SOS-Convex Lyapunov Functions with Applications to Nonlinear Switched Systems

Amir Ali Ahmadi and Raphaël M. Jungers

***Abstract*— We introduce the concept of sos-convex Lyapunov functions for stability analysis of discrete time switched systems. These are polynomial Lyapunov functions that have an algebraic certificate of convexity, and can be efficiently found by semidefinite programming. We show that sos-convex Lyapunov functions are universal (i.e., necessary and sufficient) for stability analysis of switched linear systems. On the other hand, we show via an explicit example that the minimum degree of an sos-convex Lyapunov function can be arbitrarily higher than a (non-convex) polynomial Lyapunov function, whose induced Minkowski functional is also a valid (non-polynomial) convex Lyapunov function. In the second part of the paper, we show that if the switched system is defined as the convex hull of a finite number of *nonlinear* functions, then existence of a non-convex common Lyapunov function is *not* a sufficient condition for switched stability, but existence of a convex common Lyapunov function is. This shows the usefulness of the computational machinery of sos-convex Lyapunov functions which can be applied either directly to the switched nonlinear system, or to its linearization, to provide proof of local switched stability for the convex hull of the nonlinear system. An example is given where no polynomial of degree less than 14 can provide an estimate to the region of attraction under arbitrary switching.**

I. INTRODUCTION

The most commonly used Lyapunov functions in control, namely the quadratic ones, are always convex. This convexity property is not always purposefully sought after—it is simply an artifact of the nonnegativity requirement of Lyapunov functions, which for quadratic forms coincides with convexity. If one however seeks Lyapunov functions that are polynomial functions of degree larger than two—a task which was intractable in the previous millennium but has now become widespread thanks to advances in sum of squares optimization [31]—then convexity is no longer implied by the nonnegativity requirement of the Lyapunov function. In this paper we ask the question, what do we gain (or lose) by requiring a polynomial Lyapunov function to be convex. We also present a computational methodology, based on semidefinite programming, for automatically searching for convex polynomial Lyapunov functions.

Our study of this question is motivated by, and for the purposes of this paper exclusively focused on, the stability problem for discrete time switched systems. We are concerned with an uncertain and time-varying map:

$$x_{k+1} = \tilde{f}(x_k), \tag{1}$$

where

$$\tilde{f}(x_k) \in conv\{f_1(x_k), \ldots, f_m(x_k)\}, \tag{2}$$

$f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}^n$ are m different (possibly nonlinear) maps with $f_i(0) = 0$, and $conv$ denotes the convex hull operation. The question of interest is (local or global) *asymptotic stability under arbitrary switching*; i.e., we would like to know whether the origin attracts all initial conditions for *all* possible values that $\tilde{f}$ can take at each time step k.

The special case of this problem where the maps $f_1, \ldots, f_m$ are *linear* has been and continues to be the subject of intense study in the control community, as well as the mathematics or computer science community [9], [13], [16], [22], [25], [26], [30], [39]. A switched linear system in this setting,

$$x_{k+1} \in conv\{A_i x_k\}, \quad i = 1, \ldots, m, \tag{3}$$

is defined by m real $n \times n$ matrices $A_1, \ldots, A_m$ and its (local or equivalently global) asymptotic stability under arbitrary switching is equivalent to the *joint spectral radius* of these matrices being strictly less than one.

Definition 1 (Joint Spectral Radius – JSR [37]): the *joint spectral radius* of a set of matrices $\mathcal{M}$ is defined as

$$\rho(\mathcal{M}) = \lim_{k\to\infty} \max_{A_1,\ldots,A_k \in \mathcal{M}} ||A_1 \ldots A_k||^{1/k}, \tag{4}$$

where $\|\cdot\|$ is any matrix norm on $\mathbb{R}^{n\times n}$.

Deciding whether $\rho < 1$ is notoriously difficult. No finite time procedure for this purpose is known to date, and the related problems of testing whether $\rho \leq 1$ or whether the trajectories of (3) are bounded under arbitrary switching are provably undecidable [40]. On the positive side though, a large host of sufficient conditions for this stability property are known, mostly based on the numerical construction of special Lyapunov functions, and some with theoretical guarantees in terms of their quality of approximation of the joint spectral radius [8], [19], [23], [33], [34].

It is well-known that if the switched linear system (3) is stable[1], then it admits a common *convex* Lyapunov function, in fact a norm [22]. It is also known that stable switched linear systems also admit a common *polynomial* Lyapunov function [33]. It is therefore natural to ask whether existence of a common *convex polynomial* Lyapunov function is also necessary for stability. One would in addition want to know how the degree of such convex polynomial Lyapunov function compares with the degree of a non-convex polynomial

Amir Ali Ahmadi is a Goldstine Fellow at the Department of Business Analytics and Mathematical Sciences of the IBM Watson Research Center. Email: a_a_a@mit.edu. Raphaël Jungers is an F.R.S.-FNRS Research Associate at the ICTEAM Institute, Université catholique de Louvain. E-mail: raphael.jungers@uclouvain.be.

[1]Throughout this paper, by the word "stable" we mean asymptotically stable under arbitrary switching.

Lyapunov function. We address both of these questions in this paper.

It is not difficult to show (see [22, Proposition 1.8]) that stability of the linear inclusion (3) is equivalent to stability of its "corners"; i.e. to stability of a switched system that at each time step applies one of the m matrices $A_1, \ldots, A_m$, but never a matrix strictly inside their convex hull. This statement is no longer true for the switched nonlinear system in (1)-(2); see Example 1. It turns out, however, that one can still prove switched stability of the entire convex hull by finding a common *convex* Lyapunov function for the corner systems $f_1, \ldots, f_m$. This is demonstrated in our Theorem 4.1 and Example 2, where we point out that the convexity of the Lyapunov function is a crucial requirement.

In view of that, one would like to have an efficient algorithm that automatically searches over all candidate convex polynomial Lyapunov functions of a given degree. This task, however, is unfortunately an intractable one even when one restricts attention to quartic (degree four) Lyapunov functions and switched linear systems. See our discussion below. In order to cope with this issue, we introduce the class of *sos-convex* Lyapunov functions (see Definition 2), which constitute a subset of convex polynomial Lyapunov functions whose convexity is certified through an algebraic identity. One can search over sos-convex Lyapunov functions by solving a single semidefinite program whose size is polynomial in the description size of the dynamical system. The methodology can directly handle the linear switched system in (3) or its nonlinear counterpart in (1)-(2), if the mappings $f_1, \ldots, f_m$ are *polynomial functions* or *rational functions*.[2]

We will review some results from the thesis of the first author which show that for certain dimensions and degrees, the set of convex and sos-convex Lyapunov functions coincide. In fact in relatively low dimensions and degrees, it is quite challenging to find convex polynomials that are not sos-convex [6]. This is evidence of the strength of this semidefinite relaxation and is encouraging from an application viewpoint. Nevertheless, since sos-convex polynomials are in general a strict subset of the convex ones, a more refined (and perhaps more computationally relevant) converse Lyapunov question for switched linear systems is to see whether their stability guarantees existence of an sos-convex Lyapunov function. This question is also addressed in this paper.

Finally, we shall remark that there are other classes of convex Lyapunov functions whose construction is amenable to semidefinite or linear programming. The main examples include polytopic Lyapunov functions, and piecewise quadratic Lyapunov functions that are a geometric combinations of several quadratics [12], [17], [20], [24], [25], [34], [35]. These Lyapunov functions are mostly studied for the case of linear switched systems, where they are known to be necessary and sufficient for stability. The extension of their applicability to polynomial or rational switched systems is also possible via the sum of squares relaxation. Our focus in this paper though is solely on studying the properties of sos-convex polynomial Lyapunov functions.

A. Related work

The literature on stability of switched systems is too extensive for us to review. We simply refer the interested reader to [18], [22], [39] and references therein. Closer to the specific focus of this paper is the work of Mason et al. [28], where the authors prove existence of polynomial Lyapunov functions for switched linear systems in *continuous time*. Our proof of the analogous statement in discrete time closely follows theirs. In [4], Ahmadi and Parrilo show that the Lyapunov function of Mason et al. further implies existence of a Lyapunov function that can be found with sum of squares techniques. Similar statements are proven there for polynomial differential equations. In [33], Parrilo and Jadbabaie prove that stable switched linear systems in discrete time always admit a (not necessarily convex) polynomial Lyapunov function which further can be found with sum of squares techniques. Also closely related to our work, Blanchini and Franco show in [10] that in contrast with the case of uncontrolled switching (our setting), controlled linear switched systems, both in discrete and continuous time, can be stabilized by means of a suitable switching without necessarily admitting a convex Lyapunov function.

In [15], [14], Chesi and Hung motivate several interesting applications of working with convex Lyapunov functions or Lyapunov functions with convex sublevel sets. These include establishing more regular behavior of the trajectories, ease of optimization over sublevel sets of the Lyapunov function, stability of recurrent neural networks, etc. The authors in fact propose sum of squares based conditions for imposing convexity of polynomials. However, it is shown in [5, Sect. 4] that these conditions lead to semidefinite programs of significantly larger size than those of sos-convexity, while at the same time being at least as conservative. Moreover, the works in [15], [14] offer no analysis of the performance (existence) of convex Lyapunov functions.

Finally, the reader interested in knowing more about sos-convex polynomials, their role in convex algebraic geometry and optimization, and their applications outside of control is referred to the works by Ahmadi and Parrilo [6], [7], Helton and Nie [21], and Magnani et al. [27], or to Section 3.3.3 of the edited volume [11].

B. Organization and contributions of the paper

The paper is organized as follows. In Section II, we present the mathematical and algorithmic machinery necessary for working with sos-convex Lyapunov functions through semidefinite programming. In Section III, we study switched linear systems. We show that given any homogeneous Lyapunov function, the Minkowski norm defined by the convex hull of its sublevel set is also a valid (convex) Lyapunov function. We then show that any stable switched linear system admits a convex polynomial Lyapunov function. We further strengthen this result by proving existence

[2] Extensions to broader classes of dynamical systems, e.g. trigonometric ones, is possible; see e.g. [29].

of an sos-convex Lyapunov function. An explicit family of examples is also provided to show that the minimum degree of a convex polynomial Lyapunov function can be arbitrarily larger than a non-convex one.

In Section IV, we study nonlinear switched systems. We show that stability of these systems cannot be concluded by means of a common Lyapunov function for the corner systems. However, we prove that this conclusion can be made if the Lyapunov function is convex. We give an example where an sos-convex Lyapunov function of degree 14 provides an inner estimate of the region of attraction of a nonlinear switched system.

II. SOS-CONVEX POLYNOMIALS

A multivariate polynomial $p(x) := p(x_1, \dots, x_n)$ is *nonnegative* or *positive semidefinite* (psd) if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. We say a polynomial p is a *sum of squares* (sos) if it can be written as $p = \sum_i q_i^2$, where each q_i is a polynomial. It is well-known that if p has even degree four or larger, then testing nonnegativity is NP-hard, while testing existence of a sum of squares decomposition, which provides a sufficient condition and an algebraic certificate for nonnegativity, can be done by solving a polynomially sized semidefinite program [31], [32].

A polynomial $p := p(x)$ is *convex* if its Hessian $H(x)$ (i.e., the $n \times n$ polynomial matrix of the second derivatives) forms a positive semidefinite matrix for all $x \in \mathbb{R}^n$. This is equivalent to the scalar valued polynomial $y^T H(x) y$ in $2n$ variables $(x_1, \dots, x_n, y_1, \dots, y_n)$ being nonnegative. It has recently been shown in [3] that testing if polynomial of degree four is convex is NP-hard in the strong sense. This motivates the algebraic notion of *sos-convexity*, which can be checked with semidefinite programming and provides a sufficient condition for convexity.

Definition 2: A polynomial $p := p(x)$ is *sos-convex* if its Hessian $H(x)$ can be factored as

$$H(x) = M^T(x) M(x),$$

where $M(x)$ is a (not necessarily square) polynomial matrix; i.e., a matrix with polynomial entries.

Polynomial matrices which admit a decomposition as above are called *sos matrices*. The term *sos-convex* was coined in a seminal paper of Helton and Nie [21], where they prove (among other things) that a basic semialgebraic set defined by sos-convex inequalities always has a lifted semidefinite representation. The following theorem is an algebraic analogue of a classical theorem in convex analysis and provides equivalent characterizations of sos-convexity.

Theorem 2.1 (Ahmadi and Parrilo [7]): Let $p := p(x)$ be a polynomial of degree d in n variables with its gradient and Hessian denoted respectively by $\nabla p := \nabla p(x)$ and $H := H(x)$. Let g_λ, g_∇, and g_{∇^2} be defined as

$$\begin{aligned} g_\lambda(x,y) &= (1-\lambda)p(x) + \lambda p(y) - p((1-\lambda)x + \lambda y), \\ g_\nabla(x,y) &= p(y) - p(x) - \nabla p(x)^T (y-x), \\ g_{\nabla^2}(x,y) &= y^T H(x) y. \end{aligned} \tag{5}$$

Then the following are equivalent:

(a) $g_{\frac{1}{2}}(x,y)$ is sos[3].

(b) $g_\nabla(x,y)$ is sos.

(c) $g_{\nabla^2}(x,y)$ is sos; (equivalently $H(x)$ is an sos-matrix).

The theorem above is reassuring, in the sense that it tells us that the definition of sos-convexity is independent of which characterization of convexity we apply the sos relaxation to. Since existence of an sos decomposition can be checked via semidefinite programming (SDP), any of the three equivalent conditions above, and hence sos-convexity of a polynomial, can also be checked by SDP. Even though the polynomials $g_{\frac{1}{2}}$, g_∇, g_{∇^2} above are all in $2n$ variables and have degree d, the structure of the polynomial g_{∇^2} allows for much smaller SDPs (see [5] for details). Hence, we will use the Hessian condition throughout this paper.

Semidefinite programming allows for not just checking if a given polynomial is sos-convex, but also searching and optimizing over a family of sos-convex polynomials subject to affine constraints. This allows for an automated search for convex polynomial Lyapunov functions. Of course, a Lyapunov function V also needs to satisfy other requirements, namely positivity, $V > 0$, and monotonic decrease along trajectories, $V_{k+1} < V_k$. Following the standard approach, we will also replace these inequalities with the requirement that $V_k - V_{k+1}$ has a sum of squares decomposition.

Throughout this paper what we mean by an *sos-convex Lyapunov function* is a polynomial function which satisfies all these requirements[4]. Interestingly, when the Lyapunov function can be assumed to be homogeneous—as is the case when the dynamics is homogeneous [36]—then the following lemma establishes that the convexity requirement of the polynomial automatically meets its nonnegativity requirement.

A *homogeneous polynomial* (or a *form*) is simply a polynomial where all monomials have the same degree.

Lemma 2.2: Convex forms are nonnegative and sos-convex forms are sos.

Proof: See [21, Lemma 8] or [7, Lemma 3.2]. ∎

For stability analysis of the switched linear system in (3), the requirements of a (common) sos-convex Lyapunov function V are the following:

$$\begin{array}{ll} V(x) & \text{sos-convex} \\ V(x) - V(A_i x) & \text{sos for } i = 1, \dots, m. \end{array} \tag{6}$$

Given a set of matrices $\{A_1, \dots, A_m\}$, the search for the coefficients of a (fixed degree) polynomial V satisfying the above condition amounts to solving an SDP whose size is

[3]The constant $\frac{1}{2}$ in $g_{\frac{1}{2}}(x,y)$ of condition **(a)** is arbitrary and chosen for convenience. One can show that $g_{\frac{1}{2}}$ being sos implies that g_λ is sos for any fixed $\lambda \in [0,1]$. Conversely, if g_λ is sos for some $\lambda \in (0,1)$, then $g_{\frac{1}{2}}$ is sos.

[4]Even though an sos decomposition in general merely guarantees polynomial nonnegativity, sos decompositions obtained numerically from interior point methods generically provide proofs of polynomial *positivity*; see the discussion in [1, p.41]. In this paper, whenever we prove a result about existence of a Lyapunov function satisfying certain sos conditions, we carefully make sure that the resulting inequalities are strict (if they need be).

polynomial in the description size of the matrices. If this SDP is (strictly) feasible, the switched system in (3) is stable under arbitrary switching. The same implication remains true if the sos-convexity requirement of V is replaced with the requirement that V is simply sos; see [33, Thm. 2.2]. (This is no longer true for switched nonlinear systems.)

In the next section, we will study the existence of Lyapunov functions satisfying the semidefinite conditions in (6). As a related remark, we end this section by mentioning that examples of convex polynomials that are not sos-convex are known. In fact, a complete characterization of the dimensions and the degrees for which convexity and sos-convexity are the same notion is available [7], [2, Chap. 3]. In general, finding examples of convex but not sos-convex polynomials is a challenging task [6]. From an application viewpoint, this is good news. It implies that our sos-convex Lyapunov functions are a powerful replacement for convex polynomial Lyapunov functions.

III. SOS-CONVEX LYAPUNOV FUNCTIONS AND SWITCHED LINEAR SYSTEMS

As remarked in the introduction, it is known that asymptotic stability of a switched linear system under arbitrary switching implies existence of a common convex Lyapunov function (in fact a norm) and also a common polynomial Lyapunov function. In this section, we show that one can in fact conclude existence of a convex polynomial Lyapunov function, and even more, an sos-convex Lyapunov function. Before we prove these results, we state a proposition which shows that in the particular case of switched linear systems, any common Lyapunov function (e.g. a nonconvex polynomial) can be turned into a convex one (although not necessarily an efficiently computable one). The proof is omitted.

Proposition 1: Consider the switched linear system in (3). If $\mathcal{S}$ is an invariant set for this dynamics, then so is $conv(\mathcal{S})$. Moreover, if V is a common homogeneous Lyapunov function with unit sublevel set $\mathcal{S}$, then the Minkowski norm[5] defined by $conv(\mathcal{S})$ is a convex common Lyapunov function for (3).

A. Existence of convex polynomial Lyapunov functions

In our proofs, we will need the following classical result, which was first proved in [37] to the best of our knowledge.

Theorem 3.1 (see [22], [37]): For any set of matrices $\mathcal{M} \subset \mathbb{R}^{n\times n}$, for any $\epsilon > 0$ there exists a vector norm $|\cdot|_\epsilon$ in $\mathbb{R}^n$ such that for any matrix in $\mathcal{M}$,

$$|x|_\epsilon \leq 1 \quad \Rightarrow \quad |Ax|_\epsilon \leq \rho + \epsilon.$$

Theorem 3.2: For any asymptotically stable linear switched system, there exists a convex polynomial Lyapunov function.

Our proof is inspired by the main result of [28], which proves the existence of a convex polynomial Lyapunov function for *continuous time* switching systems, but we are not aware of an equivalent statement in discrete time.

Proof: Let us denote the JSR of $\mathcal{M}$ by ρ. By assumption we have $\rho < 1$. By Theorem 3.1, there exists a norm, which we hereafter simply denote $|\cdot|$ such that

$$|x| \leq 1 \Rightarrow \quad |Ax| \leq \rho + (1-\rho)/2.$$

We denote B the unit ball of this norm, such that $\mathcal{M}B \subset ((\rho + (1-\rho)/2)B$. (We use the notation $\mathcal{M}B = \{Ax : A \in \mathcal{M} \text{ and } x \in B\}$.)

The agenda of our proof is to insert a level set of a convex polynomial between the boundary of B and the boundary of $(\rho+(1-\rho)/2)B$. This set will be an invariant set, and hence, the corresponding polynomial will be a common convex polynomial Lyapunov Function. We will first construct a polytope with this property, and then approximate it with a convex polynomial.

First step. Let us consider the set of points

$$C = \{x : |x| = (\rho + 3(1-\rho)/4)\}.$$

For any $x \in C$, we associate a dual vector $v(x)$ orthogonal to a support hyperplane of C containing x :

$$H(x) = \{y : v(x)^T y = v(x)^T x\}$$

(that is, $\forall y \in C,\ v(x)^T y \leq v(x)^T x$). Since $x \in \text{int}B$, the set

$$S(x) = \{y : v(x)^T y > v(x)^T x \text{ and } |y| = 1\}$$

is a relatively open nonempty subset of the boundary ∂B of the unit ball. Moreover,

$$x/|x| \in S(x).$$

Now, the family of sets $S(x)$ is an open covering of ∂B, and we can extract $x_1, \ldots, x_N$ such that the union of the sets $S(x_i)$ covers ∂B.

Second step. We denote $v_i \triangleq v(x_i)$ and we define a polytope

$$P = \{y : v_i^T y \leq v_i^T x_i \ \forall i = 1 \ldots N\}.$$

Observe that $0 \subset C \subset P$.

We now claim that $\mathcal{M}P \subset \text{int}P$.

First, $P \subset \text{int}B$, because for any vector y such that $|y| = 1$, there exists a vector x_i such that $y \in S(x_i)$ (indeed $\{S(x_i)\}$ covers ∂B). Thus, $v_i^T y > v_i^T x_i$, and $y \notin P$, which implies that $P \subset \text{int}B$.

Summarizing, we have

$$(\rho + (1-\rho)/2)B \subset \text{int}(\rho + 3(1-\rho)/4)B \subset P \subset \text{int}B.$$

Thus, taking any matrix in $\mathcal{M}$ and multiplying in the above inclusions, we obtain the claim.

Third step. For any natural number d, we define the polynomial function

$$p_d(y) = \sum_1^N (v_i^T y / v_i^T x_i)^{2d}. \tag{7}$$

[5]The Minkowski (or gauge) norm q defined by a symmetric convex set $\mathcal{S}$ is given by $q(x) = \inf\{t > 0 |\ x \in t\mathcal{S}\}$.

This polynomial is convex, as a sum of even powers of linear functions. Now, the level sets ∂S_d, where $S_d = \{y : p_d(y) \leq 1\}$ tend pointwise to the boundary of P as $d \to \infty$. Moreover,

$$\begin{aligned} &\forall x \in (\rho + (1-\rho)/2)B, \\ &\qquad \max_i (v_i^T y / v_i^T x_i) \quad \leq \quad \frac{\rho + (1-\rho)/2}{\rho + 3(1-\rho)/4} < 1. \end{aligned}$$

From that, we deduce that there is a natural d such that

$$(\rho + (1-\rho)/2)B \subset S_d \subset \mathrm{int}P \subset B,$$

so that we can rewrite the second step of this proof replacing P with S_d, in order to prove that $\mathcal{M}S_d \subset \mathrm{int}S_d$, that is, $\forall A \in \mathcal{M}$, $\forall x \in \mathbb{R}^n$, $p(Ax) < p(x)$. ■

B. Existence of sos-convex polynomial Lyapunov functions

The main result of this subsection is the following theorem.

Theorem 3.3: For any asymptotically stable linear switched system, there exists an sos-convex Lyapunov function; i.e. a polynomial Lyapunov function satisfying the semidefinite requirements in (6).

We will give the main idea of the proof but the details are omitted and will appear in a journal version of this work. The main component of the proof is the following powerful Positivstellensatz result due to Scheiderer.

Theorem 3.4 (Scheiderer, [38]): Given any two positive definite homogeneous polynomials h and g, there exists an integer N such that hg^N is a sum of squares.

The strategy of the proof is to start with the convex polynomial Lyapunov function p_d (for a large enough fixed d) constructed in the previous subsection and turn it into an sos-convex Lyapunov function q. It turns out that we can take

$$q(x) = p_d^k(x),$$

for some big enough integer k. Here, p_d is the polynomial given in (7). It is easy to see that linear forms are sos-convex, and that sums and even powers of sos-convex forms are sos-convex. Therefore, the polynomial q constructed this way is sos-convex. To show that the polynomials

$$q(x) - q(A_i x) = p_d^k(x) - p_d^k(A_i x)$$

are all sos, one uses the algebraic identity

$$a^k - b^k = (a-b) \sum_{l=0}^{k-1} a^{k-1-l} b^l$$

and appropriately applies Theorem 3.4.

Finally, we remark that because of the way q is constructed, all of our sos conditions imply strict positivity. So this polynomial will be a strictly feasible solution to a large enough semidefinite program.

C. Non-existence of a uniform bound on the degree of convex polynomial Lyapunov functions

Theorem 3.2 tells us in fact that one can approximate with an arbitrary accuracy the JSR of a set of matrices by restricting the family of polynomial Lyapunov functions to convex polynomials. Note that it is known that there are stable sets of matrices with polynomial Lyapunov functions of arbitrary degree, but one could wonder whether the existence of a polynomial Lyapunov function of a certain degree actually implies a bound on the degree of a *convex* Lyapunov function. We show that this is not true in the next example.

Example 3.1: Consider the set of matrices $\mathcal{A} = \{A_1, A_2\}$, with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

This is a benchmark set of matrices that has been studied in [8], [33], [4] because it gives the worst case approximation ratio of a common quadratic Lyapunov function. Indeed, it is easy to show that $\rho(\mathcal{A}) = 1$, but a common quadratic Lyapunov function can only produce an upper bound of $\sqrt{2}$.

Theorem 3.5: Consider the sets of matrices $\mathcal{M}_\gamma = \{\gamma A_1, \gamma A_2\}$. For all $\gamma < 1$ there is a degree four polynomial Lyapunov function, but for any integer d, there is a value of $\gamma < 1$ such that there is no convex polynomial Lyapunov function of degree less than d.

Proof: The first claim is proven in [33]. For the latter claim, it is sufficient to prove that the set $\{A_1, A_2\}$ has no convex invariant set defined as the level set of a polynomial. Indeed, if there were a uniform bound D on the degree of a convex polynomial Lyapunov function, by compactness it would imply the existence of an invariant set which is the level set of a convex polynomial function of degree D.
We prove our claim by contradiction. In fact, we will prove the slightly stronger fact that for these matrices, the only convex invariant set is the square $S = \{(x,y) : |x|, |y| \leq 1\}$ (or, of course, a scaling of it).

So, let us suppose that there exists a convex bivariate polynomial $p(x)$ whose level set is the boundary of an invariant set. More precisely, we suppose that

$$\forall x \in \mathbb{R}^2, \; \forall A \in \mathcal{A}, \quad p(Ax) \leq p(x). \tag{8}$$

We denote x^* the abscissa of the intersection of this level set with the main bisector:

$$p(x^*, x^*) = 1.$$

It is easy to check that the following matrices can be obtained as products of matrices in A :

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \right\} \quad \subset \mathcal{A}^*, \tag{9}$$

and this implies that

$$\begin{aligned} p(\mathbf{x}) \quad &= \quad 1 \\ \text{for } \mathbf{x} \quad &\in \quad \{(x^*, -x^*), (-x^*, -x^*), (-x^*, x^*)\} \end{aligned}$$

as well, because these points can all be mapped onto each other with matrices from (9).

Suppose that there is an $x > x^*$, $-x^* < y < x^*$ such that $p(x, y) = 1$. Then we reach a contradiction because (9) implies that (x, y) can be mapped on (x, x), which contradicts (8) because $x > x^*$.
This implies that $\forall y : -x^* < y < x^*,\ p(x^*, y) \geq 1$. However, the convexity of p, implies that $p(x^*, y) \leq 1$ for all y such that $-x^* < y < x^*$.
Thus, we have proved that $p(x^*, y) = 1$ for all $-x^* < y < x^*$. The same is true for $p(-x^*, y)$ by symmetry.

In the same vein, if there is a $y > x^*$, $-x^* < x < x^*$ such that $p(x, y) = 1$, this point can be mapped on $(-y, -y)$, which again leads to a contradiction, because $p(-x^*, -x^*) = 1$.
Hence, $p(x, x^*) = 1, p(x, -x^*) = 1$ for all $-x^* < x < x^*$, which concludes the proof. ∎

IV. SOS-CONVEX LYAPUNOV FUNCTIONS AND SWITCHED NONLINEAR SYSTEMS

In this section, we demonstrate a noteworthy application of the computational machinery of sos-convex Lyapunov functions, namely the stability analysis of switched nonlinear systems. These are the systems satisfying these equations:

$$\begin{aligned} x_{k+1} &= \tilde{f}(x_k), \\ \tilde{f}(x_k) &\in conv\{f_1(x_k), \ldots, f_m(x_k)\}. \end{aligned} \tag{10}$$

We start by showing on an example the significance of *convex* Lyapunov functions.

Example 1: Let us consider the two-dimensional nonlinear switching system (10) with $m = 2$ and

$$\begin{aligned} f_1(x) &= (x_1 x_2, 0)^T, \\ f_2(x) &= (0, x_1 x_2)^T. \end{aligned} \tag{11}$$

The function

$$V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2) \tag{12}$$

is a common Lyapunov function for both f_1 and f_2. However, the system (10) is unstable.

To see this, let us first remark that

$$V(f_i(x)) = x_1^2 x_2^2 < V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2)$$

for $i = 1, 2$, whenever $x \neq 0$.
On the other hand, (10) is unstable since in particular the dynamics

$$f(x) = \left(\frac{x_1 x_2}{2}, \frac{x_1 x_2}{2}\right) \in conv\{f_1(x_k), f_2(x_k)\}$$

is obviously unstable.

Thus, unlike for linear switching systems, one cannot resort to plain Lyapunov functions of the 'corners' to prove the stability of a nonlinear switching system (or even to prove their *robust stability*). However, we show now that *convex* Lyapunov functions are indeed a sufficient condition for switched stability.

Theorem 4.1: Consider the nonlinear switched system in (10). If the m functions f_i have a common convex Lyapunov function, then the system (10) is asymptotically stable under arbitrary switching.

Proof: Let $V(x)$ be the common convex Lyapunov function, and suppose that at step k, the function $f = \sum \lambda_i f_i$ is applied to the system. We have the inequality

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= V(\sum \lambda_i f_i(x_k)) - V(x_k) \\ &\leq \sum (\lambda_i V(f_i(x_k))) - V(x_k) \\ &\leq \sum \lambda_i (V(f_i(x_k)) - V(x_k)) \\ &< 0, \end{aligned}$$

and $V(x)$ is a Lyapunov function for the switched system as well. Note the crucial use of convexity of $V(x)$ in the first inequality. ∎

Remark 4.1: We remark that the theorem above provides an easy way of proving that a *linear* switched system defined by a finite number of matrices (i.e., at each time step, one of these matrices is applied to the system) is stable if and only if the switched system defined by the convex hull of the set of matrices is stable. Indeed, it is well known that the former system is stable if and only if there exists a common convex Lyapunov function for it (see Theorem 3.1), which directly implies that the convex hull is also stable.

A. Examples: region of attraction under nonlinear arbitrary switching

Our technique also allows for computation of inner approximations to regions of attraction when the switched nonlinear system is not globally stable. We show this on two examples.

Example 2: Let us look back at the system (11) of Example 1. It turns out that the function

$$W(x) = x_1^2 + x_2^2,$$

which is convex, is a common Lyapunov function for f_1, f_2 on the set

$$S = \{x : x_1, x_2 \leq 1\}.$$

Indeed, for $i = 1, 2$, and $x \in \mathcal{S}$,

$$\begin{aligned} W(f_i(x)) &= x_1^2 x_2^2 \\ &< x_1^2 + x_2^2 \\ &= W(x). \end{aligned}$$

Moreover, S is an invariant set. Hence, for the system (11), the set S is part of the region of attraction of the origin under arbitrary switching.

Example 3: Consider the nonlinear switched system (10) with $m = 2$ and

$$\begin{aligned} f_1(x) &= \begin{pmatrix} 0.687x_1 + 0.558x_2 - .0001x_1x_2 \\ -0.292x_1 + 0.773x_2 \end{pmatrix}, \\ f_2(x) &= \begin{pmatrix} 0.369x_1 + 0.532x_2 - .0001x_1^2 \\ -1.27x_1 + 0.12x_2 - .0001x_1x_2 \end{pmatrix}. \end{aligned} \tag{13}$$

The goal is to use sos-convex Lyapunov functions and semidefinite programming to compute an estimate of the

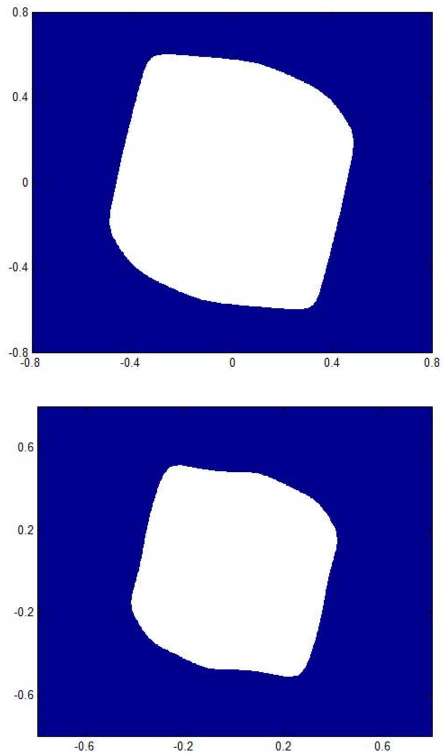

Fig. 1. Top: Sublevel set of a degree 14 sos-convex Lyapunov function. This set is provably part of the region of attraction of the origin of the switched nonlinear system (13). Bottom: Sublevel set of a non-convex polynomial Lyapunov function of degree 12. We cannot make any claims about this set being in the region of attraction under arbitrary switching.

region of attraction of this system under arbitrary switching. The linearization of this switched system is given by the pair of matrices

$$A_1 = \begin{pmatrix} 0.687 & 0.558 \\ -0.292 & 0.773 \end{pmatrix}, A_2 = \begin{pmatrix} 0.369 & 0.532 \\ -1.27 & 0.12 \end{pmatrix}. \quad (14)$$

This is an interesting pair of matrices, because it is stable under arbitrary switching, though there exists no common polynomial Lyapunov function of degree 10 or lower. The linear system does admit a common non-convex polynomial Lyapunov function of degree 12, but if convexity of the Lyapunov function is required, then the lowest possible degree is 14. A sublevel set of both of these Lyapunov functions is presented in Figure 1. These polynomials (not shown) have been found by solving a semidefinite program. The polynomial of degree 14 is an sos-convex polynomial Lyapunov function. Since all convex bivariate forms (of any degree) are sos-convex [7], and all nonnegative bivariate forms are sos, in this example we are not loosing anything by working with the sos relaxation.

The goal is now to use the Lyapunov function for the linearization to find a guaranteed region of attraction for the switched nonlinear system. We would like to find the largest level set of these Lyapunov functions that is invariant under the dynamics in (13). This is precisely when the advantage of having a convex Lyapunov function becomes clear. Indeed, if we were to work with the non-convex Lyapunov function of degree 12, then we would need to work with the convex hull of its sublevel set which is not algebraic. Finding the largest invariant level set of the Minkowski norm defined by this set is intractable. On the other hand, if we work with the convex Lyapunov function of degree 14, this task at hand simply becomes a new sos program. This program finds the largest sublevel set of the degree 14 polynomial in which the inequality $V(f_i(x)) < V(x),\ i = 1, 2$ holds. The resulting sublevel set is in fact the one plotted in Figure 1. This set is provably part of the region of attraction.

To final remarks are in place: (i) this example demonstrated the benefits of sos-convex Lyapunov functions even when applied to switched *linear* systems, and (ii) polynomials of degree $2, 4, 6, 8, 10, 12$ would completely fail to prove any nontrivial portion of the region of attraction for this example.

V. Conclusions

In this work, we have introduced the concept of sos-convex Lyapunov functions for stability analysis of switched linear and nonlinear systems. The methodology is amenable to semidefinite programming. For switched linear systems, we proved a converse Lyapunov theorem on guaranteed existence of sos-convex Lyapunov functions. We further showed that the degree of a convex polynomial Lyapunov function can be arbitrarily higher than the degree of a non-convex one. For switched nonlinear systems, we showed that sos-convex Lyapunov functions allow for computation of regions of attraction under arbitrary switching, while non-convex Lyapunov functions in general do not.

VI. Acknowledgments

Amir Ali Ahmadi is supported by a Goldstine Fellowship at IBM Research. Raphaël Jungers is supported by the Communauté française de Belgique - Actions de Recherche Concertées, and by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. Amir Ali Ahmadi is thankful to Alexandre Megretski for insightful discussions on advantages of convex Lyapunov functions.

References

[1] A. A. Ahmadi. Non-monotonic Lyapunov functions for stability of nonlinear and switched systems: theory and computation. Master's thesis, Massachusetts Institute of Technology, June 2008. Available from `http://dspace.mit.edu/handle/1721.1/44206`.

[2] A. A. Ahmadi. *Algebraic relaxations and hardness results in polynomial optimization and Lyapunov analysis*. PhD thesis, Massachusetts Institute of Technology, September 2011.

[3] A. A. Ahmadi, A. Olshevsky, P. A. Parrilo, and J. N. Tsitsiklis. NP-hardness of deciding convexity of quartic polynomials and related problems. *Mathematical Programming*, 137(1-2):453–476, 2013.

[4] A. A. Ahmadi and P. A. Parrilo. Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems. In *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008.

[5] A. A. Ahmadi and P. A. Parrilo. On the equivalence of algebraic conditions for convexity and quasiconvexity of polynomials. In *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010.

[6] A. A. Ahmadi and P. A. Parrilo. A convex polynomial that is not sos-convex. *Mathematical Programming*, 2011. DOI: 10.1007/s10107-011-0457-z.

[7] A. A. Ahmadi and P. A. Parrilo. A complete characterization of the gap between convexity and sos-convexity. *SIAM Journal on Optimization*, 2013. To appear.

[8] T. Ando and M.-H. Shih. Simultaneous contractibility. *SIAM Journal on Matrix Analysis and Applications*, 19:487–498, 1998.

[9] F. Blanchini, P. Colaneri, and M.E. Valcher. Co-positive lyapunov functions for the stabilization of positive switched systems. *Automatic Control, IEEE Transactions on*, 57(12):3038–3050, 2012.

[10] Franco Blanchini and Carlo Savorgnan. Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions. *Automatica*, 44(4):1166–1170, 2008.

[11] G. Blekherman, P. A. Parrilo, and R. Thomas (editors). *Semidefinite Optimization and Convex Algebraic Geometry*. MOS-SIAM Series on Optimization, 2012.

[12] V. D. Blondel and Yu. Nesterov. Computationally efficient approximations of the joint spectral radius. *SIAM J. Matrix Anal. Appl.*, 27(1):256–272, 2005.

[13] T. Bousch and J. Mairesse. Asymptotic height optimization for topical IFS, Tetris heaps, and the finiteness conjecture. *Journal of the American Mathematical Society*, 15(1):77–111, 2002.

[14] G. Chesi and Y. S. Hung. On the convexity of sublevel sets of polynomial and homogeneous polynomial Lyapunov functions. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 5198–5203, 2006.

[15] G. Chesi and Y. S. Hung. Establishing convexity of polynomial Lyapunov functions and their sublevel sets. *IEEE Trans. Automat. Control*, 53(10):2431–2436, 2008.

[16] J. M. Hendrickx G. Vankeerberghen and R. M. Jungers. The JSR toolbox. *Matlab Central, http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox*.

[17] R. Goebel, T. Hu, and A. R. Teel. Dual matrix inequalities in stability and performance analysis of linear differential/difference inclusions. In *Current Trends in Nonlinear Systems and Control*, pages 103–122. 2006.

[18] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[19] N. Guglielmi and V. Protasov. Exact computation of joint spectral characteristics of linear operators. *Foundations of Computational Mathematics*, pages 1–61, 2012.

[20] N. Guglielmi, F. Wirth, and M. Zennaro. Complex polytope extremality results for families of matrices. *SIAM Journal on Matrix Analysis and Applications*, 27(3):721–743, 2005.

[21] J. W. Helton and J. Nie. Semidefinite representation of convex sets. *Mathematical Programming*, 122(1, Ser. A):21–64, 2010.

[22] R. M. Jungers. The joint spectral radius, theory and applications. In *Lecture Notes in Control and Information Sciences*, volume 385. Springer-Verlag, Berlin, 2009.

[23] R. M. Jungers, N. Guglielmi, and A. Cicone. Lifted polytope methods for the asymptotic analysis of matrix semigroups. *preprint*.

[24] R. M. Jungers and V. Yu. Protasov. Counterexamples to the CPE conjecture. *SIAM Journal on Matrix Analysis and Applications*, 31(2):404–409, 2009.

[25] J. W. Lee and G. E. Dullerud. Uniform stabilization of discrete-time switched and Markovian jump linear systems. *Automatica*, 42(2):205–218, 2006.

[26] D. Liberzon. Towards robust Lie-algebraic stability conditions for switched linear systems. In *Proceedings of the IEEE CDC, special session on hybrid systems*, Atlanta, 2010.

[27] A. Magnani, S. Lall, and S. Boyd. Tractable fitting with convex polynomials via sum of squares. In *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005.

[28] P. Mason, U. Boscain, and Y. Chitour. Common polynomial Lyapunov functions for linear switched systems. *SIAM Journal on Control and Optimization*, 45(1):226–245, 2006.

[29] Alexandre Megretski. Positivity of trigonometric polynomials. In *In the Proceedings of the 42nd IEEE Conference on Decision and Control*, volume 4, pages 3814–3817, 2003.

[30] I. Morris. A rapidly-converging lower bound for the joint spectral radius via multiplicative ergodic theory. *Advances in Mathematics*, 225:3425–3445, 2010.

[31] P. A. Parrilo. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. PhD thesis, California Institute of Technology, May 2000.

[32] P. A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96(2, Ser. B):293–320, 2003.

[33] P. A. Parrilo and A. Jadbabaie. Approximation of the joint spectral radius using sum of squares. *Linear Algebra Appl.*, 428(10):2385–2402, 2008.

[34] V. Y. Protasov, R. M. Jungers, and V. D. Blondel. Joint spectral characteristics of matrices: a conic programming approach. *SIAM Journal on Matrix Analysis and Applications*, 31(4):2146–2162, 2010.

[35] V. Yu. Protasov. The geometric approach for computing the joint spectral radius. In *Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference 2005*, pages 3001–3006, 2005.

[36] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector fields. *Systems and Control Letters*, 19(6):467–473, 1992.

[37] G. C. Rota and W. G. Strang. A note on the joint spectral radius. *Indag. Math.*, 22:379–381, 1960.

[38] C. Scheiderer. A Positivstellensatz for projective real varieties. arXiv:1104.1772, 2011.

[39] Robert Shorten, Fabian Wirth, Oliver Mason, Kai Wulff, and Christopher King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49:545–592, 2007.

[40] J. N. Tsitsiklis and V.D. Blondel. The Lyapunov exponent and joint spectral radius of pairs of matrices are hard- when not impossible- to compute and to approximate. *Mathematics of Control, Signals, and Systems*, 10:31–40, 1997.