Ahmadi, Krstic, Parrilo,'11]

is GAS but has **no (global) polynomial Lyapunov function of any degree!**

- Existence of Lyapunov functions that we can efficiently search for are almost **never necessary** for stability

- Often Lyapunov functions that we find are too complicated, e.g., polynomial of high degree or piecewise quadratics with many pieces

- Recall: a polynomial in *n* variables and degree *d* has

$\binom{n+d}{d}$ coefficients! (≈460 for n=5, d=6)

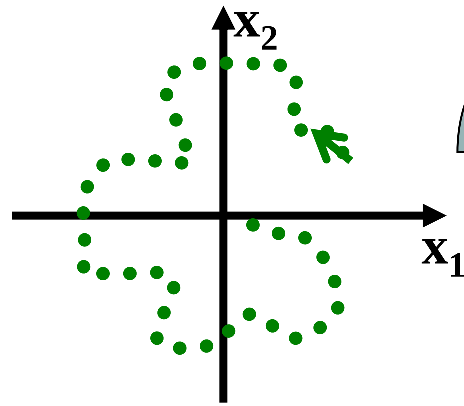
  - Explore simpler parameterizations?

  - Can we relax the conditions of Lyapunov's theorem to prove stability with simpler functions?

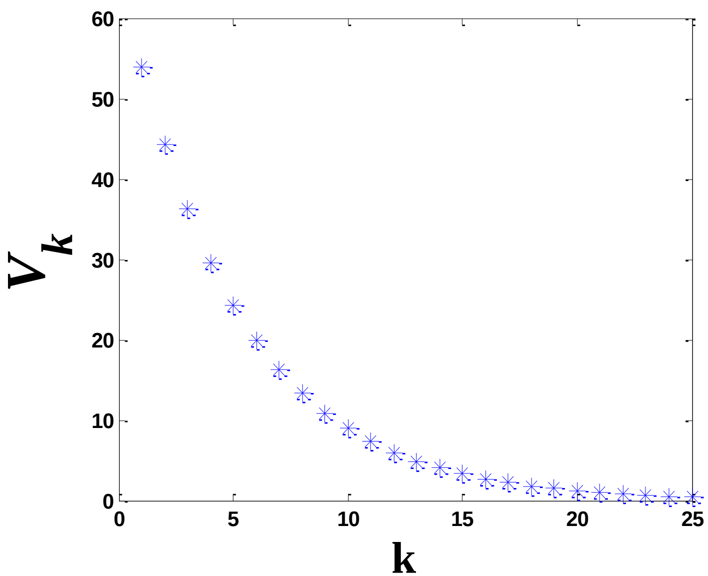
Main motivation

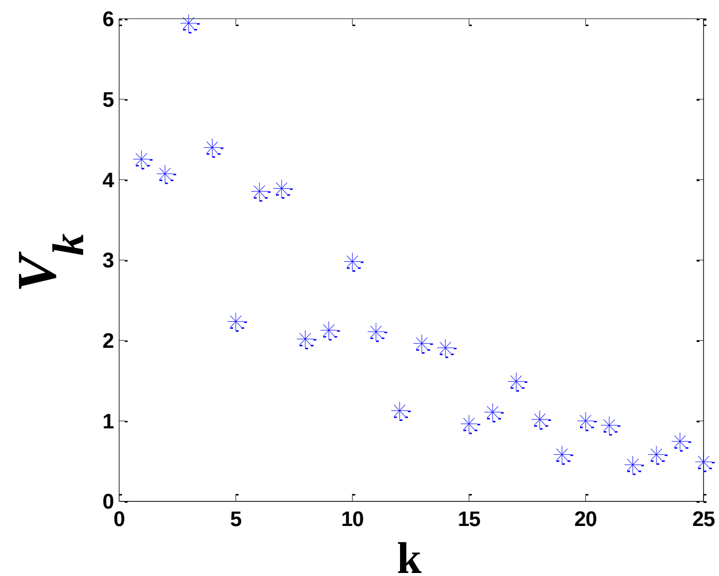
Q: If all we need is $V \rightarrow 0$,
why require a monotonic decrease?

Complicated
V



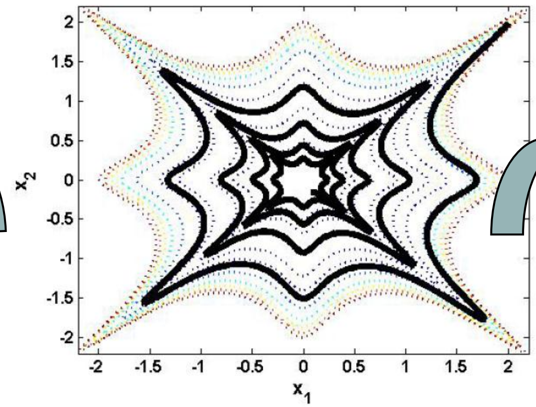
Simpler
V





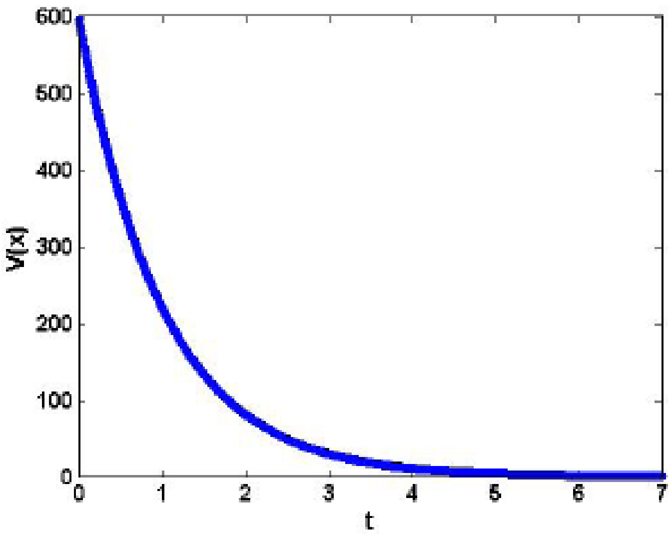
Main motivation

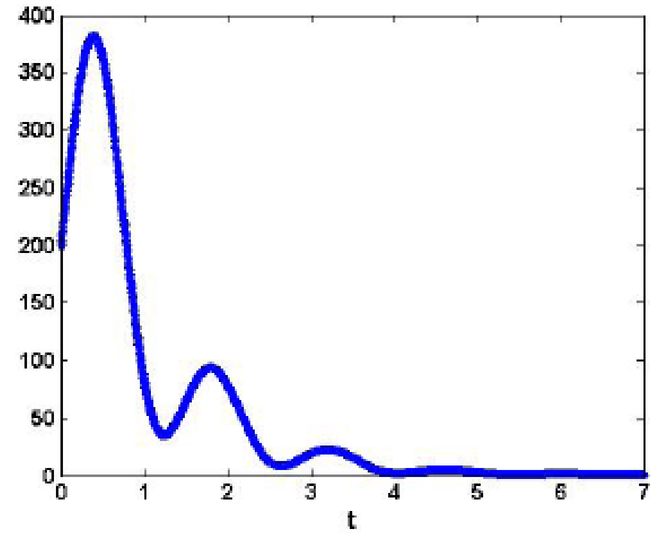
➢ **Similarly in continuous time:**



Complicated
V

Simpler
V





Questions of interest

- **Q1:** Conditions that allow the Lyapunov functions to **increase locally** but guarantee their **convergence to zero in the limit**?
- **Q2:** Can the search for **non-monotonic Lyapunov functions** satisfying the new conditions be cast as a **convex program**?
- **Q3:** Connections between non-monotonic Lyapunov functions and **standard Lyapunov functions**?

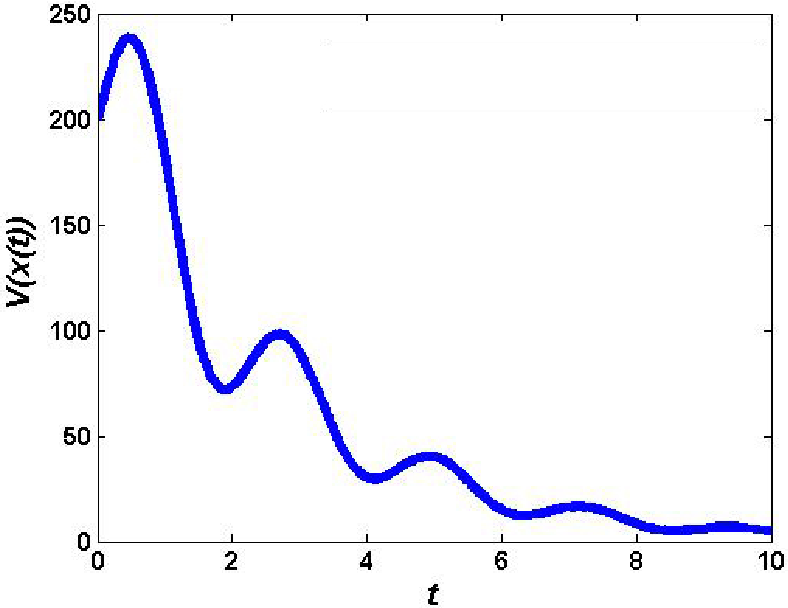
- **Discrete time (DT) – idea: use higher order differences**
  - **Q1, Q2, Q3:** [Ahmadi, Parrilo ’08]
- **Continuous time (CT) – idea: use higher order derivatives**
  - **Q1:** [Butz ’69], [Heinen, Vidyasagar ‘70], [Gunderson ’71], [Meigoli, Nikravesh ’09]
  - **Focus of this (2-page) ACC paper:**

    Simple observation on **Q2** and **Q3**

CT: relaxing monotonicity via higher order derivatives

$$\dot{x} = f(x)$$

- Allow $\dot{V} > 0$ at some points in space
- Limit the rate at which V can increase by imposing constraints on **higher order derivatives**



$$\dot{V}(x) = \langle \frac{\partial V(x)}{\partial x}, f(x) \rangle$$

$$\ddot{V}(x) = \langle \frac{\partial \dot{V}(x)}{\partial x}, f(x) \rangle$$

$$\dddot{V}(x) = \langle \frac{\partial \ddot{V}(x)}{\partial x}, f(x) \rangle$$

$$\cdots$$

- Cheap to compute
- All linear in V !

First two derivatives alone don't help for inferring stability

- A condition of type

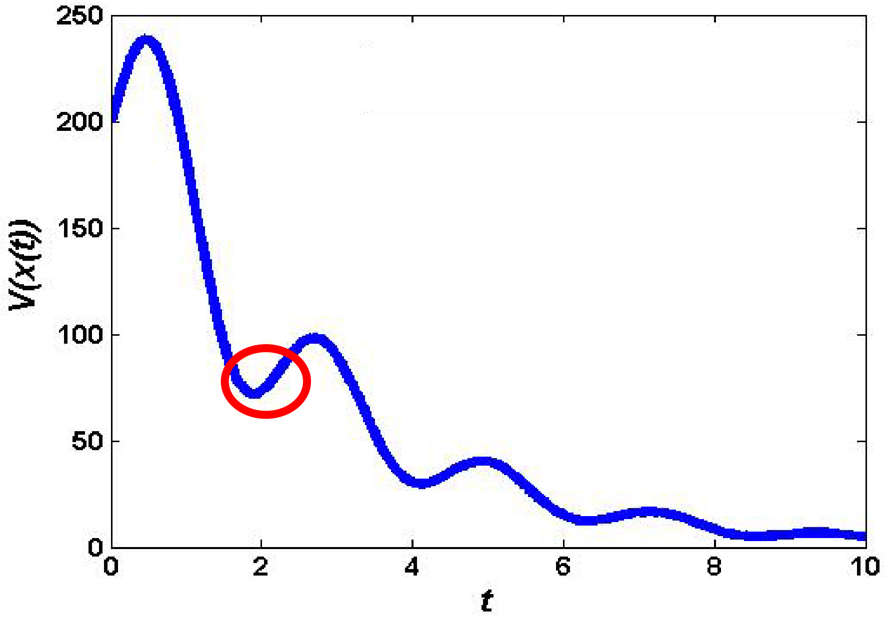
$$\min\{\dot{V}(x), \ddot{V}(x)\} < 0 \quad \forall x \neq 0$$

is vacuous. In particular,

$$\tau \ddot{V}(x) + \dot{V}(x) < 0 \ \forall x \neq 0,$$

$$for\ some\ \tau \geq 0$$

is never satisfied unless $\dot{V}(x) < 0$ everywhere [Butz,'69].

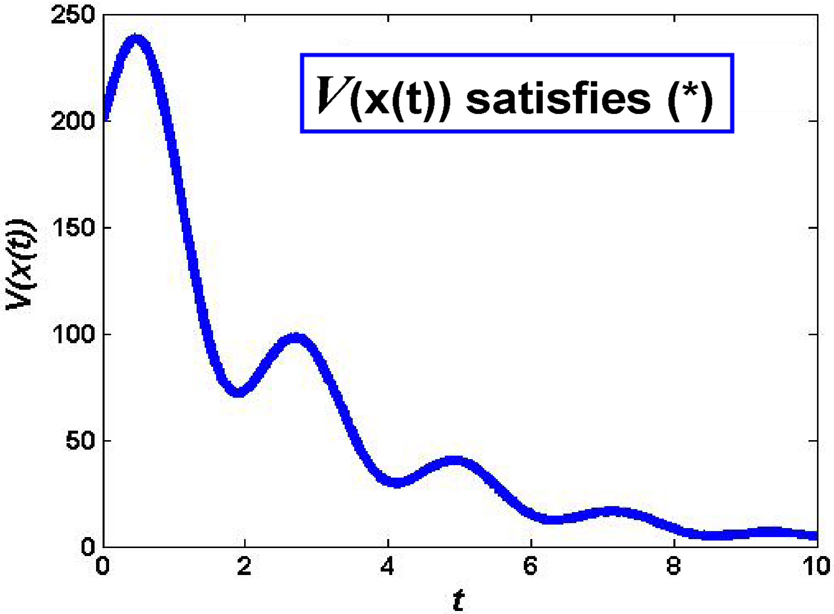


But the first three derivatives help

Thm **(Butz, 1969): existence of a positive Lyapunov function V and nonnegative scalars $\tau_{1,2}$ satisfying**

$$\tau_2 \dddot{V} + \tau_1 \ddot{V} + \dot{V} < 0 \quad (*)$$

implies (global) asymptotic stability.



- Proof by comparison lemma type arguments and basic facts about ODEs
- (*) imposed on complements of compact sets implies Lagrange stability (boundedness of trajectories) [Heinen, Vidyasagar '70]
- (*) is **nonconvex** (bilinear in decision vars. V and τ_i)
  - We will get around this issue shortly

Condition is non-vacuous

$$\tau_2 \dddot{V} + \tau_1 \ddot{V} + \dot{V} < 0 \quad (*)$$

- An example by Butz:

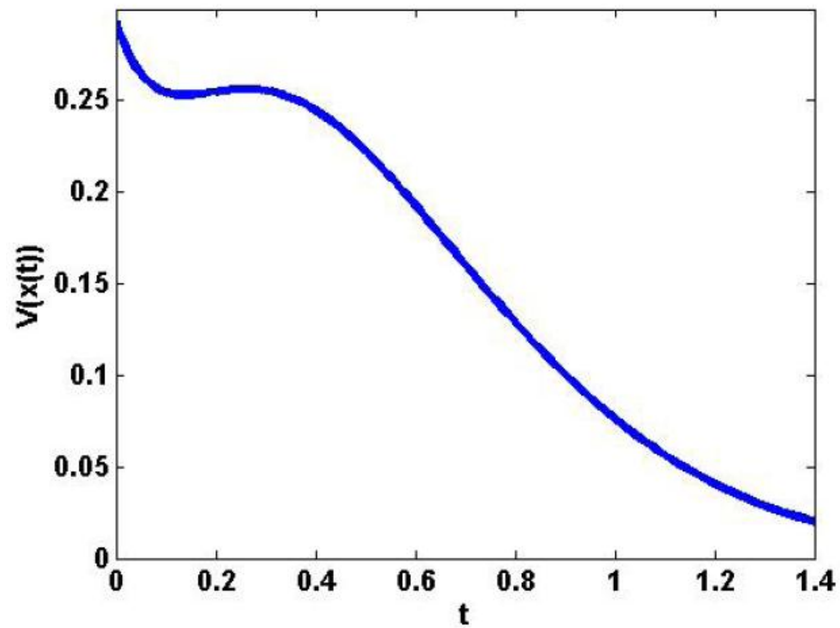
$$\dot{x} = Ax$$

$$A = \begin{bmatrix} -4 & -5 \\ 1 & 0 \end{bmatrix} \quad \text{Eig:} \quad -2 \pm j$$

$$V(x) = \frac{1}{2} x^T P x, \quad \textit{with } P = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$\dot{V}(x) = \frac{1}{2} x^T Q x, \quad \textit{with } Q = \begin{bmatrix} -7 & -6 \\ -6 & -5 \end{bmatrix}$$

not negative definite

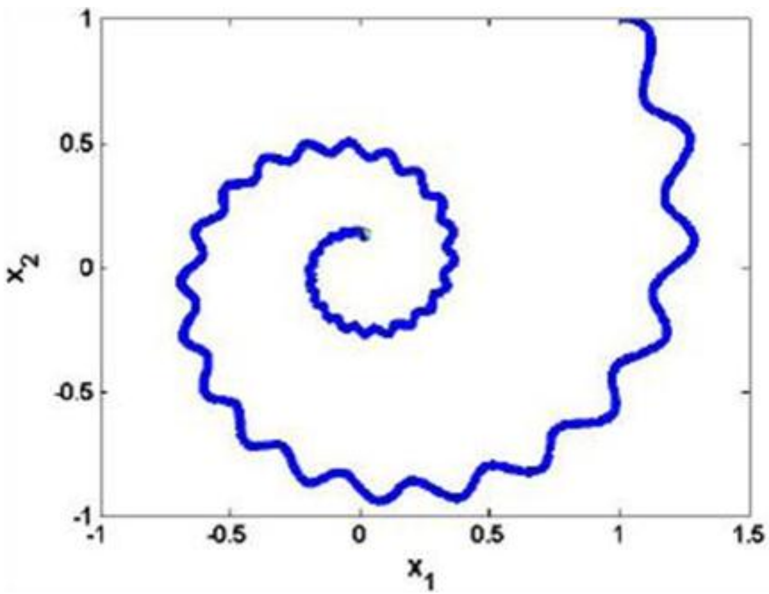


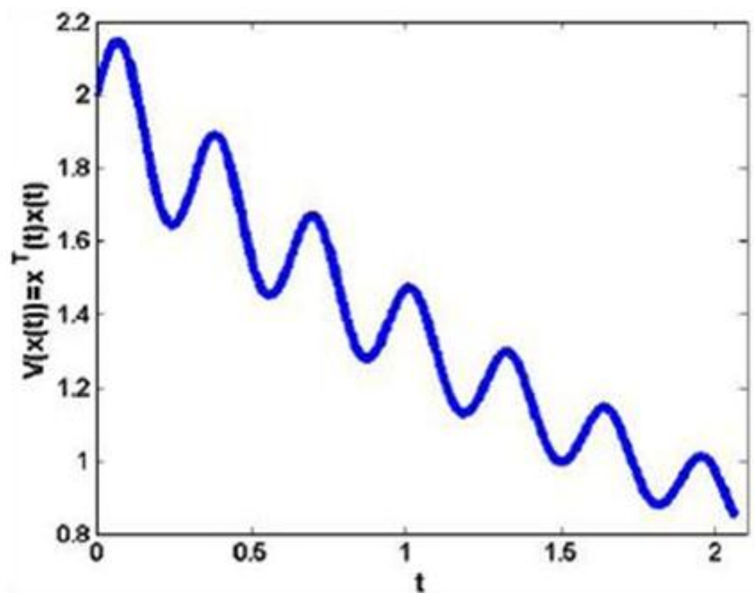
But (*) is satisfied with:

$$\tau_1 = 0$$
$$0.0021 < \tau_2 < 0.0486$$

A more interesting example

$$\dot{x}(t) = \begin{bmatrix} \cos(20t) - 0.2 & 1 \\ -1 & \cos(20t) - 0.2 \end{bmatrix} x(t)$$





Claims:

- A **time-independent standard Lyap fn.** (*if* there is one) must have a complicated structure
- But $V(x) = x_1^2 + x_2^2$ satisfies

$$\tau_1 = 0.0039 \qquad \tau_2 \dddot{V}(x) + \tau_1 \ddot{V}(x) + \dot{V}(x) < 0$$
$$\tau_2 = 0.0025$$

Generalization to derivatives of higher order

THM ([Meigoli, Nikravesh,'09-a]):

If you find $V>0$ satisfying

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0$$

with scalars τ_i such that the characteristic polynomial

$$p(s) = s^m + \tau_{m-1} s^{m-1} + \cdots + \tau_1 s$$

has all roots negative and real, then the system is (locally/globally) asymptotically stable.

Condition later relaxed to ([Meigoli, Nikravesh,'09-b]):

- $p(s)$ being Hurwitz
- $p(s)$ having nonnegative coefficients
- (generalization to time-varying dynamics also done)

Links to standard Lyapunov functions?

We make the following simple observation:

THM: No matter what conditions are placed on the function V and the scalars τ_i, if $V(0)=0$,

$$V^{(m)}(x) + \tau_{m-1}V^{(m-1)}(x) + \cdots + \tau_1\dot{V}(x) < 0$$

holds, and the system is (locally/globally) asymptotically stable, then,

$$W(x) = V^{(m-1)}(x)+\tau_{m-1}V^{(m-2)}+\cdots+\tau_2\dot{V}(x)+\tau_1 V(x)$$

is a standard Lyapunov function.

Proof: easy.

Let's revisit our example

$$\dot{x}(t) = \begin{bmatrix} \cos(20t) - 0.2 & 1 \\ -1 & \cos(20t) - 0.2 \end{bmatrix} x(t)$$

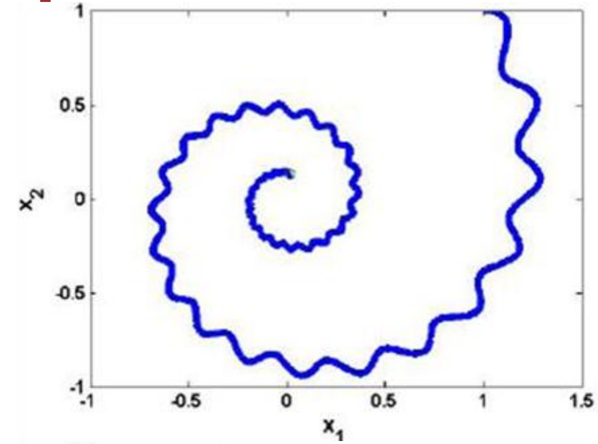
$V(x) = x_1^2 + x_2^2$ satisfies

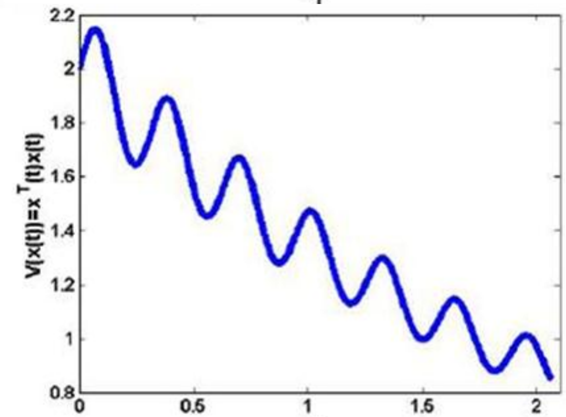
$$\tau_2 \dddot{V} + \tau_1 + \ddot{V} + \dot{V} < 0$$

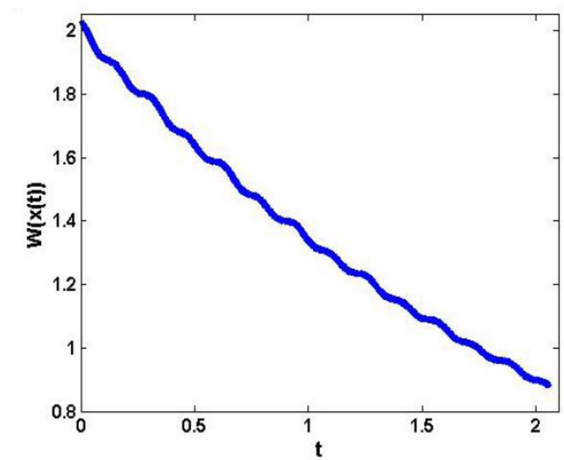
$$W(x,t) = \tau_2 \ddot{V}(x,t) + \tau_1 \dot{V}(x,t) + V(x) = x^T x\{$$
$$\tau_2[-40\sin(20t) + 4(\cos(20t) - 0.2)^2]$$
$$+\tau_1[2(\cos(20t) - 0.2)] + 1 \; \}$$

satisfies $W > 0 \quad \dot{W} < 0$

but W is time-varying and more complicated







Implications of this observation (I)

→ Non-monotonic Lyapunov functions can be interpreted as standard Lyapunov functions of a very specific structure:

$$W(x) = V^{(m-1)}(x) + \tau_{m-1} V^{(m-2)} + \cdots + \tau_2 \dot{V}(x) + \tau_1 V(x)$$

- This is a Lyapunov function that has the vector field *f(x)* and its derivatives embedded in its structure
- Reminiscent of Krasovskii's method: use *f(x)* in the parametrization of the Lyapunov function

→ Our observation does not necessarily imply that higher order derivatives are not useful

- *W* is often more complicated than *V*

Implications of this observation (II)

→ Instead of requiring

$$V(x) > 0$$
$$\tau_i \geq 0$$

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0$$

it is always less conservative to require

$$V^{(m-1)}(x) + \tau_{m-1} V^{(m-2)}(x) + \cdots + \tau_2 \dot{V}(x) + \tau_1 V(x) > 0$$

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_2 \ddot{V}(x) + \tau_1 \dot{V}(x) < 0$$

(with no condition on V or τ_i)

Implications of this observation (III)

→ With this observation, we can **convexify** the previously nonconvex condition:

- Simply search for different functions $V_1(x), \ldots, V_m(x)$

  with no conditions on them individually, such that

$$V_m^{(m-1)}(x) + V_{m-1}^{(m-2)}(x) + \cdots + \dot{V}_2(x) + V_1(x) > 0$$

$$V_m^{(m)}(x) + V_{m-1}^{(m-1)}(x) + \cdots + \ddot{V}_2(x) + \dot{V}_1(x) < 0$$

- Guaranteed to have a solution if any of the previous conditions had a feasible solution

- Specific parametrization, depends on vector field

- Can be cast as a convex program

Example

$$\begin{aligned}\dot{x}_1 &= -0.8x_1^3 - 1.5x_1x_2^2 - 0.4x_1x_2 - 0.4x_1x_3^2 - 1.1x_1\\ \dot{x}_2 &= x_1^4 + x_3^6 + x_1^2x_3^4\\ \dot{x}_3 &= -0.2x_1^2x_3 - 0.7x_2^2x_3 - 0.3x_2x_3 - 0.5x_3^3 - 0.5x_3.\end{aligned}$$

- **No quadratic standard Lyapunov function exists**
- **But**

$$\begin{aligned}V_1(x) &= 0.47x_1^2 + 0.89x_2^2 + 0.91x_3^2\\ V_2(x) &= 0.36x_2\end{aligned}$$

satisfy

$$\dot{V}_2(x) + V_1(x) \quad \text{SOS}$$

$$-(\ddot{V}_2(x) + \dot{V}_1(x)) \quad \text{SOS}$$

- **Proves GAS. If desired, can construct a standard (sextic) Lyapunov function from it:**

$$W(x) = \dot{V}^2(x) + V^1(x) = 0.36x_1^4 + 0.36x_1^2x_3^4 + 0.47x_1^2 + 0.89x_2^2 + 0.36x_3^6 + 0.91x_3^2.$$

  - **Number of decision variables saved as compared to a search for a standard sextic polynomial Lyapunov function: 68**

Messages to take home…

- Monotonicity requirement of Lyapunov's theorem can be relaxed by using **higher order differences/derivatives**

- When the higher order differential inclusions are satisfied, one **can always construct a (more complicated) standard Lyapunov function**

- This observation allows us to write conditions that are
  - **Always less conservative**
  - **Checkable with convex programs**

- Employing higher order derivatives in the structure of the Lyapunov function may lead to **proofs of stability with simpler functions**
  - Computationally, this translates to **fewer decision variables**

***V(x)* simple**	***W(x)* complicated**
Polynomial of low degree	Polynomial of high degree
Smooth	Piecewise with many pieces
Time independent	Time dependent

Thank you for your attention!

Want to know more?

Amir Ali's homepage:

http://aaa.lids.mit.edu