SOS-Convex Lyapunov Functions
with Applications to Nonlinear Switched Systems

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Abstract—We introduce the concept of sos-convex Lyapunov functions for stability analysis of discrete time switched systems. These are polynomial Lyapunov functions that have an algebraic certificate of convexity, and can be efficiently found by semidefinite programming. We show that sos-convex Lyapunov functions are universal (i.e., necessary and sufficient) for stability analysis of switched linear systems. On the other hand, we show via an explicit example that the minimum degree of an sos-convex Lyapunov function can be arbitrarily higher than a (non-convex) polynomial Lyapunov function, whose induced Minkowski functional is also a valid (non-polynomial) convex Lyapunov function. In the second part of the paper, we show that if the switched system is defined as the convex hull of a finite number of nonlinear functions, then existence of a non-convex common Lyapunov function is not a sufficient condition for switched stability, but existence of a convex common Lyapunov function is. This shows the usefulness of the computational machinery of sos-convex Lyapunov functions which can be applied either directly to the switched nonlinear system, or to its linearization, to provide proof of local switched stability for the convex hull of the nonlinear system. An example is given where no polynomial of degree less than 14 can provide an estimate to the region of attraction under arbitrary switching.

I. INTRODUCTION

The most commonly used Lyapunov functions in control, namely the quadratic ones, are always convex. This convexity property is not always purposefully sought after—it is simply an artifact of the nonnegativity requirement of Lyapunov functions, which for quadratic forms coincides with convexity. If one however seeks Lyapunov functions that are polynomial functions of degree larger than two—a task which was intractable in the previous millennium but has now become widespread thanks to advances in sum of squares optimization [31]—then convexity is no longer implied by the nonnegativity requirement of the Lyapunov function. In this paper we ask the question, what do we gain (or lose) by requiring a polynomial Lyapunov function to be convex. We also present a computational methodology, based on semidefinite programming, for automatically searching for convex polynomial Lyapunov functions.

Our study of this question is motivated by, and for the purposes of this paper exclusively focused on, the stability problem for discrete time switched systems. We are concerned with an uncertain and time-varying map:

\[ x_{k+1} = \tilde{f}(x_k), \]

where\[ \tilde{f}(x_k) \in \text{conv}\{f_1(x_k), \ldots, f_m(x_k)\}, \] (2)
\[ f_1, \ldots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}^n \] are different (possibly nonlinear) maps with \( f_i(0) = 0, \) and \( \text{conv} \) denotes the convex hull operation. The question of interest is (local or global) asymptotic stability under arbitrary switching; i.e., we would like to know whether the origin attracts all initial conditions for all possible values of \( \tilde{f} \) at each time step \( k. \)

The special case of this problem where the maps \( f_1, \ldots, f_m \) are linear has been and continues to be the subject of intense study in the control community, as well as the mathematics or computer science community [9], [13], [16], [22], [25], [26], [30], [39]. A switched linear system in this setting,

\[ x_{k+1} \in \text{conv}\{A_i x_k\}, \quad i = 1, \ldots, m, \] (3)
is defined by \( m \) real \( n \times n \) matrices \( A_1, \ldots, A_m \) and its (local or equivalently global) asymptotic stability under arbitrary switching is equivalent to the joint spectral radius of these matrices being strictly less than one.

Definition 1 (Joint Spectral Radius – JSR [37]): the joint spectral radius of a set of matrices \( \mathcal{M} \) is defined as

\[ \rho(\mathcal{M}) = \lim_{k \to \infty} \max_{A_1, \ldots, A_k \in \mathcal{M}} \|A_1 \ldots A_k\|^{1/k}, \] (4)

where \( \| \cdot \| \) is any matrix norm on \( \mathbb{R}^{n \times n} \).

Deciding whether \( \rho < 1 \) is notoriously difficult. No finite time procedure for this purpose is known to date, and the related problems of testing whether \( \rho \leq 1 \) or whether the trajectories of (3) are bounded under arbitrary switching are provably undecidable [40]. On the positive side though, a large host of sufficient conditions for this stability property are known, mostly based on the numerical construction of special Lyapunov functions, and some with theoretical guarantees in terms of their quality of approximation of the joint spectral radius [8], [19], [23], [33], [34].

It is well-known that if the switched linear system (3) is stable\(^1\), then it admits a common convex Lyapunov function, in fact a norm [22]. It is also known that stable switched linear systems also admit a common polynomial Lyapunov function [33]. It is therefore natural to ask whether existence of a common convex polynomial Lyapunov function is also necessary for stability. One would in addition want to know how the degree of such convex polynomial Lyapunov function compares with the degree of a non-convex polynomial

\(^1\)Throughout this paper, by the word “stable” we mean asymptotically stable under arbitrary switching.
Lyapunov function. We address both of these questions in this paper.

It is not difficult to show (see [22, Proposition 1.8]) that stability of the linear inclusion (3) is equivalent to stability of its “corners”; i.e. to stability of a switched system that at each time step applies one of the \( m \) matrices \( A_1, \ldots, A_m \), but never a matrix strictly inside their convex hull. This statement is no longer true for the switched nonlinear system in (1)-(2); see Example 1. It turns out, however, that one can still prove switched stability of the entire convex hull by finding a common convex Lyapunov function for the corner systems \( f_1, \ldots, f_m \). This is demonstrated in our Theorem 4.1 and Example 2, where we point out that the convexity of the Lyapunov function is a crucial requirement.

In view of that, one would like to have an efficient algorithm that automatically searches over all candidate convex polynomial Lyapunov functions of a given degree. This task, however, is unfortunately an intractable one even when one restricts attention to quartic (degree four) Lyapunov functions and switched linear systems. See our discussion below. In order to cope with this issue, we introduce the class of sos-convex Lyapunov functions (see Definition 2), which constitute a subset of convex polynomial Lyapunov functions whose convexity is certified through an algebraic identity. One can search over sos-convex Lyapunov functions by solving a single semidefinite program whose size is polynomial in the description size of the dynamical system. The methodology can directly handle the linear switched system in (3) or its nonlinear counterpart in (1)-(2), if the mappings \( f_1, \ldots, f_m \) are polynomial functions or rational functions.\(^2\)

We will review some results from the thesis of the first author which show that for certain dimensions and degrees, the set of convex and sos-convex Lyapunov functions coincide. In fact, in relatively low dimensions and degrees, it is quite challenging to find convex polynomials that are not sos-convex [6]. This is evidence of the strength of this semidefinite relaxation and is encouraging from an application viewpoint. Nevertheless, since sos-convex polynomials are in general a strict subset of the convex ones, a more refined (and perhaps more computationally relevant) converse Lyapunov function question for switched linear systems is to see whether their stability guarantees existence of an sos-convex Lyapunov function. This question is also addressed in this paper.

Finally, we shall remark that there are other classes of convex Lyapunov functions whose construction is amenable to semidefinite or linear programming. The main examples include polytopic Lyapunov functions, and piecewise quadratic Lyapunov functions that are a geometric combinations of several quadratics [12], [17], [20], [24], [25], [34], [35]. These Lyapunov functions are mostly studied for the case of linear switched systems, where they are known to be necessary and sufficient for stability. The extension of their applicability to polynomial or rational switched systems is also possible via the sum of squares relaxation. Our focus in this paper though is solely on studying the properties of sos-convex polynomial Lyapunov functions.

\(^2\)Extensions to broader classes of dynamical systems, e.g. trigonometric ones, is possible; see e.g. [29].

A. Related work

The literature on stability of switched systems is too extensive for us to review. We simply refer the interested reader to [18], [22], [39] and references therein. Closer to the specific focus of this paper is the work of Mason et al. [28], where the authors prove existence of polynomial Lyapunov functions for switched linear systems in continuous time. Our proof of the analogous statement in discrete time closely follows theirs. In [4], Ahmadi and Parrilo show that the Lyapunov function of Mason et al. further implies existence of a Lyapunov function that can be found with sum of squares techniques. Similar statements are proven there for polynomial differential equations. In [33], Parrilo and Jadbabaie prove that stable switched linear systems in discrete time always admit a (not necessarily convex) polynomial Lyapunov function which further can be found with sum of squares techniques. Also closely related to our work, Blanchini and Franco show in [10] that in contrast with the case of uncontrolled switching (our setting), controlled linear switched systems, both in discrete and continuous time, can be stabilized by means of a suitable switching without necessarily admitting a convex Lyapunov function.

In [15], [14], Chesi and Hung motivate several interesting applications of working with convex Lyapunov functions or Lyapunov functions with convex sublevel sets. These include establishing more regular behavior of the trajectories, ease of optimization over sublevel sets of the Lyapunov function, stability of recurrent neural networks, etc. The authors in fact propose sum of squares based conditions for imposing convexity of polynomials. However, it is shown in [5, Sect. 4] that these conditions lead to semidefinite programs of significantly larger size than those of sos-convexity, while at the same time being at least as conservative. Moreover, the works in [15], [14] offer no analysis of the performance (existence) of convex Lyapunov functions.

Finally, the reader interested in knowing more about sos-convex polynomials, their role in convex algebraic geometry and optimization, and their applications outside of control is referred to the works by Ahmadi and Parrilo [6], [7], Helton and Nie [21], and Magnani et al. [27], or to Section 3.3.3 of the edited volume [11].

B. Organization and contributions of the paper

The paper is organized as follows. In Section II, we present the mathematical and algorithmic machinery necessary for working with sos-convex Lyapunov functions through semidefinite programming. In Section III, we study switched linear systems. We show that given any homogeneous Lyapunov function, the Minkowski norm defined by the convex hull of its sublevel set is also a valid (convex) Lyapunov function. We then show that any stable switched linear system admits a convex polynomial Lyapunov function. We further strengthen this result by proving existence.
of an sos-convex Lyapunov function. An explicit family of
examples is also provided to show that the minimum degree of
a convex polynomial Lyapunov function can be arbitrarily
larger than a non-convex one.

In Section IV, we study nonlinear switched systems. We
show that stability of these systems cannot be concluded
by means of a common Lyapunov function for the corner
systems. However, we prove that this conclusion can be
made if the Lyapunov function is convex. We give an
example where an sos-convex Lyapunov function of degree
14 provides an inner estimate of the region of attraction of
a nonlinear switched system.

II. SOS-CONVEX POLYNOMIALS

A multivariate polynomial \( p(x) := p(x_1, \ldots, x_n) \) is non-
negative or positive semidefinite (psd) if \( p(x) \geq 0 \) for all
\( x \in \mathbb{R}^n \). We say a polynomial \( p \) is a sum of squares
(sos) if it can be written as \( p = \sum_i q_i^2 \), where each \( q_i \) is
a polynomial. It is well-known that if \( p \) has even degree
four or larger, then testing nonnegativity is NP-hard, while
testing existence of a sum of squares decomposition, which
provides a sufficient condition and an algebraic certificate
for nonnegativity, can be done by solving a polynomially sized
semidefinite program [31, 32].

A polynomial \( p := p(x) \) is convex if its Hessian \( H(x) \)
(i.e., the \( n \times n \) polynomial matrix of the second
derivatives) forms a positive semidefinite matrix for all \( x \in \mathbb{R}^n \).
This is equivalent to the scalar valued polynomial \( y^T H(x)y \)
in \( 2n \) variables \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \) being nonnegative. It
has recently been shown in [3] that testing if polynomial of
degree four is convex is NP-hard in the strong sense. This
motivates the algebraic notion of sos-convexity, which can
be checked with semidefinite programming and provides a
sufficient condition for convexity.

Definition 2: A polynomial \( p := p(x) \) is sos-convex if its
Hessian \( H(x) \) can be factored as

\[
H(x) = M^T(x)M(x),
\]

where \( M(x) \) is a (not necessarily square) polynomial matrix;
\( i.e., \), a matrix with polynomial entries.

Polynomial matrices which admit a decomposition as
above are called sos matrices. The term sos-convex was
coinined in a seminal paper of Helton and Nie [21], where
they prove (among other things) that a basic semialgebraic
set defined by sos-convex inequalities always has a lifted
semidefinite representation. The following theorem is an
algebraic analogue of a classical theorem in convex analysis
and provides equivalent characterizations of sos-convexity.

Theorem 2.1 (Ahmadi and Parrilo [7]): Let \( p := p(x) \) be
a polynomial of degree \( d \) in \( n \) variables with its gradient
and Hessian denoted respectively by \( \nabla p := \nabla p(x) \) and
\( H := H(x) \). Let \( g_\lambda, g_{\nabla^2}, \) and \( g_{\nabla^2,2} \) be defined as

\[
g_\lambda(x, y) = (1 - \lambda)p(x) + \lambda p(y) - p((1 - \lambda)x + \lambda y),
\]

\[
g_{\nabla^2}(x, y) = p(y) - p(x) - \nabla p(x)^T(y - x),
\]

\[
g_{\nabla^2,2}(x, y) = y^2 H(x)y.
\]

Then the following are equivalent:

(a) \( g_\frac{1}{2}(x, y) \) is sos\(^3\).

(b) \( g_{\nabla^2}(x, y) \) is sos.

(c) \( g_{\nabla^2,2}(x, y) \) is sos; (equivalently \( H(x) \) is an sos-matrix).

The theorem above is reassuring, in the sense that it tells us
that the definition of sos-convexity is independent of which
characterization of convexity we apply the sos relaxation to.
Since existence of an sos decomposition can be checked via
semidefinite programming (SDP), any of the three equivalent
conditions above, and hence sos-convexity of a polynomial,
can also be checked by SDP. Even though the polynomials
\( g_\frac{1}{2}, g_{\nabla^2}, g_{\nabla^2,2} \) above are all in \( 2n \) variables and have degree \( d \),
the structure of the polynomial \( g_{\nabla^2,2} \) allows for much smaller
SDPs (see [5] for details). Hence, we will use the Hessian
condition throughout this paper.

Semidefinite programming allows for not just checking if
a given polynomial is sos-convex, but also searching and
optimizing over a family of sos-convex polynomials subject
to affine constraints. This allows for an automated search for
convex polynomial Lyapunov functions. Of course, a Lya-
punov function \( V \) also needs to satisfy other requirements,
namely positivity, \( V > 0 \), and monotonic decrease along
trajectories, \( V_k < V_{k+1} \). Following the standard approach,
we will also replace these inequalities with the requirement
that \( V_k - V_{k+1} \) has a sum of squares decomposition.

Throughout this paper what we mean by an sos-convex
Lyapunov function is a polynomial function which satisfies
all these requirements\(^4\). Interestingly, when the Lyapunov
function can be assumed to be homogeneous—as is the
case when the dynamics is homogeneous [36]—then the
following lemma establishes that the convexity requirement
of the polynomial automatically meets its nonnegativity
requirement.

A homogeneous polynomial (or a form) is simply a poly-
nomial where all monomials have the same degree.

Lemma 2.2: Convex forms are nonnegative and sos-
convex forms are sos.

Proof: See [21, Lemma 8] or [7, Lemma 3.2].

For stability analysis of the switched linear system in
(3), the requirements of a (common) sos-convex Lyapunov
function \( V \) are the following:

\[
\begin{align*}
V(x) & \text{ sos-convex} \\
V(x) - V(A_ix) & \text{ sos for } i = 1, \ldots, m.
\end{align*}
\]

(6)

Given a set of matrices \( \{A_1, \ldots, A_m\} \), the search for the
coefficients of a (fixed degree) polynomial \( V \) satisfying the
above condition amounts to solving an SDP whose size is

\(^3\)The constant \( \frac{1}{2} \) in \( g_\frac{1}{2}(x, y) \) of condition (a) is arbitrary and chosen for
convenience. One can show that \( g_\frac{1}{2} \) being sos implies that \( g_\frac{1}{2} \) is sos for
any fixed \( \lambda \in [0, 1] \). Conversely, if \( g_\lambda \) is sos for some \( \lambda \in (0, 1) \), then \( g_\frac{1}{2} \) is sos.

\(^4\)Even though an sos decomposition in general merely guarantees poly-
nomial nonnegativity, sos decompositions obtained numerically from interior
point methods generally provide proofs of polynomial positivity; see the
discussion in [1, p.41]. In this paper, whenever we prove a result about
existence of a Lyapunov function satisfying certain sos conditions, we
carefully make sure that the resulting inequalities are strict (if they need
be).
polytope with this property, and then approximate it with a convex polynomial.

First step. Let us consider the set of points

\[ C = \{ x : |x| = (\rho + 3(1-\rho)/4) \}. \]

For any \( x \in C \), we associate a dual vector \( v(x) \) orthogonal to a support hyperplane of \( C \) containing \( x \):

\[ H(x) = \{ y : v(x)^T y = v(x)^T x \} \]

\( \text{that is, } \forall y \in C,\ v(x)^T y \leq v(x)^T x. \) Since \( x \in \text{int}B \), the set

\[ S(x) = \{ y : v(x)^T y > v(x)^T x \text{ and } |y| = 1 \} \]

is a relatively open nonempty subset of the boundary \( \partial B \) of the unit ball. Moreover,

\[ x/|x| \in S(x). \]

Now, the family of sets \( S(x) \) is an open covering of \( \partial B \), and we can extract \( x_1, \ldots, x_N \) such that the union of the sets \( S(x_i) \) covers \( \partial B \).

Second step. We denote \( v_i \triangleq v(x_i) \) and we define a polytope

\[ P = \{ y : v_i^T y \leq v_i^T x_i \forall i = 1 \ldots N \}. \]

Observe that \( 0 \subset C \subset P \).

We now claim that \( MP \subset \text{int}P \).

First, \( P \subset \text{int}B \), because for any vector \( y \) such that \( |y| = 1 \), there exists a vector \( x_i \) such that \( y \in S(x_i) \) (indeed \( \{ S(x_i) \} \) covers \( \partial B \)). Thus, \( v_i^T y > v_i^T x_i \), and \( y \notin P \), which implies that \( P \subset \text{int}B \).

Summarizing, we have

\[ (\rho + (1-\rho)/2)B \subset \text{int}(\rho + 3(1-\rho)/4)B \subset P \subset \text{int}B. \]

Thus, taking any matrix in \( M \) and multiplying in the above inclusions, we obtain the claim.

Third step. For any natural number \( d \), we define the polynomial function

\[ p_d(y) = \sum_{i=1}^{N} (v_i^T y/v_i^T x_i)^{2d}. \]
Moreover, sos-convex. To show that the polynomials $q$ and that sums and even powers of sos-convex forms are sos-convex, given in (7). It is easy to see that linear forms are sos-convex, for some big enough integer $N$.

Theorem 3.4 (Scheiderer, [38]): Given any two positive definite homogeneous polynomials $h$ and $g$, there exists an integer $N$ such that $h g^N$ is a sum of squares.

The strategy of the proof is to start with the convex polynomial Lyapunov function $p_d$ (for a large enough fixed $d$) constructed in the previous subsection and turn it into a sos-convex Lyapunov function $q$. It turns out that we can take

$$q(x) = p_d^2(x),$$

for some big enough integer $k$. Here, $p_d$ is the polynomial given in (7). It is easy to see that linear forms are sos-convex, and that sums and even powers of sos-convex forms are sos-convex. Therefore, the polynomial $q$ constructed this way is sos-convex. To show that the polynomials

$$q(x) - q(A;x) = p_d^2(x) - p_d^2(A;x)$$

are all sos, one uses the algebraic identity

$$a^k - b^k = (a - b) \sum_{l=0}^{k-1} a^{k-1-l} b^l$$

and appropriately applies Theorem 3.4.

Finally, we remark that because of the way $q$ is constructed, all of our sos conditions imply strict positivity. So this polynomial will be a strictly feasible solution to a large enough semidefinite program.

C. Non-existence of a uniform bound on the degree of convex polynomial Lyapunov functions

Theorem 3.2 tells us in fact that one can approximate with an arbitrary accuracy the JSR of a set of matrices by restricting the family of polynomial Lyapunov functions to convex polynomials. Note that it is known that there are stable sets of matrices with polynomial Lyapunov functions of arbitrary degree, but one could wonder whether the existence of a polynomial Lyapunov function of a certain degree actually implies a bound on the degree of a convex Lyapunov function. We show that this is not true in the next example.

Example 3.1: Consider the set of matrices $A = \{A_1, A_2\}$, with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

This is a benchmark set of matrices that has been studied in [8], [33], [4] because it gives the worst case approximation ratio of a common quadratic Lyapunov function. Indeed, it is easy to show that $\rho(A) = 1$, but a common quadratic Lyapunov function can only produce an upper bound of $\sqrt{2}$.

Theorem 3.5: Consider the set of matrices $M = \{\gamma A_1, \gamma A_2\}$ for all $\gamma < 1$ there is a degree four polynomial Lyapunov function, but for any integer $d$, there is a value of $\gamma < 1$ such that there is no convex polynomial Lyapunov function of degree less than $d$.

Proof: The first claim is proven in [33]. For the latter claim, it is sufficient to prove that the set $\{A_1, A_2\}$ has no convex invariant set defined as the level set of a polynomial. Indeed, if there were a uniform bound $D$ on the degree of a convex polynomial Lyapunov function, by compactness it would imply the existence of an invariant set which is the level set of a convex polynomial function of degree $D$.

We prove our claim by contradiction. In fact, we will prove the slightly stronger fact that for these matrices, the only convex invariant set is the square $S = \{(x,y) : |x|, |y| \leq 1\}$ (or, of course, a scaling of it).

So, let us suppose that there exists a convex bivariate polynomial $p(x)$ whose level set is the boundary of an invariant set. More precisely, we suppose that

$$\forall x \in \mathbb{R}^2, \forall A \in A, \quad p(Ax) \leq p(x). \quad (8)$$

We denote $x^*$ the abscissa of the intersection of this level set with the main bisector:

$$p(x^*, x^*) = 1.$$

It is easy to check that the following matrices can be obtained as products of matrices in $A$:

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \right\} \subset A^*,$$

and this implies that

$$p(x) = 1$$

for $x \in \{(x^*, x^*), (-x^*, -x^*), (-x^*, x^*)\}$.
as well, because these points can all be mapped onto each other with matrices from (9).

Suppose that there is an \( x > x^* \), \(-x^* < y < x^*\) such that \( p(x, y) = 1 \). Then we reach a contradiction because (9) implies that \((x, y)\) can be mapped on \((x, x)\), which contradicts (8) because \( x > x^* \).

This implies that \(\forall y : -x^* < y < x^*, \ p(x^*, y) \geq 1\). However, the convexity of \( p \) implies that \( p(x^*, y) \leq 1 \) for all \( y \) such that \(-x^* < y < x^*\).

Thus, we have proved that \( p(x^*, y) = 1 \) for all \(-x^* < y < x^*\). The same is true for \( p(-x^*, y) \) by symmetry.

In the same vein, if there is a \( y > x^*, -x^* < x < x^* \) such that \( p(x, y) = 1 \), this point can be mapped on \((-y, -y)\), which again leads to a contradiction, because \( p(-x^*, -x^*) = 1 \).

Hence, \( p(x, x^*) = 1, p(x, -x^*) = 1 \) for all \(-x^* < x < x^*\), which concludes the proof.

\[\text{IV. SOS-Convex Lyapunov Functions and Switched Nonlinear Systems}\]

In this section, we demonstrate a noteworthy application of the computational machinery of sos-convex Lyapunov functions, namely the stability analysis of switched nonlinear systems. These are the systems satisfying these equations:

\[ x_{k+1} = \tilde{f}(x_k), \]
\[ \tilde{f}(x_k) \in \text{conv}\{f_1(x_k), \ldots, f_m(x_k)\}. \]  

(10)

We start by showing on an example the significance of convex Lyapunov functions.

**Example 1:** Let us consider the two-dimensional nonlinear switching system (10) with \( m = 2 \) and

\[ f_1(x) = (x_1x_2, 0)^T, \]
\[ f_2(x) = (0, x_1x_2)^T. \]  

(11)

The function

\[ V(x) = x_1^2x_2^2 + (x_1^2 + x_2^2) \]

(12)

is a common Lyapunov function for both \( f_1 \) and \( f_2 \). However, the system (10) is unstable.

To see this, let us first remark that

\[ V(f_i(x)) = x_1^2x_2^2 < V(x) = x_1^2x_2^2 + (x_1^2 + x_2^2) \]

for \( i = 1, 2 \), whenever \( x \neq 0 \).

On the other hand, (10) is unstable since in particular the dynamics

\[ f(x) = \left( \frac{x_1x_2}{2}, \frac{x_1x_2}{2} \right) \in \text{conv}\{f_1(x_k), f_2(x_k)\} \]

is obviously unstable.

Thus, unlike for linear switching systems, one cannot resort to plain Lyapunov functions of the ‘corners’ to prove the stability of a nonlinear switching system (or even to prove their robust stability). However, we show now that convex Lyapunov functions are indeed a sufficient condition for switched stability.

**Theorem 4.1:** Consider the nonlinear switched system in (10). If the \( m \) functions \( f_i \) have a common convex Lyapunov function, then the system (10) is asymptotically stable under arbitrary switching.

**Proof:** Let \( V(x) \) be the common convex Lyapunov function, and suppose that at step \( k \), the function \( f = \sum \lambda_i f_i \) is applied to the system. We have the inequality

\[ V(x_{k+1}) - V(x_k) = V(\sum \lambda_i f_i(x_k)) - V(x_k) \leq \sum \lambda_i (V(f_i(x_k))) - V(x_k) < 0, \]

and \( V(x) \) is a Lyapunov function for the switched system as well. Note the crucial use of convexity of \( V(x) \) in the first inequality.

**Remark 4.1:** We remark that the theorem above provides an easy way of proving that a linear switched system defined by a finite number of matrices (i.e., at each time step, one of these matrices is applied to the system) is stable if and only if the switched system defined by the convex hull of the set of matrices is stable. Indeed, it is well known that the former system is stable if and only if there exists a common convex Lyapunov function for it (see Theorem 3.1), which directly implies that the convex hull is also stable.

A. Examples: region of attraction under nonlinear arbitrary switching

Our technique also allows for computation of inner approximations to regions of attraction when the switched nonlinear system is not globally stable. We show this on two examples.

**Example 2:** Let us look back at the system (11) of Example 1. It turns out that the function

\[ W(x) = x_1^2 + x_2^2, \]

which is convex, is a common Lyapunov function for \( f_1, f_2 \) on the set

\[ S = \{ x : x_1, x_2 \leq 1 \}. \]

Indeed, for \( i = 1, 2, \) and \( x \in S \),

\[ W(f_i(x)) = \begin{cases} x_1^2x_2^2 < x_1^2 + x_2^2 \quad &\text{if } x \neq 0, \\ x_1^2 + x_2^2 \quad &\text{if } x = 0 \end{cases} \]

\[ = W(x). \]

Moreover, \( S \) is an invariant set. Hence, for the system (11), the set \( S \) is part of the region of attraction of the origin under arbitrary switching.

**Example 3:** Consider the nonlinear switched system (10) with \( m = 2 \) and

\[ f_1(x) = \begin{pmatrix} 0.687x_1 + 0.558x_2 - 0.001x_1x_2 \\ -0.292x_1 + 0.773x_2 \end{pmatrix}, \]
\[ f_2(x) = \begin{pmatrix} 0.369x_1 + 0.532x_2 - 0.001x_1^2 \\ -1.27x_1 + 0.12x_2 - 0.001x_1x_2 \end{pmatrix}. \]  

(13)

The goal is to use sos-convex Lyapunov functions and semidefinite programming to compute an estimate of the
shown) have been found by solving a semidefinite program. These polynomials (not having a convex Lyapunov function becomes clear. Indeed, the dynamics in (13). This is precisely when the advantage of level set of these Lyapunov functions that is invariant under switched nonlinear system. We would like to find the largest linearization to find a guaranteed region of attraction for the switched nonlinear system. We would like to work with the sos relaxation. All forms are sos, in this example we are not loosing anything degree) are sos-convex [7], and all nonnegative bivariate degree is sos-convex polynomial. A sublevel set of both of these Lyapunov functions is presented in Figure 1. These polynomials (not shown) have been found by solving a semidefinite program. The polynomial of degree 14 is an sos-convex polynomial Lyapunov function. Since all convex bivariate forms (of any degree) are sos-convex [7], and all nonnegative bivariate forms are sos, in this example we are not loosing anything by working with the sos relaxation.

The goal is now to use the Lyapunov function for the linearization to find a guaranteed region of attraction for the switched nonlinear system. We would like to find the largest level set of these Lyapunov functions that is invariant under the dynamics in (13). This is precisely when the advantage of having a convex Lyapunov function becomes clear. Indeed, if we were to work with the non-convex Lyapunov function of degree 12, then we would need to work with the convex hull of its sublevel set which is not algebraic. Finding the largest invariant level set of the Minkowski norm defined by this set is intractable. On the other hand, if we work with the convex Lyapunov function of degree 14, this task at hand simply becomes a new sos program. This program finds the largest sublevel set of the degree 14 polynomial in which the inequality \( V(f_i(x)) < V(x), \ i = 1, 2 \) holds. The resulting sublevel set is in fact the one plotted in Figure 1. This set is provably part of the region of attraction.

To final remarks are in place: (i) this example demonstrated the benefits of sos-convex Lyapunov functions even when applied to switched linear systems, and (ii) polynomials of degree 2, 4, 6, 8, 10, 12 would completely fail to prove any nontrivial portion of the region of attraction for this example.

V. CONCLUSIONS

In this work, we have introduced the concept of sos-convex Lyapunov functions for stability analysis of switched linear and nonlinear systems. The methodology is amenable to semidefinite programming. For switched linear systems, we proved a converse Lyapunov theorem on guaranteed existence of sos-convex Lyapunov functions. We further showed that the degree of a convex polynomial Lyapunov function can be arbitrarily higher than the degree of a non-convex one. For switched nonlinear systems, we showed that sos-convex Lyapunov functions allow for computation of regions of attraction under arbitrary switching, while non-convex Lyapunov functions in general do not.

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