Testing Linear-Invariant Non-Linear Properties

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Testing Properties of Functions

$f : D \to R$ function from huge finite domain D to (small?) finite range R.

A function property is a family of functions $\mathcal{F} \subseteq \{D \to R\}$.

1-sided Property Tester

A q-local tester for function property $\mathcal{F}$ is an algorithm that, when given a parameter $\epsilon \in (0, 1)$ and oracle access to the function f,

- makes at most q oracle queries to f
- accepts with probability 1 if $f \in \mathcal{F}$
- rejects with high probability if $\min_{g \in \mathcal{F}} \Pr_{x \in D}[f(x) \neq g(x)] > \epsilon$ (a.k.a. f is ϵ-far from $\mathcal{F}$)

$\mathcal{F}$ is called q-locally testable if there is a local tester with query complexity q independent of $|D|$.

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Locally Testable Properties: Abbreviated History

- Prehistoric: Statistical Sampling
  - Is input function identically 0?
- Linearity Testing [BLR '90], Multilinearity Testing [BFL '91]
- Graph/Combinatorial Property Testing [GGR '98]
  - Is a graph "close" to being 3-colorable?
- Algebraic Testing [GLRSW, RS, FS, AKKLR, KR, JPRZ]
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Our Quest: **What makes a property testable?**

Necessary Conditions for Testability

- One-sided error and testability:
  - Suppose f is rejected by a k-query tester. Suppose queried points are $x_1, \dots, x_k \in D$ and $f(x_i) = \alpha_i$.
  - For every function $g \in \mathcal{F}$, $\langle g(x_1), \dots, g(x_k) \rangle \neq \langle \alpha_1, \dots, \alpha_k \rangle$.

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- k-Local Constraint: $C = \langle x_1, \ldots, x_k \rangle; S \subsetneq R^k$. Function g satisfies C if $\langle g(x_1), \ldots, g(x_k) \rangle \in S$.
  - k-Local Testability implies k-Local Constraints.

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- Every $f \notin \mathcal{F}$ rejected by some k-local constraint. Set of k-local constraints characterize $\mathcal{F}$:

$$\exists C_1, \ldots, C_m \text{ s.t. } f \in \mathcal{F} \Leftrightarrow f \text{ satisfies } C_j \ \forall j$$

  - Local Testability implies Local Characterization.

Local Characterizations Sufficient?

- NO! [Ben-Sasson, Harsha, Raskhodnikova '03]
  - $D = [n]$; $R = \{0, 1\}$; $\mathcal{F} =$ set of functions that satisfy some random 3-ary constraints
  - $\mathcal{F}$ not $o(n)$-locally testable
- Criticism: Random constraints too "asymmetric"
- Perhaps should consider more "symmetric" properties

Towards the Holy Grail

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The best understood class of nontrivial properties is graph properties

- Invariant under renaming of vertices
- [AFNS, Borgs et al] show that for graph properties in the "dense graph model", testability is *all about regularity*. A graph property is testable if and only if the property is "regular".

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What about testability of algebraic properties?

Linear-Invariant Algebraic Properties

- Domain D is a vector space
  - In this talk, domain $D = \mathbb{F}_2^n$ and range $R = \{0, 1\}$
- Property $\mathcal{F}$ is invariant under linear transformations of the domain.
  - If $f \in \mathcal{F}$, then $f \circ L \in \mathcal{F}$ for any linear transformation $L \in \mathbb{F}_2^{n \times n}$

Example

- Linear functions, and more generally, polynomials in $\mathbb{F}_2[x_1, \ldots, x_n]$ of degree $\leqslant d$
- Homogeneous polynomials of a given degree
- Dual-BCH codes
- $\mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1, \mathcal{F}_2$ are linear-invariant

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All the above properties known to be locally testable. Does testability arise from linear-invariance?

Linear-Invariant + Linear Properties

- Suppose that the range R has algebraic structure as a field $\mathbb{F}$.
  - Property $\mathcal{F}$ is *linear* if $f, g \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{F}$ implies $\alpha f + \beta g \in \mathcal{F}$.
- All the properties on previous slide were linear-invariant and linear.

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Theorem ([Kaufman, Sudan '07])

If $\mathcal{F}$ is a linear-invariant, linear property that is locally characterized, then $\mathcal{F}$ is locally testable.

Theorem explains the testability of diverse properties proved previously using diverse techniques.

Linear-Invariant Non-Linear Properties: An Example

- The property of triangle-freeness
  - A function $f : \mathbb{F}_2^n \rightarrow \{0, 1\}$ is triangle-free if for every $x, y \in \mathbb{F}_2^n$, $\langle f(x), f(y), f(x + y)\rangle \neq \langle 1, 1, 1\rangle$
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- Linear-invariant, yet non-linear
- Green in [Green '05] showed that triangle-freeness is testable!
- Analysis is very different from that of typical algebraic tests and is more reminiscent of graph property testing.
  - An algebraic Regularity Lemma

Matroid Freeness Properties

A more systematically defined class of linear-invariant, non-linear properties:

Definition (Matroid Freeness)

Given a binary matroid $\mathcal{M}$ represented by $v_1, \ldots, v_k \in \mathbb{F}_2^k$, the property of being $\mathcal{M}$-*free* is given by the family:

$$\{f : \mathbb{F}_2^n \to \{0,1\} \,|\, \forall \text{ linear } L : \mathbb{F}_2^k \to \mathbb{F}_2^n, \langle f(L(v_1)), \ldots, f(L(v_k))\rangle \neq 1^k\}$$

Example: Triangle freeness is described by the matroid $\mathcal{M}_\Delta = \{e_1, e_2, e_1 + e_2\}$.

- Under different linear transforms, $e_1, e_2, e_1 + e_2$ get mapped to all the triples of the form $x, y, x + y$.

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For our purposes, a matroid is a collection of vectors in $\mathbb{F}_2^k$.

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- Under different linear transforms, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2$ get mapped to all the triples of the form $x, y, x + y$.

We say f *contains* $\mathcal{M}$ at L if $\langle f(L(v_1)), \ldots, f(L(v_k))\rangle = 1^k$

Graphic Matroids

A matroid $\mathcal{M} = \{v_1, \ldots, v_k\}$ forms a graphic matroid if there exists a simple graph G on k edges with each edge associated with an element of $\mathcal{M}$ such that a set $S \subset \{v_1, \ldots, v_k\}$ has a linear dependency iff the associated set of edges contains a cycle in G.

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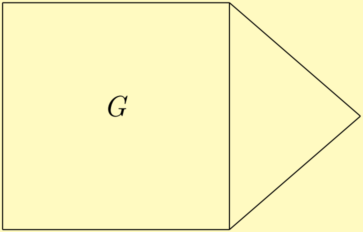

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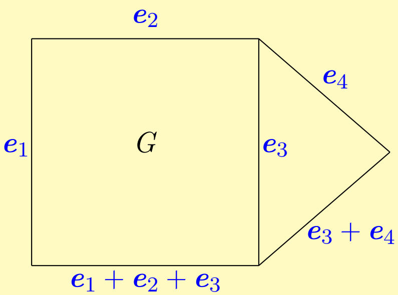

Testability of G-freeness

For any matroid $\mathcal{M} = \{v_1, \ldots, v_k\}$, there is a natural k-local $\mathcal{M}$-freeness tester: pick a linear map $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$ uniformly at random and check that f does not contain $\mathcal{M}$ at L.

[Green '05] showed that C_k-freeness is testable with natural test.

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Theorem (Main Result)

Given any graphic matroid $\mathcal{M} = \{v_1, \ldots, v_k\}$, there is a function $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ such that the natural k-local tester for $\mathcal{M}$-freeness accepts $\mathcal{M}$-free functions with probability 1 and rejects functions ϵ-far from being $\mathcal{M}$-free with probability $\tau(\epsilon)$.

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$\tau(\epsilon) = 2^{-O(W(\mathrm{poly}(1/\epsilon)))}$ where $W(t)$ is a tower of two's of height $\lceil t \rceil$.

The Family of G-free Properties

When are G_1-freeness and G_2-freeness distinct properties?

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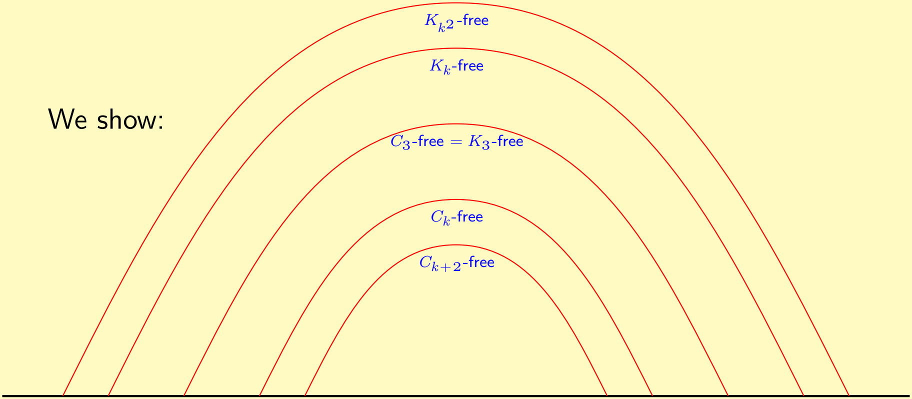
We show:



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- Future Work
  - Testability of matroid-freeness where the forbidden pattern is not 1^k but an arbitrary binary string
  - Allow for conjunctions of constraints
  - Show separation between more classes of matroid-freeness properties
  - Lower bounds for query complexity of matroid-freeness properties

Outline of Proof for Testability

Based on notion of "complexity" of matroids, introduced by [Green, Tao '06]. Complexity is an integer and the simplest matroids have "complexity one". We prove:

Lemma (Complexity of Graphic Matroids)

For all graphs G, the graphic matroid of G has complexity 1.

Lemma (Testability of 1-complex Matroid Freeness)

If $\mathcal{M} = \{v_1, \ldots, v_k\}$ is a matroid of complexity 1, then the natural k-local tester for $\mathcal{M}$-freeness accepts $\mathcal{M}$-free functions with probability 1 and rejects functions ϵ-far from $\mathcal{M}$-free with probability at least $\tau(\epsilon)$.

The Complexity of Matroids

Definition (Matroid complexity [Green, Tao '06])

Given a binary matroid $\mathcal{M} = \{v_1, \ldots, v_k\}$, $\mathcal{M}$ has complexity c if for every $i \in [k]$, the set $\{v_j\}_{j \in [k] \setminus \{i\}}$ can be partitioned into $c + 1$ classes such that v_i is not in the linear span of any of the classes.

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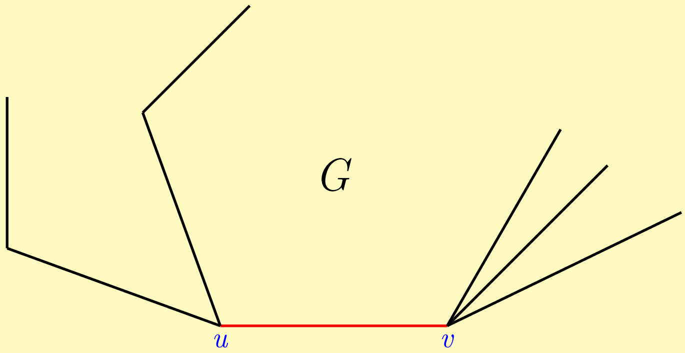
Motivating Example:

- Let $\mathcal{M} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{k-1}, \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{k-1}\}$ be the graphic matroid of C_k. Then, $\mathcal{M}$ has complexity 1.

Graphic matroids have Complexity 1



For each edge (u, v), we need to partition rest of the edges into two classes such that neither contains (u, v) as part of a cycle.

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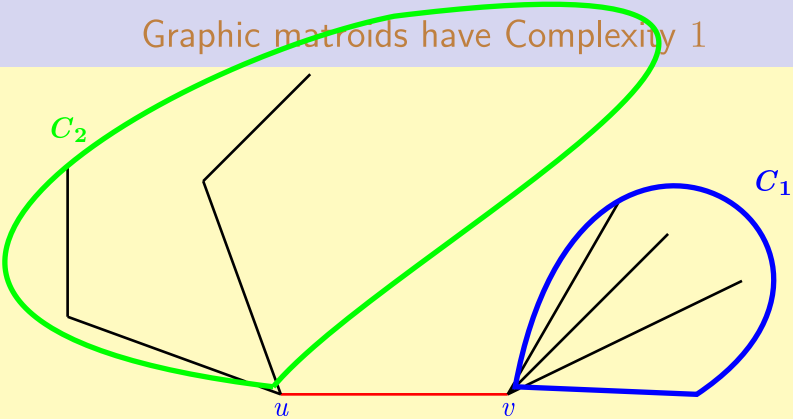

If $C_1 = \{\text{edges incident to } v\}$ and $C_2 = E(G) \setminus (C_1 \cup \{(u, v)\})$, edge (u, v) is not in the span of either of these sets because G is simple.

Hence, graphic matroids have complexity 1.

Key Property of Complexity 1 Matroids

[Green, Tao '06] showed that if a matroid $\mathcal{M} = \{v_1, \ldots, v_k\}$ has complexity 1, and if $A \subseteq \mathbb{F}_2^n$, then the number of linear maps $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$ such that $L(v_i) \in A$ for all $i \in [k]$ is controlled by the Fourier spectrum of A.

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Overview of Boolean Harmonic Analysis

For $f : \mathbb{F}_2^n \to \mathbb{R}$, define $\widehat{f} : \mathbb{F}_2^n \to \mathbb{R}$ as

$$\widehat{f}(\alpha) = \mathop{\mathbb{E}}_{x \in \{0,1\}^n} \left[f(x)(-1)^{\sum \alpha_i x_i} \right]$$

Properties:

- Fourier inversion: $f(x) = \sum_\alpha \widehat{f}(\alpha)(-1)^{\sum \alpha_i x_i}$
- Parseval: $\sum_\alpha \widehat{f}^2(\alpha) = \mathbb{E}_x[f^2(x)]$
- Zero'th coefficient: $\widehat{f}(0) = \mathbb{E}_x[f(x)]$

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Lemma (Generalized von Neumann)

Suppose the matroid $\mathcal{M} = \{v_1, \ldots, v_k\}$ *has complexity* 1 *and let* $f_1, \ldots, f_k : \mathbb{F}_2^n \to \{0,1\}$. *Then:*

$$\mathop{\mathbb{E}}_{L:\mathbb{F}_2^k \to \mathbb{F}_2^n}\left[\prod_{i=1}^{k} f_i(L(v_i))\right] \leqslant \min_{i \in [k]} \sum_{\alpha \in \mathbb{F}_2^n} \widehat{f_i}(\alpha)^4$$

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So, we need to be able to control the Fourier spectrum of f, or some functions related to it.

Green Regularity Lemma

Proved by Green in [Green '05], the Green Regularity Lemma states that for any $f : \mathbb{F}_2^n \to \{0, 1\}$, it is possible to find a subgroup $H \leqslant \mathbb{F}_2^n$ such that f restricted to most of the cosets of H is pseudo-random with respect to linear tests.

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Definition (Uniformity)

Given $\epsilon \in (0,1)$, a function $f : \mathbb{F}_2^n \to \{0,1\}$ is ϵ-*uniform* if for every nonzero $\alpha \in \mathbb{F}_2^n$, $|\widehat{f}(\alpha)| \leqslant \epsilon$.

Lemma (Regularity Lemma over $\mathbb{F}_2^n$)

Given $f : \mathbb{F}_2^n \to \{0,1\}$ and $\epsilon \in (0,1)$, there exists a subspace H of $\mathbb{F}_2^n$ of co-dimension at most $W(\epsilon^{-3})$ such that $\Pr_{g\in\mathbb{F}_2^n}[f_{g+H}$ is ϵ-uniform$] \geqslant 1-\epsilon$.

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- Use Green Regularity Lemma to obtain subspace $H \leqslant \mathbb{F}_2^n$ of co-dimension at most $W(a(\epsilon)^{-3})$, where $a(\epsilon)$ specified later.
- Define reduced function $f^R : \mathbb{F}_2^n \to \{0,1\}$. For each $g \in \mathbb{F}_2^n$, if $f_{g+H} : H \to \{0,1\}$ is $a(\epsilon)$-uniform, then:

$$f^R_{g+H}(x) = \begin{cases} 0 & \text{if } \mu(f_{g+H}) \leqslant b(\epsilon) \\ f_{g+H} & \text{otherwise.} \end{cases}$$

where $b(\epsilon)$ specified later. If f_{g+H} is not $a(\epsilon)$-uniform, then define $f^R_{g+H} = 0$.

Testability of Complexity 1 Matroids Contd.

- If $a(\epsilon) + b(\epsilon) < \epsilon$, f^R contains $\mathcal{M}$ at some linear $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$.

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- Complexity 1 implies f contains $\mathcal{M}$ at many linear maps.

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- Complexity 1 implies f contains $\mathcal{M}$ at many linear maps.
  - Because of rounding, $f_i \stackrel{\text{def}}{=} f_{L(v_i)+H}$ must be $a(\epsilon)$-uniform and $b(\epsilon)$-dense for all $i \in [k]$

Testability of Complexity 1 Matroids Contd.

- If $a(\epsilon) + b(\epsilon) < \epsilon$, f^R contains $\mathcal{M}$ at some linear $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$.
- Complexity 1 implies f contains $\mathcal{M}$ at many linear maps.
  - Because of rounding, $f_i \stackrel{\text{def}}{=} f_{L(v_i)+H}$ must be $a(\epsilon)$-uniform and $b(\epsilon)$-dense for all $i \in [k]$
  - Lower-bound number of linear maps $\phi : \mathbb{F}_2^k \to H$ such that f contains $\mathcal{M}$ at $L + \phi$:

$$\mathop{\mathbb{E}}_{\phi:\mathbb{F}_2^k \to H}\left[\prod_{i\in[k]} f_i(\phi(v_i))\right] = \mathop{\mathbb{E}}_{\phi:\mathbb{F}_2^k \to H}\left[\prod_{i\in[k]} \left(\widehat{f_i}(\emptyset) + (f_i(\phi(v_i)) - \widehat{f_i}(\emptyset))\right)\right]$$

$$\geqslant \underbrace{b(\epsilon)^k}_{\text{(density)}} - \underbrace{(2^k - 1)a(\epsilon)^2}_{\text{(uniformity)}}$$

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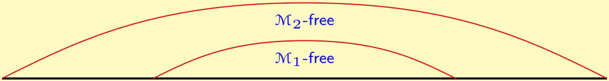
- Set $a(\epsilon) = \left(\frac{\epsilon}{2}\right)^k$, $b(\epsilon) = \frac{\epsilon}{2}$. Rejection probability is $2^{-O(W(\text{poly}(1/\epsilon)))}$. □

Separation between Matroid Freeness Properties



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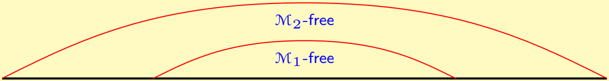

Given two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose we want to show that there exist functions that are far from $\mathcal{M}_1$-free yet contain $\mathcal{M}_2$.

General strategy:

- Based on matroid $\mathcal{M}_1$, we construct a "canonical" function $f_{\mathcal{M}_1} : \mathbb{F}_2^n \to \{0, 1\}$ that is far from $\mathcal{M}_1$-free
- If the canonical function $f_{\mathcal{M}_1}$ contains $\mathcal{M}_2$ at some linear map, then there is a "matroid homomorphism" from $\mathcal{M}_2$ to $\mathcal{M}_1$
- Based upon properties of $\mathcal{M}_1$ and $\mathcal{M}_2$, we show impossibility of such matroid homomorphisms

Canonical Function that is far from $\mathcal{M}_1$-free

Given $\mathcal{M}_1$ represented by $v_1, \ldots, v_k \in \mathbb{F}_2^k$ and integer $n \geqslant k$, define the *canonical function* $f_{\mathcal{M}_1} : \mathbb{F}_2^n \to \{0, 1\}$ as $f_{\mathcal{M}_1}(x, y) = 1$ iff $x \in \{v_1, \ldots, v_k\}$, where $x \in \mathbb{F}_2^k$ and $y \in \mathbb{F}_2^{n-k}$.

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Lemma

The canonical function $f_{\mathcal{M}_1}$ is 2^{-k}-far from $\mathcal{M}_1$-free.

Proof Sketch:

$f_{\mathcal{M}_1}$ contains $\mathcal{M}_1$ at the linear map $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$ that sends x to $\langle x, 0 \rangle$. So, $f_{\mathcal{M}_1}$ is clearly not $\mathcal{M}_1$-free. Let g be a function 2^{-k}-close to f. Choose a random linear map $L_1 : \mathbb{F}_2^k \to \mathbb{F}_2^{n-k}$, and let $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$ send x to $\langle x, L_1(x) \rangle$. With probability > 0, g contains $\mathcal{M}_1$ at L. $\square$

Matroid Homomorphisms

Definition

Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ matroids given by $v_1, \dots, v_k \in \mathbb{F}_2^k$ and $w_1, \dots, w_\ell \in \mathbb{F}_2^\ell$. A map $\phi : \{w_1, \dots, w_\ell\} \to \{v_1, \dots, v_\ell\}$ is a *matroid homomorphism* from $\mathcal{M}_2$ to $\mathcal{M}_1$ if for every set $T \subseteq [\ell]$ such that $\sum_{i \in T} w_i = 0$, then $\sum_{i \in T} \phi(w_i) = 0$.

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Lemma

If $f_{\mathcal{M}_1}$ contains $\mathcal{M}_2$, then $\mathcal{M}_2$ has a homomorphism to $\mathcal{M}_1$.

Proof Sketch:

Suppose $f_{\mathcal{M}_1}$ contains $\mathcal{M}_2$ at L. If $\pi : \mathbb{F}_2^n \to \mathbb{F}_2^k$ is the map sending $\langle x, y \rangle$ to x, then $\pi \circ L$ is the desired homomorphism. □

Homomorphism Impossibility Proof

Lemma

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the graphic matroids of C_a and C_b. If a, b are odd and $b < a$, then there is no homomorphism from $\mathcal{M}_2$ to $\mathcal{M}_1$.

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Proof Sketch:

Suppose ϕ is such a homomorphism. Then, if $v_1, \ldots, v_b$ represent $\mathcal{M}_2$, $v_1 + \cdots + v_b = 0$; so $\phi(v_1) + \cdots + \phi(v_b) = 0$ in $\mathcal{M}_1$. But then $b \geqslant a$ because there is no odd-sized set of dependent elements in $\mathcal{M}_1$ of size smaller than a.

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Theorem (Separation Theorem 1)

For every odd k, there exist functions g that are C_k-free but far from C_{k+2}-free.

Homomorphism Impossibility Proof

Lemma

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the graphic matroids of K_a and K_b. If $a \geqslant 3$ and $b \geqslant \binom{a}{2} + 2$, then there is no homomorphism from $\mathcal{M}_2$ to $\mathcal{M}_1$.

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Proof Sketch:

Suppose ϕ is such a homomorphism. By pigeonhole principle, there must exist incident edges e_i, e_j in K_b such that $\phi(e_i) = \phi(e_j)$. If f forms triangle with e_i and e_j in K_b, then because $e_i + e_j + f = 0$ in $\mathcal{M}_2$, $\phi(f) = \phi(e_i) + \phi(e_j) = 0 \notin \mathcal{M}_1$, an impossibility. □

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Theorem (Separation Theorem 2)

If $k \geqslant 3$ and $\ell \geqslant \binom{k}{2} + 2$, then there is a function that is K_ℓ-free but far from K_k-free.

Conclusions

- Goal is to find the "smallest" set of invariances and other conditions that explain testability.
- We show linearity is not a necessary condition for testability and suggest that linear invariance solely might be sufficient for properties with small description.
- We show that the class of testable matroid-freeness properties is provably infinite.