Stability of Polynomial Differential Equations: Complexity and Converse Lyapunov Questions

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Abstract

Stability analysis of polynomial differential equations is a central topic in nonlinear dynamics and control which in recent years has undergone major *algorithmic* developments due to advances in optimization theory. Notably, the last decade has seen a widespread interest in the use of sum of squares (sos) based semidefinite programs that can automatically find polynomial Lyapunov functions and produce explicit certificates of stability. However, despite their popularity, the converse question of whether such algebraic, efficiently constructable certificates of stability always exist has remained elusive. Naturally, an algorithmic question of this nature is closely intertwined with the fundamental computational complexity of proving stability. In this paper, we make a number of contributions to the questions of (i) complexity of deciding stability, (ii) existence of polynomial Lyapunov functions, and (iii) existence of sos Lyapunov functions.

(i) We show that deciding local or global asymptotic stability of cubic vector fields is strongly NP-hard. Simple variations of our proof are shown to imply strong NP-hardness of several other decision problems: testing local attractivity of an equilibrium point, stability of an equilibrium point in the sense of Lyapunov, invariance of the unit ball, boundedness of trajectories, convergence of all trajectories in a ball to a given equilibrium point, existence of a quadratic Lyapunov function, local collision avoidance, and existence of a stabilizing control law.

(ii) We present a simple, explicit example of a globally asymptotically stable quadratic vector field on the plane which does not admit a polynomial Lyapunov function (joint work with M. Krstic and presented here without proof). For the subclass of homogeneous vector fields, we conjecture that asymptotic stability implies existence of a polynomial Lyapunov function, but show that the minimum degree of such a Lyapunov function can be arbitrarily large even for vector fields in fixed dimension and degree. For the same class of vector fields, we further establish that there is no monotonicity in the degree of polynomial Lyapunov functions.

(iii) We show via an explicit counterexample that if the degree of the polynomial Lyapunov function is fixed, then sos programming may fail to find a valid Lyapunov function even though one exists. On the other hand, if the degree is allowed to increase, we prove that existence of a polynomial Lyapunov function for a planar or a homogeneous vector field implies existence of a polynomial Lyapunov function that is sos and that the negative of its derivative is also sos.

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1 Introduction

The diversity of application domains in science and engineering where complex dynamical systems are modelled as polynomial differential equations is unsurpassed. Dynamics of population growth in ecology, prices and business cycles in economics, chemical reactions in cell biology, spread of epidemics in network science, and motion of a wide range of electromechanical systems in control engineering, are but few examples of areas where slight deviations from the traditional "simplifying assumptions of linearity" leave us with polynomial vector fields. Polynomial systems also enjoy much added interest stemming from the fact that various other types of dynamical systems are often approximated by their Taylor expansions of some order around equilibrium points. A prime example of this is in the field of robotics where equations of motion (described by the manipulator equations) give rise to trigonometric differential equations which are then commonly approximated by polynomials for analysis and control. Aside from practice, polynomial vector fields have also always been at the heart of diverse branches of mathematics: Some of the earliest examples of chaos (e.g. the Lorenz attractor) arise from simple low degree polynomial vector fields [47]; the still-open Hilbert's 16th problem concerns polynomial vector fields on the plane [43]; Shannon's General Purpose Analog Computer is equivalent in simulation power to systems of polynomial differential equations [27]; and the list goes on.

In many application areas, the question of interest is often not to solve for particular solutions of these differential equation numerically, but rather understand certain qualitative properties of the trajectories as a whole. Among the different qualitative questions one can ask, the question of *stability* of equilibrium points is arguably the most ubiquitous. Will deviations from equilibrium prices in the market be drawn back to the equilibrium? Will the epidemic die out or become widespread? We are concerned for the most of part of this paper with this stability question, which we study from an algorithmic viewpoint. Our goal is to shed light on the complexity of this question and to understand the power/limitations of some of the most promising optimization-based computational techniques currently available for tackling this problem.

Almost universally, the study of stability in dynamical systems leads to Lyapunov's second method or one of its many variants. An outgrowth of Lyapunov's 1892 doctoral dissertation [48], Lyapunov's second method tells us, roughly speaking, that if we succeed in finding a *Lyapunov function*—an energy-like function of the state that decreases along trajectories—then we have proven that the dynamical system in question is stable. In the mid 1900s, a series of *converse Lyapunov theorems* were developed which established that any stable system indeed has a Lyapunov function (see [37, Chap. 6] for an overview). Although this is encouraging, except for the simplest classes of systems such as linear systems, converse Lyapunov theorems do not provide much practical insight into how one may go about finding a Lyapunov function.

In the last few decades however, advances in the theory and practice of convex optimization and in particular semidefinite programming (SDP) have rejuvenated Lyapunov theory. The approach has been to parameterize a class of Lyapunov functions with restricted complexity (e.g., quadratics, pointwise maximum of quadratics, polynomials, etc.) and then pose the search for a Lyapunov function as a convex feasibility problem. A widely popular example of this framework which we study in this paper is the method of sum of squares (sos) Lyapunov functions [52],[54]. Expanding on the concept of sum of squares decomposition of polynomials and its relation to semidefinite programming, this technique allows one to formulate SDPs that automatically search for polynomial Lyapunov functions for polynomial dynamical systems. Over the last decade, the applicability of sum of squares Lyapunov functions has been explored in many directions and numerous extensions have been developed to tackle a wide range of problems in systems and control. We refer the reader to the by no means exhaustive list of works [41], [28], [44], [59], [61], [29], [7], [14], and references therein. Despite the wealth of research in this area, however, the converse question of whether a proof of stability via sum of squares Lyapunov functions is always possible has remained elusive.

Our paper speaks to the premise that an algorithmic approach to Lyapunov theory naturally calls for new converse theorems. Indeed, classical converse Lyapunov theorems only guarantee existence of Lyapunov functions within very broad classes of functions (e.g. the class of continuously differentiable functions) that are a priori not amenable to computation. So there is the need to know whether Lyapunov functions belonging to certain more restricted classes of functions that can be computationally searched over also exist. For example, do stable polynomial systems admit Lyapunov functions that are polynomial? What about polynomial functions that can be found with sum of squares techniques?

As one would expect, questions of this nature are intrinsically related to the computational complexity of deciding stability. As an example, consider the following fundamental question which to the best of our knowledge is open: Given a polynomial vector field (with rational coefficients), is there an algorithm that can decide whether the origin is a (locally or globally) asymptotically stable equilibrium point? A well-known conjecture of Arnold [13] (see Section 3) states that there cannot be such an algorithm; i.e., that the problem is *undecidable*. Suppose now that one could prove that (local or global) asymptotic stability of polynomial vector fields implies existence of a polynomial Lyapunov function, together with a *computable upper bound* on its degree. Such a result would imply that the question of stability is *decidable*. Indeed, given a polynomial vector field and an integer d, one could e.g. use the quantifier elimination theory of Tarski and Seidenberg [70], [67] to test, in finite time, whether the vector field admits a polynomial Lyapunov function of degree d.

Just as a proof of undecidability for stability would rule out existence of polynomial Lyapunov functions of a priori known degree, a hardness result on *time complexity* (e.g. an NP-hardness result) would inform us that we should not always expect to find "efficiently constructable" Lyapunov functions (e.g. those based on convex optimization). In fact, much of the motivation behind sum of squares Lyapunov functions as a replacement for polynomial Lyapunov functions is computational efficiency. Even though polynomial functions are finitely parameterized, the computational problem of finding a polynomial that satisfies the Lyapunov inequalities (nonnegativity of the function and nonpositivity of its derivative) is intractable. (This is due to the fact that even the task of checking if a given quartic polynomial is nonnegative is NP-hard [51], [55]). By contrast, when one replaces the two Lyapunov inequalities by the requirement of having a sum of squares decomposition, the search for this (sum of squares) Lyapunov function becomes a semidefinite program, which can be solved efficiently e.g. using interior point methods [71].

It is important to also keep in mind the distinctions between establishing hardness results for stability versus answering questions on existence of Lyapunov functions. On one hand, complexity results have much stronger implications in the sense that they rule out the possibility of *any* (efficient) algorithms for deciding stability, not just those that may be based on a search for a particular family of Lyapunov functions, or based on Lyapunov theory to begin with for that matter. On the other hand, complexity results only imply that "hard instances" of the problem should exist asymptotically (when the number of variables is large), often without giving rise to an explicit example or determining whether such instances also exist within reasonable dimensions. For these reasons, a separate study of both questions is granted, but the connections between the two are always to be noted.

1.1 Contributions and organization of the paper

We begin this paper by a short section on preliminaries (Section 2), where we formally define some basic notions of interest, such as sum of squares Lyapunov functions, homogeneous vector fields, etc. The next three sections respectively include our main contributions on (i) complexity of testing stability, (ii) existence of polynomial Lyapunov functions, and (iii) existence of sos Lyapunov functions. These contributions are outlined below. We end the paper in Section 6 with a list of open problems.

Complexity results (Section 3). After a brief account of a conjecture of Arnold on undecidability of testing stability, we present the following NP-hardness results:

- We prove that deciding (local or global) asymptotic stability of cubic vector fields is strongly NP-hard even when the vector fields are restricted to be homogeneous (Theorem 3.1). Degree three is the minimum degree possible for such a hardness result as far as homogeneous systems are concerned (Lemma 3.2). The main challenge in establishing this result is to find a way around the seemingly hopeless task of relating solutions of a combinatorial problem to trajectories of cubic differential equations, to which we do not even have explicit access.

- By modifying our reduction appropriately, we further prove that the following decision problems which arise in numerous areas of systems and control theory are also strongly NP-hard (Theorem 3.5):

  - testing local attractivity of an equilibrium point,
  - testing stability of an equilibrium point in the sense of Lyapunov,
  - testing boundedness of trajectories,
  - testing convergence of all trajectories in a ball to a given equilibrium point,
  - testing existence of a quadratic Lyapunov function,
  - testing local collision avoidance,
  - testing existence of a stabilizing control law,
  - testing invariance of the unit ball,
  - testing invariance of a basic semialgebraic set under linear dynamics.

(Non)-existence of polynomial Lyapunov functions (Section 4). In this section we study whether asymptotic stability implies existence of a polynomial Lyapunov function, and if so whether an upper bound on the degree of such a Lyapunov function can be given.

- We present a simple, explicit example of a quadratic differential equation in two variables and with rational coefficients that is globally asymptotically stable but does not have a polynomial Lyapunov function of any degree (Theorem 4.1). This is joint work with Miroslav Krstic [5] and is presented here without proof.

- Unlike the general case, we conjecture that for the subclass of homogeneous vector fields (see Section 2 for a definition) existence of a (homogeneous) *polynomial* Lyapunov function is necessary and sufficient for asymptotic stability. This class of vector fields has been extensively studied in the literature on nonlinear control [64], [11], [36], [16], [39], [63], [49]. Although a polynomial Lyapunov function is conjectured to always exist, we show that the degree of such a Lyapunov function cannot be bounded as a function of the degree and the dimension of the vector field only. (This is in contrast with stable linear systems which always admit quadratic Lyapunov functions.) We prove this by presenting a family of globally asymptotically stable cubic homogeneous vector fields in two variables for which the minimum degree of a polynomial Lyapunov function can be arbitrarily large (Theorem 4.3).

- For homogeneous vector fields of degree as low as three, we show that there is no monotonicity in the degree of polynomial Lyapunov functions that prove asymptotic stability; i.e., a homogeneous cubic vector field with no homogeneous polynomial Lyapunov function of some degree d can very well have a homogeneous polynomial Lyapunov function of degree less than d (Theorem 4.4).

(Non)-existence of sum of squares Lyapunov functions (Section 5). The question of interest in this section is to investigate whether existence of a polynomial Lyapunov function for a polynomial vector field implies existence of a sum of squares Lyapunov function (see Definition 2.1). Naturally this question comes in two variants: (i) does an sos Lyapunov function of the same degree as the original polynomial Lyapunov function exist? (ii) does an sos Lyapunov function of possibly higher degree exist? Both questions are studied in this section. We first state a well-known result of Hilbert on the gap between nonnegative and sum of squares polynomials and explain why our desired result does not follow from Hilbert's work. Then, we present the following results:

- In Subsection 5.2, we show via an explicit counterexample that existence of a polynomial Lyapunov function does not imply existence of a sum of squares Lyapunov function of the same degree. This is done by proving infeasibility of a certain semidefinite program.
- By contrast, we show in Subsection 5.3 that existence of a polynomial Lyapunov function always implies existence of a sum of squares Lyapunov function of possibly higher degree in the case where the vector field is homogeneous (Theorem 5.3) or when it is planar and an additional mild assumption is met (Theorem 5.5). The proofs of these two theorems are quite simple and rely on powerful Positivstellensatz results due to Scheiderer (Theorems 5.2 and 5.4).

Parts of this paper have previously appeared in several conference papers [8], [5], [4], [6], and in the PhD thesis of the first author [3, Chap. 4].

2 Preliminaries

We are concerned in this paper with a continuous time dynamical system

$$\dot{x} = f(x), \tag{1}$$

where each component of $f : \mathbb{R}^n \to \mathbb{R}^n$ is a multivariate polynomial. We assume throughout and without loss of generality that f has an equilibrium point at the origin, i.e., $f(0) = 0$. We say that the origin is *stable in the sense of Lyapunov* if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $||x(0)|| < \delta \implies ||x(t)|| < \epsilon, \forall t > 0$. The origin is said to be *locally asymptotically stable* (LAS) if it is stable in the sense of Lyapunov and there exists $\epsilon > 0$ such that $||x(0)|| < \epsilon \implies \lim_{t\to\infty} x(t) = 0$. If the origin is stable in the sense of Lyapunov and $\forall x(0) \in \mathbb{R}^n$, we have $\lim_{t\to\infty} x(t) = 0$, then we say that the origin is *globally asymptotically stable* (GAS). This is the notion of stability that we are mostly concerned with in this paper. We know from standard Lyapunov theory (see e.g. [45, Chap. 4]) that if we find a radially unbounded (Lyapunov) function $V(x) : \mathbb{R}^n \to \mathbb{R}$ that vanishes at the origin and satisfies

$$V(x) \quad > \quad 0 \quad \forall x \neq 0 \tag{2}$$

$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle \quad < \quad 0 \quad \forall x \neq 0, \tag{3}$$

then the origin of (1) is GAS. Here, $\dot{V}$ denotes the time derivative of V along the trajectories of (1), $\nabla V(x)$ is the gradient vector of V, and $\langle ., . \rangle$ is the standard inner product in $\mathbb{R}^n$. Similarly, existence of a Lyapunov function satisfying the two inequalities above on some ball around the origin would imply LAS. A real valued function V satisfying the inequality in (2) is said to be *positive definite*. We say that V is *negative definite* if $-V$ is positive definite.

As discussed in the introduction, for stability analysis of polynomial systems it is most common (and quite natural) to search for Lyapunov functions that are polynomials themselves. When such a candidate Lyapunov function is sought, the conditions in (2) and (3) are two polynomial positivity conditions that V needs to satisfy. The computational intractability of testing existence of such V (when at least one of the two polynmials in (2) or (3) have degree four or larger) leads us to the notion of sum of squares Lyapunov functions. Recall that a polynomial p is a *sum of squares* (sos), if it can be written as a sum of squares of polynomials, i.e., $p(x) = \sum_{i=1}^{r} q_i^2(x)$ for some integer r and polynomials q_i.

Definition 2.1. *We say that a polynomial V is a* sum of squares Lyapunov function *for the polynomial vector field f in (1) if the following two sos decomposition constraints are satisfied:*

$$V \quad sos \tag{4}$$

$$-\dot{V} = -\langle \nabla V, f \rangle \quad sos. \tag{5}$$

A sum of squares decomposition is a sufficient condition for polynomial nonnegativity[1] that can be checked with semidefinite programming. We do not present here the semidefinite program that decides if a given polynomial is sos since it has already appeared in several places. The unfamiliar reader is referred to [53], [18]. For a fixed degree of a polynomial Lyapunov candidate V, the search for the coefficients of V subject to the constraints (4) and (5) is also a semidefinite program (SDP). The size of this SDP is polynomial in the size of the coefficients of the vector field f. Efficient algorithms for solving this SDP are available, for example those base on interior point methods [71].

We emphasize that Definition 2.1 is the sensible definition of a sum of squares Lyapunov function and not what the name may suggest, which is a Lyapunov function that is a sum of squares. Indeed, the underlying semidefinite program will find a Lyapunov function V if and only if V satisfies *both* conditions (4) and (5). We shall also remark that we think of the conditions in (4) and (5) as sufficient conditions for the inequalities in (2) and (3). Even though an sos polynomial can be merely nonnegative (as opposed to positive definite), when the underlying semidefinite programming feasibility problems are solved by interior point methods, solutions that are returned are generically positive definite; see the discussion in [2, p. 41]. Later in the paper, when we want to prove results on existence of sos Lyapunov functions, we certainly require the proofs to imply existence of *positive definite* sos Lyapunov functions.

Finally, we recall some basic facts about homogeneous vector fields. A real-valued function p is said to be *homogeneous* (of degree d) if it satisfies $p(\lambda x) = \lambda^d p(x)$ for any scalar $\lambda \in \mathbb{R}$. If p is a polynomial, this condition is equivalent to all monomials of p having the same degree d. It is easy to see that homogeneous polynomials are closed under sums and products and that the gradient of a homogeneous polynomial has entries that are homogeneous polynomials. A polynomial vector field $\dot{x} = f(x)$ is *homogeneous* if all entries of f are homogeneous polynomials of the same degree (see the vector field in (19) for an example). Linear systems, for instance, are homogeneous polynomial vector fields of degree one. There is a large literature in nonlinear control theory on homogeneous vector fields [64], [11], [36], [16], [39], [63], [49], and some of the results of this paper (both negative and positive) are derived specifically for this class of systems. (Note that a negative result established for homogeneous polynomial vector fields is stronger than a negative result derived

[1] A polynomial p is *nonnegative* if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

for general polynomial vector fields.) A basic fact about homogeneous systems is that for these vector fields the notions of local and global stability are equivalent. Indeed, a homogeneous vector field of degree d satisfies $f(\lambda x) = \lambda^d f(x)$ for any scalar λ, and therefore the value of f on the unit sphere determines its value everywhere. It is also well-known that an asymptotically stable homogeneous system admits a homogeneous Lyapunov funciton [37],[63].

3 Complexity of deciding stability for polynomial vector fields

The most basic algorithmic question one can ask about stability of equilibrium points of polynomial vector fields is whether this property can be decided in finite time. This is in fact a well-known question of Arnold that appears in [13]:

> "Is the stability problem for stationary points algorithmically decidable? The well-known Lyapounov theorem[2] solves the problem in the absence of eigenvalues with zero real parts. In more complicated cases, where the stability depends on higher order terms in the Taylor series, there exists no algebraic criterion.
>
> Let a vector field be given by polynomials of a fixed degree, with rational coefficients. Does an algorithm exist, allowing to decide, whether the stationary point is stable?"

Later in [30], the question of Arnold is quoted with more detail:

> "In my problem the coefficients of the polynomials of known degree and of a known number of variables are written on the tape of the standard Turing machine in the standard order and in the standard representation. The problem is whether there exists an algorithm (an additional text for the machine independent of the values of the coefficients) such that it solves the stability problem for the stationary point at the origin (i.e., always stops giving the answer "stable" or "unstable").
>
> I hope, this algorithm exists if the degree is one. It also exists when the dimension is one. My conjecture has always been that there is no algorithm for some sufficiently high degree and dimension, perhaps for dimension 3 and degree 3 or even 2. I am less certain about what happens in dimension 2. Of course the nonexistence of a general algorithm for a fixed dimension working for arbitrary degree or for a fixed degree working for an arbitrary dimension, or working for all polynomials with arbitrary degree and dimension would also be interesting."

To our knowledge, there has been no formal resolution to these questions, neither for the case of stability in the sense of Lyapunov, nor for the case of asymptotic stability (in its local or global version). In [30], da Costa and Doria show that if the right hand side of the differential equation contains elementary functions (sines, cosines, exponentials, absolute value function, etc.), then there is no algorithm for deciding whether the origin is stable or unstable. They also present a dynamical system in [31] where one cannot decide whether a Hopf bifurcation will occur or whether there will be parameter values such that a stable fixed point becomes unstable. In earlier work, Arnold himself demonstrates some of the difficulties that arise in stability analysis of polynomial systems by presenting a parametric polynomial system in 3 variables and degree 5, where the boundary between stability and instability in parameter space is not a semialgebraic set [12]. Similar approaches have been taken in the systems theory literature to show so-called "algebraic unsolvability" or "rational undecidability" of some fundamental stability questions in controls, such as that of simultaneous

[2] The theorem that Arnold is referring to here is the indirect method of Lyapunov related to linearization. This is not to be confused with Lyapunov's direct method (or the second method), which is what we are concerned with in sections that follow.

stabilizability of three linear systems [19], or stability of a pair of discrete time linear systems under arbitrary switching [46]. A relatively larger number of undecidability results are available for questions related to other properties of polynomial vector fields, such as reachability [38] or boundedness of domain of definition [35], or for questions about stability of hybrid systems [21], [24], [23], [20]. We refer the interested reader to the survey papers in [26], [38], [68], [22], [25].

While we are also unable to give a proof of undecidability of testing stability, we can establish the following result which gives a lower bound on the complexity of the problem.

Theorem 3.1. *Deciding (local or global) asymptotic stability of cubic polynomial vector fields is strongly NP-hard. This is true even when the vector field is restricted to be homogeneous.*

The implication of the NP-hardness of this problem is that unless P=NP, it is impossible to design an algorithm that can take as input the (rational) coefficients of a homogeneous cubic vector field, have running time bounded by a polynomial in the number of bits needed to represent these coefficients, and always output the correct yes/no answer on asymptotic stability. Moreover, the fact that our NP-hardness result is in the strong sense (as opposed to weakly NP-hard problems such as KNAPSACK, SUBSET SUM, etc.) implies that the problem remains NP-hard even if the size (bit length) of the coefficients is $O(\log n)$, where n is the dimension. For a strongly NP-hard problem, even a pseudo-polynomial time algorithm cannot exist unless P=NP. See [34] for precise definitions and more details. It would be very interesting to settle whether for the restricted class of homogeneous vector fields, the stability testing problem is decidable.

Theorem 3.1 in particular suggests that unless P=NP, we should not expect sum of squares Lyapunov functions of "low enough" degree to always exist, even when the analysis is restricted to cubic homogeneous vector fields. The semidefinite program arising from a search for an sos Lyapunov function of degree $2d$ for such a vector field in n variables has size in the order of $\binom{n+d}{d+1}$. This number is polynomial in n for fixed d (but exponential in n when d grows linearly in n). Therefore, unlike the case of linear systems for which sos quadratic Lyapunov functions always exist, we should not hope to have a bound on the degree of sos Lyapunov functions that is independent of the dimension. We postpone our study of existence of sos Lyapunov functions to Section 5 and continue for now with our complexity explorations.

Before the proof of Theorem 3.1 is presented, let us remark that this hardness result is minimal in the degree as far as homogeneous vector fields are concerned. It is not hard to show that for quadratic homogeneous vector fields (and in fact for all even degree homogeneous vector fields), the origin can never be asymptotically stable; see e.g. [37, p. 283]. This leaves us with homogeneous polynomial vector fields of degree one (i.e., linear systems) for which asymptotic stability can be decided in polynomial time, e.g. as shown in the following folklore lemma.

Lemma 3.2. *There is a polynomial time algorithm for deciding asymptotic stability of the equilibrium point of the linear system $\dot{x} = Ax$.*[3]

Proof. Consider the following algorithm. Given the matrix A, solve the following linear system (the Lyapunov equation) for the symmetric matrix P:

$$A^T P + PA = -I,$$

where I here is the identity matrix of the same dimension as A. If there is no solution to this system, output "no". If a solution matrix P is found, test whether P is a positive definite matrix. If yes,

[3]This of course is equivalent to the matrix A being Hurwitz, i.e., having all eigenvalues in the open left half complex plane. Theorem 3.1 suggests that no simple characterization of this type should be possible for cubic vector fields.

output "yes". If not, output "no". The fact that this algorithm is correct is standard. Moreover, the algorithm runs in polynomial time (in fact in $O(n^3)$) since linear systems can be solved in polynomial time, and we can decide if a matrix is positive definite e.g. by checking whether all its n leading principal minors are positive rational numbers. Each determinant can be computed in polynomial time. $\square$

Corollary 3.3. *Local exponential stability*[4] *of polynomial vector fields of any degree can be decided in polynomial time.*

Proof. A polynomial vector field is locally exponentially stable if and only if its linearization is Hurwitz [45]. The linearization matrix can trivially be written down in polynomial time and its Hurwitzness can be checked by the algorithm presented in the previous lemma. $\square$

Note that Theorem 3.1 and Corollary 3.3 draw a sharp contrast between the complexity of checking local asymptotic stability versus local exponential stability. Observe also that homogeneous vector fields of degree larger than one can never be exponentially stable (since their linearization matrix is the zero matrix). Indeed, the difficulty in deciding asymptotic stability initiates when the linearization of the vector field has eigenvalues on the imaginary axis, so that Lyapunov's linearization test cannot conclusively be employed to test stability.

We now proceed to the proof of Theorem 3.1. The main intuition behind this proof is the following idea: We will relate the solution of a combinatorial problem not to the behavior of the trajectories of a cubic vector field that are hard to get a handle on, but instead to properties of a Lyapunov function that proves asymptotic stability of this vector field. As we will see shortly, insights from Lyapunov theory make the proof of this theorem relatively simple. The reduction is broken into two steps:

$$\begin{array}{c}\text{ONE-IN-THREE 3SAT}\\ \downarrow\\ \text{positivity of quartic forms}\\ \downarrow\\ \text{asymptotic stability of cubic vector fields}\end{array}$$

3.1 Reduction from ONE-IN-THREE 3SAT to positivity of quartic forms

It is well-known that deciding nonnegativity (i.e., positive semidefiniteness) of quartic forms is NP-hard. The proof commonly cited in the literature is based on a reduction from the matrix copositivity problem [51]: given a symmetric $n \times n$ matrix Q, decide whether $x^TQx \geq 0$ for all x's that are elementwise nonnegative. Clearly, a matrix Q is copositive if and only if the quartic form z^TQz, with $z_i := x_i^2$, is nonnegative. The original reduction [51] proving NP-hardness of testing matrix copositivity is from the subset sum problem and only establishes weak NP-hardness. However, reductions from the stable set problem to matrix copositivity are also known [32], [33] and they result in NP-hardness in the strong sense.

For reasons that will become clear shortly, we are interested in showing hardness of deciding *positive definiteness* of quartic forms as opposed to positive semidefiniteness. This is in some sense even easier to accomplish. A very straightforward reduction from 3SAT proves NP-hardness of deciding positive definiteness of polynomials of degree 6. By using ONE-IN-THREE 3SAT instead, we will reduce the degree of the polynomial from 6 to 4.

[4] See [45] for a definition.

Proposition 3.4. *It is strongly*[5] *NP-hard to decide whether a homogeneous polynomial of degree* 4 *is positive definite.*

Proof. We give a reduction from ONE-IN-THREE 3SAT which is known to be NP-complete [34, p. 259]. Recall that in ONE-IN-THREE 3SAT, we are given a 3SAT instance (i.e., a collection of clauses, where each clause consists of exactly three literals, and each literal is either a variable or its negation) and we are asked to decide whether there exists a $\{0,1\}$ assignment to the variables that makes the expression true with the additional property that each clause has *exactly one* true literal.

To avoid introducing unnecessary notation, we present the reduction on a specific instance. The pattern will make it obvious that the general construction is no different. Given an instance of ONE-IN-THREE 3SAT, such as the following

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_4), \tag{6}$$

we define the quartic polynomial p as follows:

$$\begin{array}{rcl} p(x) & = & \sum_{i=1}^{5} x_i^2(1-x_i)^2 \\ & & +(x_1+(1-x_2)+x_4-1)^2+((1-x_2)+(1-x_3)+x_5-1)^2 \\ & & +((1-x_1)+x_3+(1-x_5)-1)^2+(x_1+x_3+x_4-1)^2. \end{array} \tag{7}$$

Having done so, our claim is that $p(x) > 0$ for all $x \in \mathbb{R}^5$ (or generally for all $x \in \mathbb{R}^n$) if and only if the ONE-IN-THREE 3SAT instance is not satisfiable. Note that p is a sum of squares and therefore nonnegative. The only possible locations for zeros of p are by construction among the points in $\{0,1\}^5$. If there is a satisfying Boolean assignment x to (6) with exactly one true literal per clause, then p will vanish at point x. Conversely, if there are no such satisfying assignments, then for any point in $\{0,1\}^5$, at least one of the terms in (7) will be positive and hence p will have no zeros.

It remains to make p homogeneous. This can be done via introducing a new scalar variable y. If we let

$$p_h(x,y) = y^4 p(\tfrac{x}{y}), \tag{8}$$

then we claim that p_h (which is a quartic form) is positive definite if and only if p constructed as in (7) has no zeros.[6] Indeed, if p has a zero at a point x, then that zero is inherited by p_h at the point $(x,1)$. If p has no zeros, then (8) shows that p_h can only possibly have zeros at points with $y=0$. However, from the structure of p in (7) we see that

$$p_h(x,0) = x_1^4 + \cdots + x_5^4,$$

which cannot be zero (except at the origin). This concludes the proof. $\square$

3.2 Reduction from positivity of quartic forms to asymptotic stability of cubic vector fields

We now present the second step of the reduction and finish the proof of Theorem 3.1.

[5]The NP-hardness results of this section will all be in the strong sense. From here on, we will drop the prefix "strong" for brevity.

[6]In general, homogenization does not preserve positivity. For example, as shown in [62], the polynomial $x_1^2 + (1-x_1x_2)^2$ has no zeros, but its homogenization $x_1^2y^2 + (y^2 - x_1x_2)^2$ has zeros at the points $(1,0,0)^T$ and $(0,1,0)^T$. Nevertheless, positivity is preserved under homogenization for the special class of polynomials constructed in this reduction, essentially because polynomials of type (7) have no zeros at infinity.

Proof of Theorem 3.1. We give a reduction from the problem of deciding positive definiteness of quartic forms, whose NP-hardness was established in Proposition 3.4. Given a quartic form $V := V(x)$, we define the polynomial vector field

$$\dot{x} = -\nabla V(x). \tag{9}$$

Note that the vector field is homogeneous of degree 3. We claim that the above vector field is (locally or equivalently globally) asymptotically stable if and only if V is positive definite. First, we observe that by construction

$$\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle = -||\nabla V(x)||^2 \leq 0. \tag{10}$$

Suppose V is positive definite. By Euler's identity for homogeneous functions,[7] we have $V(x) = \frac{1}{4}x^T \nabla V(x)$. Therefore, positive definiteness of V implies that $\nabla V(x)$ cannot vanish anywhere except at the origin. Hence, $\dot{V}(x) < 0$ for all $x \neq 0$. In view of Lyapunov's theorem (see e.g. [45, p. 124]), and the fact that a positive definite homogeneous function is radially unbounded, it follows that the system in (9) is globally asymptotically stable.

For the converse direction, suppose (9) is GAS. Our first claim is that global asymptotic stability together with $\dot{V}(x) \leq 0$ implies that V must be positive semidefinite. This follows from the following simple argument, which we have also previously presented in [10] for a different purpose. Suppose for the sake of contradiction that for some $\hat{x} \in \mathbb{R}^n$ and some $\epsilon > 0$, we had $V(\hat{x}) = -\epsilon < 0$. Consider a trajectory $x(t; \hat{x})$ of system (9) that starts at initial condition $\hat{x}$, and let us evaluate the function V on this trajectory. Since $V(\hat{x}) = -\epsilon$ and $\dot{V}(x) \leq 0$, we have $V(x(t; \hat{x})) \leq -\epsilon$ for all $t > 0$. However, this contradicts the fact that by global asymptotic stability, the trajectory must go to the origin, where V, being a form, vanishes.

To prove that V is positive definite, suppose by contradiction that for some nonzero point $x^* \in \mathbb{R}^n$ we had $V(x^*) = 0$. Since we just proved that V has to be positive semidefinite, the point x^* must be a global minimum of V. Therefore, as a necessary condition of optimality, we should have $\nabla V(x^*) = 0$. But this contradicts the system in (9) being GAS, since the trajectory starting at x^* stays there forever and can never go to the origin.[8] □

3.3 Complexity of deciding other qualitative properties of the trajectories

We next establish NP-hardness of several other decision questions associated with polynomial vector fields, which arise in numerous areas of systems and control theory. Most of the results will be rather straightforward corollaries of Theorem 3.1. In what follows, whenever a property has to do with an equilibrium point, we take this equilibrium point to be at the origin. The norm $||.||$ is always the Euclidean norm, and the notation B_r denotes the ball of radius r; i.e., $B_r := \{x \mid ||x|| \leq r\}$.

Theorem 3.5 (see also [6]). *For polynomial differential equations of degree d (with d specified below), the following properties are NP-hard to decide:*
(a) $d = 3$, Inclusion of a ball in the region of attraction of an equilibrium point*:* $\forall x(0)$ *with* $||x(0)|| \leq 1$,

$$x(t) \to 0, \text{ as } t \to \infty.$$

[7] Euler's identity, $V(x) = \frac{1}{d}x^T \nabla V(x)$, is easily derived by differentiating both sides of the equation $V(\lambda x) = \lambda^d V(x)$ with respect to λ and setting $\lambda = 1$.

[8] After a presentation of this work at UCLouvain, P.-A. Absil kindly brought to our attention that similar connections between local optima of real analytic functions and local stability of their gradient systems appear in [1].

(b) $d = 3$*,* Local attractivity of an equilibrium point*:* $\exists \delta > 0$ *such that* $\forall x(0) \in B_\delta$*,*

$$x(t) \to 0, \;\; as \;\; t \to \infty.$$

(c) $d = 4$*,* Stability of an equilibrium point in the sense of Lyapunov*:* $\forall \epsilon > 0$*,* $\exists \delta = \delta(\epsilon)$ *such that*

$$||x(0)|| < \delta \;\Rightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0.$$

(d) $d = 3$*,* Boundedness of trajectories*:* $\forall x(0)$*,* $\exists r = r(x(0))$ *such that*

$$||x(t)|| < r, \quad \forall t \geq 0.$$

(e) $d = 3$*,* Existence of a local quadratic Lyapunov function*:* $\exists \delta > 0$ *and a quadratic Lyapunov function* $V(x) = x^T P x$ *such that* $V(x) > 0$ *for all* $x \in B_\delta,\ x \neq 0$ *(or equivalently* $P \succ 0$*), and*

$$\dot{V}(x) < 0, \quad \forall x \in B_\delta, \; x \neq 0.$$

(f) $d = 4$*,* Local collision avoidance*:* $\exists \delta > 0$ *such that* $\forall x(0) \in B_\delta$*,*

$$x(t) \notin \mathcal{S}, \quad \forall t \geq 0,$$

where $\mathcal{S}$ *is a given polytope.*
(g) $d = 3$*,* Existence of a stabilizing controller*: There exists a particular (e.g. smooth, or polynomial of fixed degree) control law* $u(x)$ *that makes the origin of*

$$\dot{x} = f(x) + g(x)u(x)$$

locally asymptotically stable, where f *and* $g \neq 0$ *here have degrees* 3.[9]
(h) $d = 3$*,* Invariance of a ball*:* $\forall x(0)$ *with* $||x(0)|| \leq 1$*,*

$$||x(t)|| \leq 1, \quad \forall t \geq 0.$$

(i) $d = 1$*,* Invariance of a basic semialgebraic set defined by a quartic polynomial*:* $\forall x(0) \in \mathcal{S}$*,*

$$x(t) \in \mathcal{S}, \quad \forall t \geq 0,$$

where $\mathcal{S} := \{x \mid p(x) \leq 1\}$ *and* p *is a given form of degree four.*

Proof. The proofs of (a)-(g) will be via reductions from the problem of testing local asymptotic stability (LAS) of cubic vector fields established in Theorem 3.1. These reductions in fact depend on the specific structure of the vector field constructed in the proof of Theorem 3.1:

$$\dot{x} := f(x) = -\nabla V(x), \tag{11}$$

where V is a quartic form. We recall some facts:

[9] The size of g and u here are respectively $n \times k$ and $k \times 1$ and the statement is true for any $k \in \{1, \ldots, n\}$. We present the proof for $k = n$; the other proofs are identical.

1. The vector field in (11) is homogeneous and therefore it is locally asymptotically stable if and only if it is globally asymptotically stable.

2. The vector field is (locally or globally) asymptotically stable if and only if V is positive definite. Hence, the system, if asymptotically stable, by construction always admits a quartic Lyapunov function.

3. If the vector field is *not* asymptotically stable, then there always exists a nonzero point $\bar{x} \in \{0,1\}^n$ such that $f(\bar{x}) = 0$; i.e., $\bar{x}$ is a nonzero equilibrium point.

We now proceed with the proofs. In what follows, $f(x)$ will always refer to the vector field in (11).

(a) The claim is an obvious implication of the homogeneity of f. Since $f(\lambda x) = \lambda^3 f(x)$, for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, the origin is LAS if and only if for any r, all trajectories in B_r converge to the origin.[10]

(b) If f is LAS, then of course it is by definition locally attractive. On the other hand, if f is not LAS, then $f(\bar{x}) = 0$ for some nonzero $\bar{x} \in \{0,1\}^n$. By homogeneity of f, this implies that $f(\alpha\bar{x}) = 0, \ \forall \alpha \geq 0$. Therefore, arbitrarily close to the origin we have stationary points and hence the origin cannot be locally attractive.

(c) Let $x^4 := (x_1^4, \ldots, x_n^4)^T$. Consider the vector field

$$\dot{x} = f(x) + x^4. \tag{12}$$

We claim that the origin of (12) is stable in the sense of Lyapunov if and only if the origin of (11) is LAS. Suppose first that (11) is not LAS. Then we must have $f(\alpha\bar{x}) = 0$ for some nonzero $\bar{x} \in \{0,1\}^n$ and $\forall \alpha \geq 0$. Therefore for the system (12), trajectories starting from *any* nonzero point on the line connecting the origin to $\bar{x}$ shoot out to infinity while staying on the line. (This is because on this line, the dynamics are simply $\dot{x} = x^4$.) As a result, stability in the sense of Lyapunov does not hold as there exists an $\epsilon > 0$ (in fact for any $\epsilon > 0$), for which $\nexists \delta > 0$ such that trajectories starting in B_δ stay in B_ϵ. Indeed, as we argued, arbitrarily close to the origin we have points that shoot out to infinity.

Let us now show the converse. If (11) is LAS, then V is indeed a strict Lyapunov function for it; i.e. it is positive definite and has a negative definite derivative $-||\nabla V(x)||^2$. Using the same Lyapunov function for the system in (12), we have

$$\dot{V}(x) = -||\nabla V(x)||^2 + \langle \nabla V(x), x^4 \rangle.$$

Note that the first term in this expression is a homogeneous polynomial of degree 6 while the second term is a homogeneous polynomial of degree 7. Negative definiteness of the lower order term implies that there exists a positive real number δ such that $\dot{V}(x) < 0$ for all nonzero $x \in B_\delta$. This together with positive definiteness of V implies via Lyapunov's theorem that (12) is LAS and hence stable in the sense of Lypunov.

(d) Consider the vector field

$$\dot{x} = f(x) + x. \tag{13}$$

We claim that the trajectories of (13) are bounded if and only if the origin of (11) is LAS. Suppose (11) is not LAS. Then, as in the previous proof, there exists a line connecting the origin to a point

[10] For a general cubic vector field, validity of property (a) for a particular value of r is of course not necessary for local asymptotic stability. The reader should keep in mind that the class of homogeneous cubic vector fields is a subset of the class of all cubic vector fields, and hence any hardness result for this class immediately implies the same hardness result for all cubic vector fields.

$\bar{x} \in \{0,1\}^n$ such that trajectories on this line escape to infinity. (In this case, the dynamics on this line is governed by $\dot{x} = x$.) Hence, not all trajectories can be bounded. For the converse, suppose that (11) is LAS. Then V (resp. $-||\nabla V(x)||^2$) must be positive (resp. negative) definite. Now if we consider system (13), the derivative of V along its trajectories is given by

$$\dot{V}(x) = -||\nabla V(x)||^2 + \langle \nabla V(x), x \rangle.$$

Since the first term in this expression has degree 6 and the second term degree 4, there exists an r such that $\dot{V} < 0$ for all $x \notin B_r$. This condition however implies boundedness of trajectories; see e.g. [40].

(e) If f is not LAS, then there cannot be any local Lyapunov functions, in particular not a quadratic one. If f is LAS, then we claim the quadratic function $W(x) = ||x||^2$ is a valid (and in fact global) Lyapunov function for it. This can be seen from Euler's identity

$$\dot{W}(x) = \langle 2x, -\nabla V(x) \rangle = -8V(x),$$

and by noting that V must be positive definite.

(f) We define our dynamics to be the one in (12), and the polytope $\mathcal{S}$ to be

$$\mathcal{S} = \{x \mid x_i \geq 0, 1 \leq \sum_{i=1}^{n} x_i \leq 2\}.$$

Suppose first that f is not LAS. Then by the argument given in (e), the system in (12) has trajectories that start out arbitrarily close to the origin (at points of the type $\alpha\bar{x}$ for some $\bar{x} \in \{0,1\}^n$ and for arbitrarily small α), which exit on a straight line to infinity. Note that by doing so, such trajectories must cross $\mathcal{S}$; i.e. there exists a positive real number $\bar{\alpha}$ such that $1 \leq \sum_{i=1}^{n} \bar{\alpha}\bar{x}_i \leq 2$. Hence, there is no neighborhood around the origin whose trajectories avoid $\mathcal{S}$.

For the converse, suppose f is LAS. Then, we have shown while proving (e) that (12) must also be LAS and hence stable in the sense of Lyapunov. Therefore, there exists $\delta > 0$ such that trajectories starting from B_δ do not leave $B_{\frac{1}{2}}$—a ball that is disjoint from $\mathcal{S}$.

(g) Let f be as in (11) and $g(x) = (x_1 x_2^2 - x_1^2 x_2)\mathbf{1}\mathbf{1}^T$, where $\mathbf{1}$ denotes the vector of all ones. The following simple argument establishes the desired NP-hardness result irrespective of the type of control law we may seek (e.g. linear control law, cubic control law, smooth control law, or anything else). If f is LAS, then of course there exists a stabilizing controller, namely $u = 0$. If f is not LAS, then it must have an equilibrium point at a nonzero point $\bar{x} \in \{0,1\}^n$. Note that by construction, g vanishes at all such points. Since g is homogeneous, it also vanishes at all points $\alpha\bar{x}$ for any scalar α. Therefore, arbitrarily close to the origin, there are equilibrium points that the control law $u(x)$ cannot possibly remove. Hence there is no controller that can make the origin LAS.

The proofs of parts (h) and (i) of the theorem are based on a reduction from the polynomial nonnegativity problem for quartics: given a (homogeneous) degree-4 polynomial $p : \mathbb{R}^n \to \mathbb{R}$, decide whether $p(x) \geq 0, \forall x \in \mathbb{R}^n$? (See the opening paragraph of Subsection 3.1 for references on NP-hardness of this problem.)

(h) Given a quartic form p, we construct the vector field

$$\dot{x} = -\nabla p(x).$$

Note that the vector field has degree 3 and is homogeneous. We claim that the unit ball B_1 is invariant under the trajectories of this system if and only if p is nonnegative. This of course establishes the desired NP-hardness result. To prove the claim, consider the function $V(x) := ||x||^2$.

Clearly, B_1 is invariant under the trajectories of $\dot{x} = -\nabla p(x)$ if and only if $\dot{V}(x) \leq 0$ for all x with $||x|| = 1$. Since $\dot{V}$ is a homogeneous polynomial, this condition is equivalent to having $\dot{V}$ nonpositive for all $x \in \mathbb{R}^n$. However, from Euler's identity we have

$$\dot{V}(x) = \langle 2x, -\nabla p(x) \rangle = -8p(x).$$

(i) Once again, we provide a reduction from the problem of checking nonnegativity of quartic forms. Given a quartic form p, we let the set $\mathcal{S}$ be defined as $\mathcal{S} = \{x \mid p(x) \leq 1\}$. Let us consider the linear dynamical system

$$\dot{x} = -x.$$

We claim that $\mathcal{S}$ is invariant under the trajectories of this linear system if and only if p is nonnegative. To see this, consider the derivative $\dot{p}$ of p along the trajectories of $\dot{x} = -x$ and note its homogeneity. With the same reasoning as in the proof of part (h), $\dot{p}(x) \leq 0$ for all $x \in \mathbb{R}^n$ if and only if the set $\mathcal{S}$ is invariant. From Euler's identity, we have

$$\dot{p}(x) = \langle \nabla p(x), -x \rangle = -4p(x).$$

Note that the role of the dynamics and the gradient of the "Lyapunov function" are swapped in the proofs of (h) and (i).

□

Remark 3.1. Arguments similar to the one presented in the proof of (g) above can be given to show NP-hardness of deciding existence of a controller that establishes several other properties, e.g., boundedness of trajectories, inclusion of the unit ball in the region of attraction, etc. In the statement of (f), the fact that the set $\mathcal{S}$ is a polytope is clearly arbitrary. This choice is only made because "obstacles" are most commonly modeled in the literature as polytopes. We also note that a related problem of interest here is that of deciding, given two polytopes, whether all trajectories starting in one avoid the other. This question is the complement of the usual reachability question, for which claims of undecidability have already appeared [38]; see also [17].

Remark 3.2. In [6], NP-hardness of testing local asymptotic stability is established also for vector fields that are *trigonometric*. Such vector fields appear commonly in the field of robotics.

4 Non-existence of polynomial Lyapunov functions or degree bounds

4.1 Non-existence of polynomial Lyapunov functions

As mentioned in the introduction, the question of global asymptotic stability of polynomial vector fields is commonly addressed by seeking a Lyapunov function that is polynomial itself. This approach has become further prevalent over the past decade due to the fact that we can use sum of squares techniques to algorithmically search for such Lyapunov functions. The question therefore naturally arises as to whether existence of polynomial Lyapunov functions is necessary for global stability of polynomial systems. This question appears explicitly, e.g., in [57, Sect. VIII]. In this section, we present a remarkably simple counterexample which shows that the answer to this question is negative. In view of the fact that globally asymptotically stable linear systems always admit quadratic Lyapunov functions, it is quite interesting to observe that the following vector field that is arguably "the next simplest system" to consider does not admit a polynomial Lyapunov function of any degree.

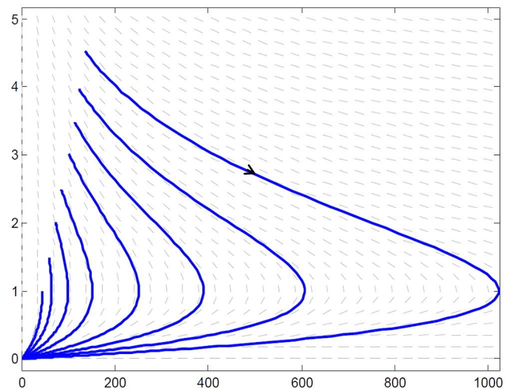


Figure 1: Typical trajectories of the vector field in (14) starting from initial conditions in the nonnegative orthant.

Theorem 4.1 (Ahmadi, Krstic, Parrilo, [5]). *Consider the polynomial vector field*

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= -y. \end{aligned} \tag{14}$$

The origin is a globally asymptotically stable equilibrium point, but the system does not admit a polynomial Lyapunov function.

The proof of this theorem is presented in [5] and omitted from here. Global asymptotic stability is established by presenting a Lyapunov function that involves the logarithm function. Non-existence of polynomial Lyapunov functions has to do with "fast growth rates" far away from the origin; see some typical trajectories of this vector field in Figure 1 and reference [5] for more details.

Example of Bacciotti and Rosier. An independent counterexample to existence of polynomial Lyapunov functions appears in a book by Bacciotti and Rosier [15, Prop. 5.2]. The differences between the two counterexamples are explained in some detail in [5], the main one being that the example of Bacciotti and Rosier relies crucially on a coefficient appearing in the vector field being an *irrational* number and is not robust to perturbations. (In practical applications where computational techniques for searching over Lyapunov functions on finite precision machines are used, such issues with irrationality of the input cannot occur.) On the other hand, the example of Bacciotti and Rosier rules out existence of a polynomial (or even analytic) Lyapunov function *locally*, as the argument is based on slow decay rates arbitrarily close to the origin. In [56], Peet shows that locally exponentially stable polynomial vector fields admit polynomial Lyapunov functions on compact sets. The example of Bacciotti and Rosier implies that the assumption of exponential stability indeed cannot be dropped.

4.2 Homogeneous vector fields: non-existence of a uniform bound on the degree of polynomial Lyapunov functions in fixed dimension and degree

In this subsection, we restrict attention to polynomial vector fields that are homogeneous. Although the results of the last subsection imply that existence of polynomial Lyapunov functions is not necessary for global asymptotic stability of polynomial vector fields, the situation seems to be different for homogeneous vector fields. Although we have not been able to formally prove this, we conjecture the following:

Existence of a *homogeneous polynomial Lyapunov function* is necessary and sufficient for global (or equivalently local) asymptotic stability of a *homogeneous polynomial vector field.*

One reason why we are interested in proving this conjecture is that in Section 5, we will show that for homogeneous polynomial vector fields, existence of a homogeneous polynomial Lyapunov function implies existence of a homogeneous *sum of squares* Lyapunov function. Hence, the two statements put together would imply that stability of homogeneous vector fields can always be checked via sum of squares techniques (and hence semidefinite programming).

In this subsection, however, we build on the work of Bacciotti and Rosier [15, Prop. 5.2] to prove that the minimum degree of a polynomial Lyapunov function for an asymptotically stable homogeneous vector field can be *arbitrarily large* even when the degree and dimension are fixed respectively to 3 and 2.

Proposition 4.2 ([15, Prop. 5.2–a])**.** *Consider the vector field*

$$\begin{array}{rcl} \dot{x} & = & -2\lambda y(x^2+y^2) - 2y(2x^2+y^2) \\ \dot{y} & = & 4\lambda x(x^2+y^2) + 2x(2x^2+y^2) \end{array} \tag{15}$$

parameterized by the scalar $\lambda > 0$. For all values of λ the origin is a center for (15), but for any irrational value of λ there exists no polynomial function V satisfying $\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 0$.

Theorem 4.3. *Let λ be a positive irrational real number and consider the following homogeneous cubic vector field parameterized by the scalar θ:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -2\lambda y(x^2+y^2) - 2y(2x^2+y^2) \\ 4\lambda x(x^2+y^2) + 2x(2x^2+y^2) \end{pmatrix}. \tag{16}$$

Then for any even degree d of a candidate polynomial Lyapunov function, there exits a $\theta > 0$ small enough such that the vector field in (16) is asymptotically stable but does not admit a polynomial Lyapunov function of degree $\leq d$.

Proof. Consider the (non-polynomial) positive definite Lyapunov function

$$V(x,y) = (2x^2+y^2)^{\lambda}(x^2+y^2)$$

whose derivative along the trajectories of (16) is equal to

$$\dot{V}(x,y) = -\sin(\theta)(2x^2+y^2)^{\lambda-1}(\dot{x}^2+\dot{y}^2).$$

Since $\dot{V}$ is negative definite for $0 < \theta < \pi$, it follows that for θ in this range, the origin of (16) is asymptotically stable.

To establish the claim in the theorem, suppose for the sake of contradiction that there exists an upper bound $\bar{d}$ such that for all $0 < \theta < \pi$ the system admits a (homogeneous) polynomial Lyapunov function of degree $d(\theta)$ with $d(\theta) \leq \bar{d}$. Let $\hat{d}$ be the least common multiplier of the degrees $d(\theta)$ for $0 < \theta < \pi$. (Note that $d(\theta)$ can at most range over all even positive integers less than or equal to $\bar{d}$.) Since positive powers of Lyapunov functions are valid Lyapunov functions, it follows that for every $0 < \theta < \pi$, the system admits a homogeneous polynomial Lyapunov function W_θ of degree $\hat{d}$. By rescaling, we can assume without loss of generality that all Lyapunov functions W_θ have unit area on the unit sphere. Let us now consider the sequence $\{W_\theta\}$ as $\theta \to 0$. We think of this sequence as residing in a compact subset of $\mathbb{R}^{\hat{d}+1}$ associated with the set $P_{2,\hat{d}}$ of (coefficients of) all nonnegative bivariate homogeneous polynomials of degree $\hat{d}$ with unit area on the unit sphere. Since every bounded sequence has a converging subsequence, it follows that there must exist a subsequence of $\{W_\theta\}$ that converges (in the coefficient sense) to some polynomial W_0 belonging to $P_{2,\hat{d}}$. Since convergence of this subsequence also implies convergence of the associated gradient vectors, we get that

$$\dot{W}_0(x,y) = \frac{\partial W_0}{\partial x}\dot{x} + \frac{\partial W_0}{\partial y}\dot{y} \leq 0.$$

On the other hand, when $\theta = 0$, the vector field in (16) is the same as the one in (15) and hence the trajectories starting from any nonzero initial condition go on periodic orbits. This however implies that $\dot{W} = 0$ everywhere and in view of Proposition 4.2 we have a contradiction. □

Remark 4.1. Unlike the result in [15, Prop. 5.2], it is easy to establish the result of Theorem 4.3 without having to use irrational coefficients in the vector field. One approach is to take an irrational number, e.g. π, and then think of a sequence of vector fields given by (16) that is parameterized by both θ and λ. We let the k-th vector field in the sequence have $\theta_k = \frac{1}{k}$ and λ_k equal to a rational number representing π up to k decimal digits. Since in the limit as $k \to \infty$ we have $\theta_k \to 0$ and $\lambda_k \to \pi$, it should be clear from the proof of Theorem 4.3 that for any integer d, there exists an AS bivariate homogeneous cubic vector field with *rational* coefficients that does not have a polynomial Lyapunov function of degree less than d.

4.3 Lack of monotonicity in the degree of polynomial Lyapunov functions

In the last part of this section, we point out yet another difficulty associated with polynomial Lyapunov functions: lack of monotonicity in their degree. Among other scenarios, this issue can arise when one seeks homogeneous polynomial Lyapunov functions (e.g., when analyzing stability of homogeneous vector fields). Note that unlike the case of (non-homogeneous) polynomials, the set of homogeneous polynomials of degree d is not a subset of the set of homogeneous polynomials of degree $\geq d$.

If a dynamical system admits a quadratic Lyapunov function V, then it also admits a polynomial Lyapunov function of any higher even degree (e.g., simply given by V^k for $k = 2, 3, \ldots$). However, our next theorem shows that for homogeneous systems that do not admit a quadratic Lyapunov function, such a monotonicity property in the degree of polynomial Lyapunov functions may not hold.

Theorem 4.4. *Consider the following homogeneous cubic vector field parameterized by the scalar θ:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\begin{pmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} x^3 \\ y^3 \end{pmatrix}. \tag{17}$$

There exists a range of values for the parameter $\theta > 0$ for which the vector field is asymptotically stable, has no homogeneous polynomial Lyapunov function of degree 6*, but admits a homogeneous polynomial Lyapunov function of degree* 4*.*

Proof. Consider the positive definite Lyapunov function

$$V(x,y) = x^4 + y^4. \tag{18}$$

The derivative of this Lyapunov function is given by

$$\dot{V}(x,y) = -4\sin(\theta)(x^6 + y^6),$$

which is negative definite for $0 < \theta < \pi$. Therefore, when θ belongs to this range, the origin of (16) is asymptotically stable and the system admits the degree 4 Lyapunov function given in

(18). On the other hand, we claim that for θ small enough, the system does not admit a degree 6 (homogeneous) polynomial Lyapunov function. To argue by contradiction, we suppose that for arbitrarily small and positive values of θ the system admits sextic Lyapunov functions W_θ. Since the vector field satisfies the symmetry (equivariance),

$$\begin{pmatrix} \dot{x}(y,-x) \\ \dot{y}(y,-x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix},$$

we can assume that the Lyapunov functions W_θ satisfy the symmetry $W_\theta(y,-x) = W_\theta(x,y)$. [11] This means that W_θ can be parameterized with no odd monomials, i.e., in the form

$$W_\theta(x,y) = c_1x^6 + c_2x^2y^4 + c_3x^4y^2 + c_4y^6,$$

where the coefficients $c_1,\ldots,c_4$ may depend on θ. Since by our assumption $\dot{W}_\theta$ is negative definite for θ arbitrarily small, an argument identical to the one used in the proof of Theorem 4.3 implies that as $\theta \to 0$, W_θ converges to a nonzero sextic homogeneous polynomial W_0 whose derivative $\dot{W}_0$ along the trajectories of (17) (with $\theta = 0$) is non-positive. However, note that when $\theta = 0$, the trajectories of (17) go on periodic orbits tracing the level sets of the function $x^4 + y^4$. This implies that

$$\dot{W}_0 = \frac{\partial W_0}{\partial x} y^3 + \frac{\partial W_0}{\partial y}(-x^3) = 0.$$

If we write out this equation, we obtain

$$\dot{W}_0 = (6c_1 - 4c_2)x^5y^3 + 2c_2xy^7 - 2c_3x^7y + (4c_3 - 6c_4)x^3y^5 = 0,$$

which implies that $c_1 = c_2 = c_3 = c_4 = 0$, hence a contradiction. $\square$

Remark 4.2. We have numerically computed the range $0 < \theta < 0.0267$, for which the conclusion of Theorem 4.4 holds. This bound has been computed via sum of squares relaxation and semidefinite programming (SDP) by using the SDP solver SeDuMi [69]. What allows the search for a Lyapunov function for the vector field in (17) to be exactly cast as a semidefinite program is the fact that all nonnegative bivariate forms are sums of squares.

5 (Non)-existence of sum of squares Lyapunov functions

In this section, we suppose that the polynomial vector field at hand admits a polynomial Lyapunov function, and we would like to investigate whether such a Lyapunov function can be found with sos programming. In other words, we would like to see whether the constrains in (4) and (5) are more conservative than the true Lyapunov inequalities in (2) and (3).

In 1888, Hilbert [42] showed that for polynomials in n variables and of degree d, the notions of nonnegativity and sum of squares are equivalent if and only if $n = 1$, $d = 2$, or (n,d)=$(2,4)$. A homogeneous version of the same result states that nonnegative homogeneous polynomials in n variables and of degree d are sums of squares if and only if $n = 2$, $d = 2$, or (n,d)=$(3,4)$. The first explicit example of a nonnegative polynomial that is not sos is due to Motzkin [50] and appeared nearly 80 years after the paper of Hilbert; see the survey in [62]. Still today, finding examples of such polynomials is a challenging task, especially if additional structure is required on the polynomial; see e.g. [9]. This itself is a premise for the powerfulness of sos techniques at least in low dimensions and degrees.

[11] To see this, note that any Lyapunov function V_θ for this system can be made into one satisfying this symmetry by letting $W_\theta(x,y) = V_\theta(x,y) + V_\theta(y,-x) + V_\theta(-x,-y) + V_\theta(-y,x)$.

Remark 5.1. Existence of nonnegative polynomials that are not sums of squares does not imply on its own that the sos conditions in (4) and (5) are more conservative than the Lyapunov inequalities in (2) and (3). Since Lyapunov functions are not in general unique, it could happen that within the set of valid polynomial Lyapunov functions of a given degree, there is always at least one that satisfies the sos conditions (4) and (5). Moreover, many of the known examples of nonnegative polynomials that are not sos have multiple zeros and local minima [62] and therefore cannot serve as Lyapunov functions. Indeed, if a function has a local minimum other than the origin, then its value evaluated on a trajectory starting from the local minimum would not be decreasing.

5.1 A motivating example

The following is a concrete example of the use of sum of squares techniques for finding Lyapunov functions. It will also motivate the type of questions that we would like to study in this section; namely, if sos programming fails to find a polynomial Lyapunov function of a particular degree, then what are the different possibilities for existence of polynomial Lyapunov functions?

Example 5.1. Consider the dynamical system

$$\begin{array}{rcl} \dot{x_1} & = & -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 \\ & & +14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7 \\ \\ \dot{x_2} & = & -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 \\ & & +807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7. \end{array} \tag{19}$$

A typical trajectory of the system that starts from the initial condition $x_0 = (2,2)^T$ is plotted in Figure 2. Our goal is to establish global asymptotic stability of the origin by searching for a polynomial Lyapunov function. Since the vector field is homogeneous, the search can be restricted to homogeneous Lyapunov functions [37], [63]. To employ the sos technique, we can use the software package SOSTOOLS [60] to search for a Lyapunov function satisfying the sos conditions (4) and (5). However, if we do this, we will not find any Lyapunov functions of degree 2, 4, or 6. If needed, a certificate from the dual semidefinite program can be obtained, which would prove that no polynomial of degree up to 6 can satisfy the sos requirements (4) and (5).

At this point we are faced with the following question. Does the system really not admit a Lyapunov function of degree 6 that satisfies the true Lyapunov inequalities in (2), (3)? Or is the failure due to the fact that the sos conditions in (4), (5) are more conservative?

Note that when searching for a degree 6 Lyapunov function, the sos constraint in (4) is requiring a homogeneous polynomial in 2 variables and of degree 6 to be a sum of squares. The sos condition (5) on the derivative is also a condition on a homogeneous polynomial in 2 variables, but in this case of degree 12. (This is easy to see from $\dot{V} = \langle \nabla V, f \rangle$.) Recall from our earlier discussion that nonnegativity and sum of squares are equivalent notions for homogeneous bivariate polynomials, irrespective of the degree. Hence, we now have a proof that this dynamical system truly does not have a Lyapunov function of degree 6 (or lower).

This fact is perhaps geometrically intuitive. Figure 2 shows that the trajectory of this system is stretching out in 8 different directions. So, we would expect the degree of the Lyapunov function to be at least 8. Indeed, when we increase the degree of the candidate function to 8, SOSTOOLS and the SDP solver SeDuMi [69] succeed in finding the following Lyapunov function:

$$\begin{array}{rcl} V(x) & = & 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 \\ & & -3.517x_1^4x_2^4 + 0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 \\ & & -0.015x_1x_2^7 + 0.02x_2^8. \end{array}$$

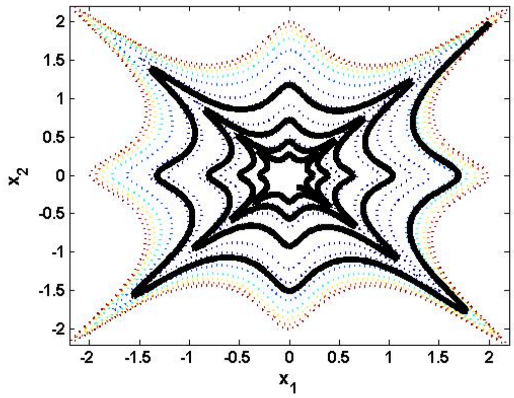


Figure 2: A typical trajectory of the vector filed in Example 5.1 (solid), level sets of a degree 8 polynomial Lyapunov function (dotted).

The level sets of this Lyapunov function are plotted in Figure 2 and are clearly invariant under the trajectory. $\triangle$

5.2 A counterexample to existence of sos Lyapunov functions

Unlike the scenario in the previous example, we now show that a failure in finding a Lyapunov function of a particular degree via sum of squares programming can also be due to the gap between nonnegativity and sum of squares. What will be conservative in the following counterexample is the sos condition on the derivative.

Consider the dynamical system

$$\begin{aligned} \dot{x_1} &= -x_1^3x_2^2 + 2x_1^3x_2 - x_1^3 + 4x_1^2x_2^2 - 8x_1^2x_2 + 4x_1^2 \\ &\quad -x_1x_2^4 + 4x_1x_2^3 - 4x_1 + 10x_2^2 \\ \dot{x_2} &= -9x_1^2x_2 + 10x_1^2 + 2x_1x_2^3 - 8x_1x_2^2 - 4x_1 - x_2^3 \\ &\quad +4x_2^2 - 4x_2. \end{aligned} \tag{20}$$

One can verify that the origin is the only equilibrium point for this system, and therefore it makes sense to investigate global asymptotic stability. If we search for a quadratic Lyapunov function for (20) using sos programming, we will not find one. It will turn out that the corresponding semidefinite program is infeasible. We will prove shortly why this is the case, i.e, why no quadratic function V can satisfy

$$\begin{array}{rl} V & \text{sos} \\ -\dot{V} & \text{sos}. \end{array} \tag{21}$$

Nevertheless, we claim that

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{22}$$

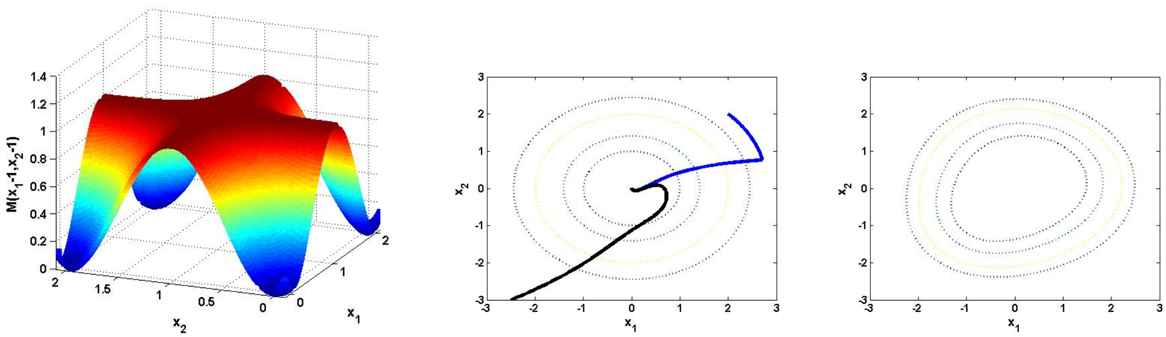
is a valid Lyapunov function. Indeed, one can check that

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -M(x_1 - 1, x_2 - 1), \tag{23}$$

where $M(x_1, x_2)$ is the Motzkin polynomial [50]:

$$M(x_1, x_2) = x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2 + 1.$$

This polynomial is just a dehomogenized version of the Motzkin form presented before, and it has the property of being nonnegative but not a sum of squares. The polynomial $\dot{V}$ is strictly negative everywhere, except for the origin and three other points $(0,2)^T$, $(2,0)^T$, and $(2,2)^T$, where $\dot{V}$ is zero. However, at each of these three points we have $\dot{x} \neq 0$. Once the trajectory reaches any of these three points, it will be kicked out to a region where $\dot{V}$ is strictly negative. Therefore, by LaSalle's invariance principle (see e.g. [45, p. 128]), the quadratic Lyapunov function in (22) proves global asymptotic stability of the origin of (20).



(a) Shifted Motzkin polynomial is nonnegative but not sos. This polynomial is $-\dot{V}$; see (23).

(b) Typical trajectories of (20) (solid), level sets of V (dotted).

(c) Level sets of a quartic Lyapunov function found through sos programming.

Figure 3: The quadratic polynomial $\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ is a valid Lyapunov function for the vector field in (20) but it is not detected through sos programming.

The fact that $\dot{V}$ is zero at three points other than the origin is not the reason why sos programming is failing. After all, when we impose the condition that $-\dot{V}$ should be sos, we allow for the possibility of a non-strict inequality. The reason why our sos program does not recognize (22) as a Lyapunov function is that the shifted Motzkin polynomial in (23) is nonnegative but it is not a sum of squares. This sextic polynomial is plotted in Figure 3(a). Trajectories of (20) starting at $(2,2)^T$ and $(-2.5,-3)^T$ along with level sets of V are shown in Figure 3(b).

So far, we have shown that V in (22) is a valid Lyapunov function but does not satisfy the sos conditions in (21). We still need to show why no other quadratic Lyapunov function

$$U(x) = c_1x_1^2 + c_2x_1x_2 + c_3x_2^2 \tag{24}$$

can satisfy the sos conditions either.[12] We will in fact prove the stronger statement that V in (22) is the only valid quadratic Lyapunov function for this system up to scaling, i.e., any quadratic function U that is not a scalar multiple of $\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ cannot satisfy $U \geq 0$ and $-\dot{U} \geq 0$. It will even be the case that no such U can satisfy $-\dot{U} \geq 0$ alone. (The latter fact is to be expected since global asymptotic stability of (20) together with $-\dot{U} \geq 0$ would automatically imply $U \geq 0$; see [10, Theorem 1.1].)

So, let us show that $-\dot{U} \geq 0$ implies U is a scalar multiple of $\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Because Lyapunov functions are closed under positive scalings, without loss of generality we can take $c_1 = 1$. One can check that

$$-\dot{U}(0,2) = -80c_2,$$

[12]Since we can assume that the Lyapunov function U and its gradient vanish at the origin, linear or constant terms are not needed in (24).

so to have $-\dot{U} \geq 0$, we need $c_2 \leq 0$. Similarly,

$$-\dot{U}(2,2) = -288c_1 + 288c_3,$$

which implies that $c_3 \geq 1$. Let us now look at

$$\begin{array}{rcl} -\dot{U}(x_1,1) & = & -c_2x_1^3 + 10c_2x_1^2 + 2c_2x_1 - 10c_2 - 2c_3x_1^2 \\ & & +20c_3x_1 + 2c_3 + 2x_1^2 - 20x_1. \end{array} \tag{25}$$

If we let $x_1 \to -\infty$, the term $-c_2x_1^3$ dominates this polynomial. Since $c_2 \leq 0$ and $-\dot{U} \geq 0$, we conclude that $c_2 = 0$. Once c_2 is set to zero in (25), the dominating term for x_1 large will be $(2 - 2c_3)x_1^2$. Therefore to have $-\dot{U}(x_1, 1) \geq 0$ as $x_1 \to \pm\infty$ we must have $c_3 \leq 1$. Hence, we conclude that $c_1 = 1, c_2 = 0, c_3 = 1$, and this finishes the proof.

Even though sos programming failed to prove stability of the system in (20) with a quadratic Lyapunov function, if we increase the degree of the candidate Lyapunov function from 2 to 4, then SOSTOOLS succeeds in finding a quartic Lyapunov function

$$\begin{array}{rcl} W(x) & = & 0.08x_1^4 - 0.04x_1^3 + 0.13x_1^2x_2^2 + 0.03x_1^2x_2 \\ & & +0.13x_1^2 + 0.04x_1x_2^2 - 0.15x_1x_2 \\ & & +0.07x_2^4 - 0.01x_2^3 + 0.12x_2^2, \end{array}$$

which satisfies the sos conditions in (21). The level sets of this function are close to circles and are plotted in Figure 3(c).

Motivated by this example, it is natural to ask whether it is always true that upon increasing the degree of the Lyapunov function one will find Lyapunov functions that satisfy the sum of squares conditions in (21). In the next subsection, we will prove that this is indeed the case, at least for planar systems such as the one in this example, and also for systems that are homogeneous.

5.3 Converse sos Lyapunov theorems

In [57], [58], it is shown that if a system admits a polynomial Lyapunov function, then it also admits one that is a sum of squares. However, the results there do not lead to any conclusions as to whether the negative of the derivative of the Lyapunov function is sos, i.e, whether condition (5) is satisfied. As we remarked before, there is therefore no guarantee that the semidefinite program can find such a Lyapunov function. Indeed, our counterexample in the previous subsection demonstrated this very phenomenon.

The proof technique used in [57],[58] is based on approximating the solution map using the Picard iteration and is interesting in itself, though the actual conclusion that a Lyapunov function that is sos exists has a far simpler proof which we give in the next lemma.

Lemma 5.1. *If a polynomial dynamical system has a positive definite polynomial Lyapunov function V with a negative definite derivative $\dot{V}$, then it also admits a positive definite polynomial Lyapunov function W which is a sum of squares.*

Proof. Take $W = V^2$. The negative of the derivative $-\dot{W} = -2V\dot{V}$ is clearly positive definite (though it may not be sos). □

We will next prove a converse sos Lyapunov theorem that guarantees the derivative of the Lyapunov function will also satisfy the sos condition, though this result is restricted to homogeneous systems. The proof of this theorem relies on the following Positivstellensatz result due to Scheiderer.

Theorem 5.2 (Scheiderer, [66]). *Given any two positive definite homogeneous polynomials p and q, there exists an integer k such that pq^k is a sum of squares.*

Theorem 5.3. *Given a homogeneous polynomial vector field, suppose there exists a homogeneous polynomial Lyapunov function V such that V and $-\dot{V}$ are positive definite. Then, there also exists a homogeneous polynomial Lyapunov function W such that W is sos and $-\dot{W}$ is sos.*

Proof. Observe that V^2 and $-2V\dot{V}$ are both positive definite and homogeneous polynomials. Applying Theorem 5.2 to these two polynomials, we conclude the existence of an integer k such that $(-2V\dot{V})(V^2)^k$ is sos. Let

$$W = V^{2k+2}.$$

Then, W is clearly sos since it is a perfect even power. Moreover,

$$-\dot{W} = -(2k+2)V^{2k+1}\dot{V} = -(k+1)2V^{2k}V\dot{V}$$

is also sos by the previous claim.[13] □

Finally, we develop a similar theorem that removes the homogeneity assumption from the vector field, but instead is restricted to vector fields on the plane. For this, we need another result of Scheiderer.

Theorem 5.4 (Scheiderer, [65, Cor. 3.12]). *Let $p := p(x_1, x_2, x_3)$ and $q := q(x_1, x_2, x_3)$ be two homogeneous polynomials in three variables, with p positive semidefinite and q positive definite. Then, there exists an integer k such that pq^k is a sum of squares.*

Theorem 5.5. *Given a (not necessarily homogeneous) polynomial vector field in two variables, suppose there exists a positive definite polynomial Lyapunov function V, with $-\dot{V}$ positive definite, and such that the highest degree homogeneous component of V has no zeros*[14]*. Then, there also exists a polynomial Lyapunov function W such that W is sos and $-\dot{W}$ is sos.*

Proof. Let $\tilde{V} = V + 1$. So, $\dot{\tilde{V}} = \dot{V}$. Consider the (non-homogeneous) polynomials $\tilde{V}^2$ and $-2\tilde{V}\dot{\tilde{V}}$ in the variables $x := (x_1, x_2)$. Let us denote the (even) degrees of these polynomials respectively by d_1 and d_2. Note that $\tilde{V}^2$ is nowhere zero and $-2\tilde{V}\dot{\tilde{V}}$ is only zero at the origin. Our first step is to homogenize these polynomials by introducing a new variable y. Observing that the homogenization of products of polynomials equals the product of homogenizations, we obtain the following two trivariate forms:

$$y^{2d_1}\tilde{V}^2(\tfrac{x}{y}), \tag{26}$$

$$-2y^{d_1}y^{d_2}\tilde{V}(\tfrac{x}{y})\dot{\tilde{V}}(\tfrac{x}{y}). \tag{27}$$

Since by assumption the highest order term of V has no zeros, the form in (26) is positive definite. The form in (27), however, is only positive semidefinite. In particular, since $\dot{\tilde{V}} = \dot{V}$ has to vanish at the origin, the form in (27) has a zero at the point $(x_1, x_2, y) = (0, 0, 1)$. Nevertheless, since Theorem 5.4 allows for positive semidefiniteness of one of the two forms, by applying it to the forms in (26) and (27), we conclude that there exists an integer k such that

$$-2y^{d_1(2k+1)}y^{d_2}\tilde{V}(\tfrac{x}{y})\dot{\tilde{V}}(\tfrac{x}{y})\tilde{V}^{2k}(\tfrac{x}{y}) \tag{28}$$

[13] Note that W constructed in this proof proves GAS since $-\dot{W}$ is positive definite and W itself being homogeneous and positive definite is automatically radially unbounded.

[14] This requirement is only slightly stronger than the requirement of radial unboundedness, which is imposed on V by Lyapunov's theorem anyway.

is sos. Let $W = \tilde{V}^{2k+2}$. Then, W is clearly sos. Moreover,

$$-\dot{W} = -(2k+2)\tilde{V}^{2k+1}\dot{\tilde{V}} = -(k+1)2\tilde{V}^{2k}\tilde{V}\dot{\tilde{V}}$$

is also sos because this polynomial is obtained from (28) by setting $y = 1$.[15] □

6 Summary and some open questions

We studied the basic problem of testing stability of equilibrium points of polynomial differential equations and asked some basic questions: What is the computational complexity of this problem? What kind of "converse Lyapunov theorems" can we expect to establish on existence of polynomial Lyapunov functions, upper bounds on their degrees, and guaranteed success of techniques based on sum of squares relaxation and semidefinite programming for finding these Lyapunov functions? Our contributions to these questions are listed in Subsection 1.1.

Some problems of interest that our work leaves open are listed below.

Open questions regarding complexity. Of course, the most interesting problem here is to formally answer the questions of Arnold on undecidability of determining stability for polynomial vector fields. As far as NP-hardness is concerned, our work leaves open the question of establishing NP-hardness of testing asymptotic stability for *quadratic* vector fields. (Recall that such a result cannot be restricted to homogeneous quadratic vector fields, but it is quite likely that the problem for all quadratic vector fields is hard.) The complexity of all problems considered in Theorem 3.5 is also open for quadratic vector fields. In general, one can reduce the degree of any vector field to two by introducing polynomially many new variables (see [38]). However, this operation may or may not preserve the property of the vector field which is of interest.

Open questions regarding existence of (sos) polynomial Lyapunov functions. Mark Tobenkin asked whether globally exponentially stable polynomial vector fields always admit polynomial Lyapunov functions. Our counterexample with Krstic in Section 4, though GAS and locally exponentially stable, is not globally exponentially stable because of exponential growth rates in the large. The counterexample of Bacciotti and Rosier in [15] is not even locally exponentially stable. Another problem left open is to prove our conjecture that GAS homogeneous polynomial vector fields admit homogeneous polynomial Lyapunov functions. This, together with Theorem 5.3, would imply that asymptotic stability of homogeneous polynomial systems can always be decided via sum of squares programming. Also, it is not clear to us whether the assumption of homogeneity and planarity can be removed from Theorems 5.3 and 5.5 on existence of sos Lyapunov functions. Finally, another research direction would be to obtain upper bounds on the degree of polynomial Lyapunov functions when they do exist. Recall that our Theorem 4.3 has already established that bounds depending only on dimension and degree of the vector field are impossible. So the questions is whether one can derive bounds that are computable from the coefficients of the vector field. Some degree bounds are known for Lyapunov analysis of locally exponentially stable systems [58], but they depend on uncomputable properties of the solution such as convergence rate. As far as sos Lyapunov functions are concerned, degree bounds on Positivstellensatz result of the type in Theorems 5.2 and 5.4 are known, but typically exponential in size and not very encouraging for practical purposes.

[15] Once again, we note that the function W constructed in this proof is radially unbounded, achieves its global minimum at the origin, and has $-\dot{W}$ positive definite. Therefore, W proves global asymptotic stability.

7 Acknowledgements

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