A Globally Asymptotically Stable Polynomial Vector Field with no Polynomial Lyapunov Function

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***Abstract*— We give a simple, explicit example of a two-dimensional polynomial vector field that is globally asymptotically stable but does not admit a polynomial Lyapunov function.**

I. Introduction and Main Result

Given a particular class of differential equations, a question of fundamental importance in stability analysis is to determine a class of Lyapunov functions whose existence is necessary and sufficient for proving stability. Arguably, the class of polynomial differential equations are among the most widely encountered in engineering and sciences. For these systems, it is most common (and most natural) to search for Lyapunov functions that are polynomials themselves. This approach has become further prevalent over the past decade due to the fact that techniques from sum of squares optimization [1] have provided for algorithms that given a polynomial system can efficiently search for a polynomial Lyapunov function ([1], [2]).

The question therefore naturally arises as to whether the existence of polynomial Lyapunov functions is necessary for stability of polynomial systems. Since polynomials can approximate smooth functions with arbitrary accuracy on compact regions, one can expect the answer to this question to be positive if certain notions of stability on compact sets are of interest. Indeed, Peet has formally proven a result of this type in [3]. On the other hand, to the best of our knowledge, the question of whether globally asymptotically stable (GAS) polynomial systems admit polynomial Lyapunov functions has been open. In fact, a recent reference in the controls literature ends with the following statement [4], [5]:

> "Still unresolved is the fundamental question of whether *globally* stable vector fields will also admit sum-of-squares Lyapunov functions."

Of course, the fundamental question referred to here is on existence of a polynomial Lyapunov function. If one were to exist, then we could simply square it to get another polynomial Lyapunov function that is a sum of squares. In this paper, we settle the question by giving a remarkably simple counterexample. In view of the fact that globally asymptotically stable linear systems always admit quadratic Lyapunov functions, it is quite interesting to observe that the following vector field that is arguably "the next simplest system" to consider does not admit a polynomial Lyapunov function of any degree.

Theorem 1.1: Consider the polynomial vector field

$$\begin{array}{rcl} \dot{x} & = & -x+xy \\ \dot{y} & = & -y. \end{array} \qquad (1)$$

The origin is a globally asymptotically stable equilibrium point, but the system does not admit a polynomial Lyapunov function.

Proof: Let us first show that the system is GAS. Consider the Lyapunov function

$$V(x,y)=\ln(1+x^2)+y^2,$$

which clearly vanishes at the origin, is strictly positive for all $(x,y)\neq(0,0)$, and is radially unbounded. The derivative of $V(x,y)$ along the trajectories of (1) is given by

$$\begin{array}{rcl} \dot{V}(x,y) & = & \frac{\partial V}{\partial x}\dot{x}+\frac{\partial V}{\partial y}\dot{y} \\ & = & \frac{2x^2(y-1)}{1+x^2}-2y^2 \\ & = & -\frac{x^2+2y^2+x^2y^2+(x-xy)^2}{1+x^2}, \end{array}$$

which is obviously strictly negative for all $(x,y)\neq(0,0)$. In view of classical Lyapunov stability theorems (see e.g. [6, p. 124]), this shows that the origin is globally asymptotically stable.

Let us now prove that no positive definite polynomial Lyapunov function (of any degree) can decrease along the trajectories of system (1). The proof will be based on simply considering the value of a candidate Lyapunov function at two specific points. We will look at trajectories on the nonnegative orthant, with initial conditions on the line $(k,\alpha k)$ for some constant $\alpha>0$, and then observe the location of the crossing of the trajectory with the horizontal line $y=\alpha$. We will argue that by taking k large enough, the trajectory will have to travel "too far east" (see Figure 1) and this will make it impossible for any polynomial Lyapunov function to decrease.

To do this formally, we start by noting that we can explicitly solve for the solution $(x(t),y(t))$ of the vector field in (1) starting from any initial condition $(x(0),y(0))$:

$$\begin{array}{rcl} x(t) & = & x(0)e^{[y(0)-y(0)e^{-t}-t]} \\ y(t) & = & y(0)e^{-t}. \end{array} \qquad (2)$$

Consider initial conditions

$$(x(0),y(0))=(k,\alpha k)$$

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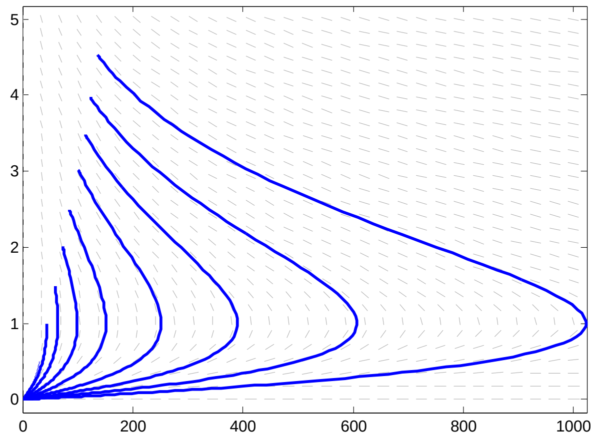

Fig. 1. Typical trajectories of the vector field in (1) starting from initial conditions in the nonnegative orthant.

parameterized by $k > 1$ and for some fixed constant $\alpha > 0$. From the explicit solution in (2) we have that the time t^* it takes for the trajectory to cross the line $y = \alpha$ is

$$t^* = \ln(k),$$

and that the location of this crossing is given by

$$(x(t^*), y(t^*)) = (e^{\alpha(k-1)}, \alpha).$$

Consider now any candidate nonnegative polynomial function $V(x, y)$ that depends on both x and y (as any Lyapunov function should). Since $k > 1$ (and thus, $t^* > 0$), for $V(x, y)$ to be a valid Lyapunov function, it must satisfy $V(x(t^*), y(t^*)) < V(x(0), y(0))$, i.e.,

$$V(e^{\alpha(k-1)}, \alpha) < V(k, \alpha k).$$

However, this inequality cannot hold for k large enough, since for a generic fixed α, the left hand side grows exponentially in k whereas the right hand side grows only polynomially in k. The only subtlety arises from the fact that $V(e^{\alpha(k-1)}, \alpha)$ could potentially be a constant for some particular choices of α. However, for any polynomial $V(x, y)$ with nontrivial dependence on y, this may happen for at most finitely many values of α. Therefore, any generic choice of α would make the argument work. ■

II. Conclusions

We showed that existence of a polynomial Lyapunov function is not necessary for global asymptotic stability of a polynomial vector field. A related converse Lyapunov question that is motivated by the use of computational techniques for analysis via polynomial Lyapunov functions is the following:

Question: Suppose a polynomial vector field has a polynomial Lyapunov function. Does this imply that sum of squares optimization will succeed in finding a polynomial Lyapunov function and proving stability?

Two results on this question are given in a recent paper of the first and third authors [7].

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