SOME ASPECTS OF THE STRUCTURE AND REPRESENTATION THEORY OF ALGEBRAIC GROUPS

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ABSTRACT. This expository paper discusses a few of the results concerning the structure theory and representation theory of algebraic groups over a fixed algebraically closed field K, such as the linearization of affine groups, the Jordan-Chevalley decomposition, and the Lie-Kolchin theorem.

0. Introduction

Algebraic groups have a relatively well-behaved decomposition. First, note that a disconnected (equivalently reducible) algebraic group has a normal subgroup which is the irreducible component of the identity, and the quotient is finite. Any connected algebraic group G has a normal subgroup G_1 such that G/G_1 is projective as a variety (hence called an abelian variety), and G_1 is affine as a variety. Next, G_1 has a normal subgroup G_2 such that G_1/G_2 is semisimple (in other words, it is a direct product of simple groups up to isogeny), and G_2 is solvable. Next, G_2 has a normal subgroup G_3 such that G_2/G_3 is a torus (isomorphic to a direct product of multiple copies of $K^\times$), and G_3 is unipotent, meaning it is a closed subgroup of the group of unipotent matrices of some dimension (matrices equal to the identity plus a nilpotent matrix, or equivalently matrices that in some basis are upper triangles with ones along the diagonal).

The decomposition is in some sense unique, and we can characterize various types of groups (e.g. unipotent groups, tori, semisimple groups) as groups such that their other components are trivial. Furthermore, the structure greatly influences the representations. For example, a finite-dimensional representations of a semisimple algebraic group decomposes uniquely as a direct sum of irreducible representations, and all representations of unipotent groups consist entirely of unipotent transformations.

We will not prove entirely this decomposition, but we will prove some of the ideas which underlie it, particularly the structure theory for linear solvable algebraic groups.

Due to restrictions on space and time, we present some of the most important proofs and leave out other, more routine results. We tend to at least state results which we use and give references. We assume all the basic algebraic geometry in [Sha94], as well as some basic facts about abstract varieties and complete varieties found in [Hum75]. This means that the paper assumes basic definitions about algebraic groups (e.g. the definition of one, the Hopf algebra on its coordinate ring) and their consequences. We also assume many standard algebraic notions (e.g., basic group theory, module theory, semisimple modules and algebras).

1. Linearization of Affine Groups

Definition 1.1. An *affine* (algebraic) group is an algebraic group whose underlying variety is affine.

We have been using the term linear and affine interchangeably in the introduction. A linear group is a closed subgroup of the general linear group of some vector space. It's clear that a linear group is affine. It turns out that the converse is also true. We prove this after a series of lemmas.

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We will prove this by showing that G acts faithfully on a finite-dimensional vector subspace of its affine coordinate ring $K[G]$. Suppose G has a left action on a variety X. This can be turned into a right action by causing g to act as g^{-1}, i.e. $g(x) = g^{-1}x$. Now, the comorphism of each element of G is an automorphism of the affine algebra, meaning G has a *left* action on $K[X]$. Explicitly, $\tau_g(f)(x) = f(g^{-1}x)$ for $f \in K[X]$, $g \in G$, and $x \in X$, where τ_g denotes the action of g on $K[X]$. For example, G acts on itself by left translation, i.e. $x(y) = xy$ for $x, y \in G$. In addition, G acts by right translation by inverse elements, which we denote by $\rho_x(f)(y) = f(yx)$.

In this way, we get a linear *representation* of G (however, we have not yet proven it is rational).

We begin with a lemma which will allow us to show that all affine groups are linear.

Lemma 1.2. *Let G act morphically on an affine variety X, and let W be a finite-dimensional subspace of $K[X]$.*

(1) W is stable under the action of G iff W is sent into $K[G] \otimes_K W$ under the comorphism of the map $G \times X \to X$.

(2) There exists a finite-dimensional subspace V of $K[X]$ containing W which is table under the action of G.

Proof. (1) If W is mapped into $K[G] \otimes_K W$, then $f(g^{-1}x) = \sum_i f_i(x)g_i(g^{-1})$ for $f_i \in W$, so W is stable. Conversely, let f_i be a basis of $K[X]$, where i runs over the positive integers, such that $f_1, \cdots, f_d$ is a basis of W. Then if $f \in W$, write $f = \sum_{i=1} g_i \otimes f_i$, and since $f(g^{-1}x)$ is always in W, we must have that $g_i(g) = 0$ for all $i > d$ and $g \in G$, so f is mapped into $K[G] \otimes_K W$.

(2) We suppose that W is one-dimensional, spanned by f. Suppose f maps to $\sum g_i \otimes f_i$, where $f_i \in K[X]$, and the g_i are linearly independent. The subspace of $K[X]$ spanned by the f_i contains all translates of f under the action of G, so it is fixed by G, and it is clearly finite-dimensional since there are finitely many f_i. If W is not one-dimensional, it is the sum of one-dimensional spaces, and the sum of the G-invariant spaces associated to these contains W.

□

We also need a technical lemma which ensures that the image of a morphism of algebraic groups is closed.

Definition 1.3. A subset of a variety is said to be *constructible* if it is a finite union of intersections of closed sets with open sets.

Lemma 1.4. *The image of a morphism $\phi : X \to Y$ of varieties is constructible.*

Proof. We may assume that the image is dense. We first assume that X, Y are irreducible. We induct on the dimension of Y. If Y has dimension 0, all subsets are constructible (even clopen). Now $\phi(X)$ contains an open subset U of Y by I.5, Theorem 6 of [Sha94], and let $Y_1, \cdots, Y_t$ be the irreducible components of $Y \setminus U$. The restrictions of ϕ to the components Z_{ij} of $\phi^{-1}(Z_i)$ have constructible images by the induction hypothesis, so the union of these and U, which is $\phi(X)$, is constructible.

If $Y = \bigcup Y_i$ with the Y_i irreducible, and Z_{ij} are the irreducible components of $\phi^{-1}(Y_i)$, then each $\phi(Z_{ij})$ is constructible, and their union is $\phi(X)$, so $\phi(X)$ is constructible. □

Corollary 1.5. *The image of a morphism of algebraic groups is closed.*

Proof. It suffices to show that a constructible subgroup of an algebraic group is closed. Thus it contains a dense open subset U of its closure, which is an algebraic group. Now U^{-1} is dense because inversion is an automorphism of the variety, hence a homeomorphism, so U meets xU^{-1} for any x in the closure, so x is a product of two elements of U, meaning that it is in the image. □

We are now prepared to prove the following:

Theorem 1.6. *If G is an affine algebraic group, then it is isomorphic to a closed subgroup of some* $\mathrm{GL}(V)$.

Proof. Let $x_1, \cdots, x_n$ generate $K[G]$ as an algebra. Let V be a G-invariant subspace containing the x_i. Let $f_1, \cdots, f_n$ be a basis of V (hence it generates $K[G]$ as an algebra). Suppose f_i maps to $\sum m_{ij} \otimes f_j$, with $m_{ij} \in K[G]$. Thus $\tau_x(f_i)(y) = \sum m_{ij}(x^{-1})f_j(y)$, so since the m_{ij} are morphisms, this representation is rational.

Note that $f_i(x) = \sum m_{ij}(x)f_j(e)$, so the m_{ij} generate $K[G]$. If an element of G is sent to the identity matrix under m_{ij}, then it has the same value as e on all of $K[G]$, so it equals e. This means the representation is an injective map to the general linear group. Furthermore, if we denote the image by G', then G' is closed, since it is a constructible subgroup. Now $K[G']$ includes into $K[G]$, and the image contains the m_{ij}, so it is all of $K[G]$, i.e. $K[G] \cong K[G']$. Thus G is isomorphic as an algebraic group to G', and we are done. □

2. Jordan Decomposition

Throughout the rest of this paper, all algebraic groups are assumed to be linear, and all representations are rational.

2.1. **Some Linear Algebra.** We state a few basic results from linear algebra:

Definition/Theorem 2.1. *A linear transformation is said to be nilpotent iff some power of it is equal to* 0*. Equivalently, all its eigenvalues are* 0*, or it is upper triangular with* 0*s along the diagonal in some basis.*

Proof. To see why, if it is nilpotent, its eigenvalues are clearly 0. If the latter is true, its Jordan decomposition is in the required form. If it is upper triangular, then it's clear that some power of it is 0. □

Definition/Theorem 2.2. *A linear transformation is said to be semisimple iff it is diagonalizable in some basis. Equivalently, its minimal polynomial has no multiple roots, or the algebra it generates is semisimple.*

Proof. If it is semisimple, we can diagonalize, and then the polynomial having all its eigenvalues as roots is its minimal polynomial. If its minimal polynomials has distinct roots, then the algebra it generates is $K[x]/(f(x))$, where $f(x)$ is the minimal polynomial, which is a direct sum of copies of K, so it is semisimple. Now the algebra $K[x]/(x-a)^k$ for $k \geq 2$ and $a \in K$ is not semisimple, since the ideal $x-a$ is not a direct summand, so if the algebra it generates is semisimple, then its minimal polynomial has distinct roots. Finally, if it is not diagonalizable, its Jordan blocks contain powers of irreducible polynomials with multiple roots, so its minimal polynomial has multiple roots, and the converse is what we want. □

We cite the following important fact:

Fact 2.3 (Additive Jordan Decomposition)**.** A linear transformation x on a finite-dimensional vector space can be written uniquely as the sum of a semisimple and a nilpotent transformation, x_s and x_n, both of which are polynomials in x. In fact, it is unique if we only require x_s and x_n to commute.

Definition/Theorem 2.4. *A linear transformation is said to be unipotent if all its eigenvalues are* 1*, or equivalently it is a nilpotent transformation plus the identity transformation.*

Proposition 2.5 (Multiplicative Jordan Decomposition)**.** *A nonsingular linear transformation x can be written uniquely as the product of a unipotent transformation and a semisimple transformation, x_s and x_u, both of which are rational functions in x.*

Proof. We have $x = x_s(1 + x_s^{-1}x_n)$. But since x_s and x_n commute, $x_s^{-1}x_n$ is nilpotent, so this decomposition gives a unipotent tranformation and a semisimple one. If we had $x = x_s x_u$ another way, then $x = x_s + x_s(x_u - 1)$, so by uniqueness of the additive decomposition, this decomposition is also unique. □

Notice that this is essentially a kind of decomposition for the algebraic group GL_n. One can also interpret the additive decomposition as the corresponding decomposition for its Lie algebra. If we have an arbitrary linear group, we can declare an element semisimple or unipotent if this is true of the linear transformation it represents. What is lacking is an assurance that this is independent of the particular embedding into $\mathrm{GL}(V)$. We will, however, discover that this is the case (i.e. that it is independent).

Definition 2.6. We say that an operator is *semisimple*, *nilpotent*, or *unipotent* on an arbitrary (possibly infinite-dimensional) vector space V if V is a union of finite-dimensional subspaces V_i stable under that operator such that the operator restricts to that type of operator on each of the V_i.

Note that this is well-defined because the restriction of such a type of operator is again of that type, and if that operator is such on a finite set of subspaces, then it is so on their union.

From this, Jordan decomposition exists in the general setting above. This follows because we can find the Jordan decomposition on each of the V_i, then note that the decomposition matches up on the intersections $V_i \cap V_j$ because of its uniqueness, and finally piece together these decompositions to get one on the whole space V.

2.2. **Some Lemmas.** We would now like to prove a kind of Jordan decomposition for an aribtrary algebraic group (in other words, the idea that the Jordan decomposition when embedded in $\mathrm{GL}(V)$ is independent of the embedding).

First, if G is an algebraic group, H a closed subgroup, we develop a nice criterion for an element $g \in G$ to be in H. We recall the action ρ_g on $K[G]$ given by $\rho_g(f)(g') = f(g'g)$.

Lemma 2.7. *Let $H \subseteq G$ be a closed subgroup with ideal $I \subseteq K[G]$. Then $g \in G$ is in H iff $\rho_g(I) \subseteq I$.*

Proof. First, suppose $g \in H$. Then if $f \in I$, it is clear that $\rho_g(f)$ vanishes on H, so ρ_g sends I into I. Conversely, suppose ρ_g leaves I stable. Then $\rho_g(f)(e) = 0 = f(g)$ for $f \in I$, so $g \in H$. □

We will use this to show that the semisimple and unipotent parts of an element of GL_n lie in any closed subgroup containing that element. To do this, we wish to say something about the semisimple and unipotent parts of the action of $g \in \mathrm{GL}_n$ on $K[\mathrm{GL}_n]$. As a first step, we prove the following lemma:

Lemma 2.8. *Let $G = \mathrm{GL}(V)$ where V has dimension n. Then $\rho_g = \rho_{g_s}\rho_{g_u}$ is the multiplicative Jordan decomposition of the linear transformation ρ_g on $K[G]$. Note that this is defined because $K[G]$ is a union of finite-dimensional subspaces invariant under the action of G by ρ_g (the proof of this fact is the same as that of Lemma 1.2).*

Proof. Since $g \mapsto \rho_g$ defines a representation of G, we know that $\rho_g = \rho_{g_s}\rho_{g_u} = \rho_{g_u}\rho_{g_s}$. Thus we must show that ρ_{g_s} and ρ_{g_u} act as semisimple and unipotent endomorphisms, respectively.

Note that G has a right action on $E = End_K(V)$, which gives a left action on E^*. Let $e_1, \cdots, e_n$ be a basis which diagonalizes g_s with $g_s(e_i) = a_i e_i$, and let T_{ij} be the corresponding dual basis for the space of matrices E. In this basis, $e_{ij}(yg_s) = a_j e_{ij}(y)$, so the action of g_s on E^* is semisimple. Now if $g_u = 1 + n$ where n is nilpotent, then n acts as a nilpotent transformation on E, hence on E^*, and 1 acts as the identity, so g_u is unipotent. By the decomposition $Sym^k(U \oplus W) = \sum_{i+j=k} Sym^i(U) \otimes Sym^j(W)$, g_s acts on the symmetric algebra $Sym(E^*)$ as a semisimple operator, and g_u as a unipotent operator.

Now the symmetric algebra defines a set of maps from E to K which are precisely the algebraic maps in its affine coordinate ring, i.e. $Sym(E^*) = K[E]$, and the action of G on this symmetric algebra is the same as its action on $K[E] \subseteq K[G]$. Finally, since G is the open subset defined by $d \neq 0$, where $d \in K[E]$ is the determinant, we find $K[G] = d^{-1}K[E]$ (i.e. the ring of fractions).

Now note that d is an eigenvector of both g_s and g_u, and since g_u has determinant 1, it has eigenvalue 1 on g_u. This means that d is a direct summand in the decomposition of g_s on $K[E]$, and as a result we get a diagonalization of g_s on $K[G]$. As for g_u, if f is any eigenvector of g_u in $K[E]$, then $d^m f$ is also an eigenvector for positive integer m, so for sufficiently large m it is in $K[E]$ and hence has eigenvalue 1, meaning that f has eigenvalue 1. This shows that g_s and g_u act as semisimple and unipotent operators on $K[G]$, respectively, so we are done. □

We first note that $x \mapsto \rho_x$ possesses a kind of functoriality. Specifically, in the category of pointed algebraic groups (G, x) and regular homomorphisms, the map $\rho_x : K[G] \to K[G]$ is a natural transformation from the contravariant functor $G \mapsto K[G]$ to itself.

2.3. The Theorem.

Finally, we can now use our two lemmas to establish our decomposition. Specifically, we have the following:

Theorem 2.9 (Jordan-Chevalley Decomposition)**.** *Let G be an algebraic group. Then for any $g \in G$, there exists $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$, ρ_{g_s} is semsimple, and ρ_{g_u} is unipotent. Furthermore, these are preserved under morphisms $G \to G'$.*

Proof. We embed G in a general linear group GL_n. Each element decomposes into a product of two elements of G, which are semisimple and unipotent, respectively, in GL_n and which commute. Since they are semisimple and unipotent on $K[\mathrm{GL}_n]$, and they preserve the ideal of G, they have an induced action on $K[G]$, which must also be semisimple and unipotent (note that a quotient of the $K[X]$-module associated to a semisimple endomorphism is also semisimple, since the induced endomorphism also satisfies the minimal polynomial of the original endomorphism). It is clear that they commute.

Furthermore, if we had two other endomorphisms such that ρ_{g_s} and ρ_{g_u} were semisimple and unipotent, then those endomorphisms of $K[G]$ would be equal. But G has a faithful representation on $K[G]$ (since if $f(xg) = f(x)$ for all $x \in G$, $f \in K[G]$, then $f(g) = f(e)$, so $g = e$), so the decomposition is unique.

If we have a morphism $\phi : G \to G'$, then this gives us a comorphism $K[G'] \to K[G]$ which commutes with ρ. It suffices to show that $\rho_{\phi(g_s)}$ and $\rho_{\phi(g_u)}$ are semisimple and unipotent, or equivalently that ρ_{g_u} and ρ_{g_u} are semisimple and unipotent when restricted to $K[G']$. We can do this in the case when ϕ is injective and the case when ϕ is an embedding and the case when ϕ is surjective, the general case being a combination of these two.

If ϕ is an embedding, then the comorphism is surjective. Since G', hence G, acts as semisimple operators on $K[G']$, it does so too on $K[G]$ (which is a quotient of $K[G']$ not just as a ring but as a G-module), and same for unipotent, hence the result is true.

If ϕ is a surjection, then the comorphism is injective, and restriction to a subspace is still semisimple, so the result is again true, and we are done. □

Corollary 2.10. *The decomposition in the theorem is the same as the multiplicative Jordan decomposition for* GL_n.

Proof. Follows from Lemma 2.8. □

Corollary 2.11. *If $G \to \mathrm{GL}(V)$ is any representation of G, then semisimple and unipotent elements of G act as such on V.*

Proof. Follows from the theorem and the previous corollary. □

Corollary 2.12. *If an element is semisimple or unipotent in any faithful representation of G, then it is so in G.*

Proof. If it is semisimple in the representation, then its unipotent part is trivial, so since the representation is faithful, it is semisimple in G, and vice versa. Note that this is also true for an infinite-dimensional representation which is a sum of finite-dimensional ones, in particular on $K[G]$. □

2.4. **Commutative Algebraic Groups.** It is not hard, using this result, to classify commutative (affine) algebraic groups. We begin with a lemma which is a generalization of the theorem that a set of commuting semisimple operators is simultaneously diagonalizable.

Lemma 2.13. *Let M be a commuting subset of $End(V)$. Then there exists an invertible matrix x such that xMx^{-1} is all upper triangular. If $N \subseteq M$ consists of semisimple matrices, then we can choose x so that N is diagonal.*

Proof. The theorem is equivalent to showing that there is a complete flag preserved by M. First, we show that the matrices have a common eigenvector by induction on the dimension of V. In dimension 1, this is obvious. In general, we assume the matrices are not all scalars, or else we are done. Then we can find $a \in K$ and $m \in M$ such that the kernel of $m - a$ is smaller than V. Furthermore, if $m' \in M$, and $mx = ax$, then $m(m'x) = m'(mx) = m'(ax) = am'x$, so this kernel is stable under M. By the induction hypothesis, this kernel contains an eigenvector common to all of M, call this vector v_1. Then all of M restricts to an action on V/v_1, and we can find another eigenvector, whose pullback generates, along with v_1, another subspace stable under the elements of M. We continue, until we have a complete flag preserved by M.

Now, the subalgebra A_N of $\mathrm{GL}(V)$ generated by the elements of N is semisimple. Let V_k be the subspace of V generated by the v_i for $i \leq k$. Then V_{n-1}, where $n = \dim(V)$, is a direct summand of V as an A_N-module, so there exists an eigenvector of N generating V/V_{n-1}. Next, we similarly find that there is an eigenvector of N contained in V_{n-1} which generates V_{n-1}/V_{n-2}, and so on, until we have a set of eigenvectors for N representing this flag. Thus N can be diagonalized at the same time.

This is essentially the theorem that an irreducible representation of a commutative (associative) algebra over a field is one-dimensional. □

Notice that the set G_u of elements g such that $g = g_u$, called the set of *unipotent elements*, is closed, since the set of unipotent matrices in GL_n is closed. Note that the unipotent part of an element is always unipotent in this sense. The same is not true about the corresponding set of semisimple elements, at least in general. However, we have the following results.

Theorem 2.14. *Let G be a commutative algebraic group. Then G_u, G_s are closed subgroups, connected if the same is true of G, such that $G = G_s \times G_u$.*

Proof. We have already seen that G_s is closed. If we have an embedding of G in $\mathrm{GL}(V)$, we can find a basis in which G_s is diagonalized. Since the subset of diagonal matrices is closed, it follows that G_s is closed, and it is clear that it is a subgroup (as the product of two diagonal matrices is diagonal.

We construct the map $G_s \times G_u \to G$ defined by multiplication. It is clear that this map is a group isomorphism by the Jordan-Chevalley Decomposition. The inverse map sends g to (g_s, g_u). In order to show that this inverse is a morphism of varieties, it suffices to show that the maps sending g to each of its components are morphisms. Note that $g_u = g_s^{-1}g$ so we need only show that $g \mapsto g_s$ is regular. By Lemma 2.13, we can assume that G is contained in the group of upper triangular matrices. Thus the semisimple part consists simply of the diagonal elements, which is a regular map, so we are done. □

Thus, we have proven the structure of commutative groups, which is a special case of the decomposition of linear algebraic groups discussed in the introduction.

3. Lie-Kolchin Theorem

3.1. **Commutators.**

Definition 3.1. If A, B are two subgroup of G, the subgroup generated by $(x, y) := xyx^{-1}y^{-1}$ for $x \in A$, $y \in B$ is denoted by (A, B) and called the *commutator* of A and B. Each (x, y) is itself called a *commutator.*

Lemma 3.2. *The application of an inner automorphism to a commutator is a commutator.*

Proof. This follows at once from the identity $a(bcb^{-1}c^{-1})a^{-1} = (aba^{-1})(aca^{-1})(aba^{-1})^{-1}(aca^{-1})^{-1}$. □

Corollary 3.3. (G, G) *is normal in* G.

We define $D^i(G)$ by $D^0(G) = G$, and $D^{i+1}(G) = (D^i(G), D^i(G))$, called the *derived series* of G. If this eventually becomes trivial, we say that G is *solvable.* We state a basic fact from group theory:

Fact 3.4. If $0 \to G \to G' \to G'' \to 0$ is an exact sequence of groups, then G' is solvable iff the same is true of both G and G''. Subgroups of solvable groups are solvable. Finally, if $A, B \leq G$ are normal and solvable, and $G = AB$, then G is solvable.

We now need a couple more (nonstandard) facts that are purely group theoretic:

Proposition 3.5. *If* $[G : Z(G)] = n < \infty$, *then* (G, G) *is finite.*

Proof. Let R be the set of commutators which generate (G, G). Notice that if we multiply either x or y by an element of $Z(G)$, (x, y) will not be changed, so this means that S is finite. Notice that $(x, y)^{n+1} = y^{-1}(x, y)^n y(x, y) = y^{-1}(x, y)^{n-1}(x, y^2)y$ since $(x, y)^n \in Z(G)$. This means that if we have a product of $n^3 + 1$ commutators, one of them must appear $n + 1$ times. We can apply an inner automorphism to make all of them adjacent, i.e. we get an $n + 1$st power of one of the commutators. But this, as we have shown, is a product of n commutators, so we get a product of n^3 commutators. This means that all of (G, G) is a product of n^3 commutators, so (G, G) is finite. □

Proposition 3.6. *If* A, B *are normal subgroups of* G, *and the set* $\{xyx^{-1}y^{-1} \mid x \in A, y \in B\}$ *is finite, then* (A, B) *is finite.*

Proof. We can let $G' = AB$. Then G' sends elements of S to elements of S. Let H be the subgroup of G' which acts trivially on S. Then clearly H is normal, and $[G' : H]$ is finite since S is finite. Furthermore, H is contained in the centralizer of (A, B). Thus $H \cap (A, B)$ is in the center of (A, B) and has finite index. By Proposition 3.5, $((A, B), (A, B))$ is finite. It is also normal in G', since the same is true of (A, B). Thus if we show that (A, B) is finite in $G'/((A, B), (A, B))$, we will have shown that it is finite. Notice that (A, B) abelian in this quotient. Now if $x \in A$, $y \in (A, B)$, then (x, y) always lies with S and commutes with other such commutators. Since (A, B) is abelian, we have $(x, y)^2 = (xyx^{-1})^2y^{-2} = (x, y^2)$. This means that $(A, (A, B))$ is finite and normal in $G'/((A, B), (A, B))$. Modding out by $(A, (A, B))$, we find that A is in the center of (A, B). This means that the square of an aribitrary commutator is again a commutator, so that (A, B) is finite. □

3.2. Commutators of Algebraic Groups.

Lemma 3.7. *Let A, B be closed subgroups of an algebraic group G. Then if A is connected, (A, B) is closed and connected, and if A and B are normal, then (A, B) is closed in G.*

Proof. For $b \in B$, the morphism $\nu_b : A \to G$ defined by $a \mapsto aba^{-1}b^{-1}$. Note that if A is connected, then $\nu_b(A)$ is connected and contains e. By Proposition 7.5 in [Hum75], the group generated by all the $\nu_b(A)$ is closed and connected, and this is (A, B). If A and B are normal, then (C, B) and (A, D) are closed and connected (and normal by pure group theory), where C and D denoted the connected component of the identity in A and B, respectively. It is clear that $(C, B)(A, D)$ is constructible as it is the image of $(C, B) \times (A, D)$, so it is closed by the proof of Corollary 1.5. Now in $G' = G/((C, B)(A, D))$, the image of C centralizes the image of B, and the image of D centralizes the image of A. Since A/C and B/D are finite, this means that (A, B) is finite in G' by Proposition 3.6, so (A, B) is closed. □

This means that $D^1(G)$ is connected and closed if G is a connected algebraic group and that $D^1(G)$ has smaller dimension if G is also solvable.

3.3. A Bit of Representation Theory.

We need one basic fact from the representation theory of algebraic groups. We begin with a definition.

Definition 3.8. A *character* of an algebraic group G is a morphism $G \to K^\times$.

Definition 3.9. If V is a representation of G, a nonzero vector $v \in V$ which spans a subrepresentation of V is called a *semi-invariant* of G. Equivalently, it is an eigenvector for all of G.

It is clear that if $\chi(g)v = g(v) \forall g \in G$, then χ defines a character of G.

Definition 3.10. If χ is a character of G, we define $V_\chi = \{v \in V \mid g(v) = \chi(g)v \forall g \in G$. Every nonzero element of V_χ is a semi-invariant.

Lemma 3.11. *If V is a representation of G, then the subspaces $V_\chi \subseteq V$ are linearly independent. Since V is finite-dimensional, only finitely many of them are nonzero.*

Proof. Suppose we have a relation $\sum_{i=1}^n v_i = 0$, with $v_i \in V_{\chi_i} \setminus \{0\}$, $n \geq 2$ minimal, and the χ_i distinct. Then $\chi_1(g) \neq \chi_2(g)$ for some $g \in G$. Thus $0 = (g - \chi_1(g))(0) = (g - \chi_1(g))(\sum_{i=1}^n v_i)$ is a relation with a smaller number of vectors, contradicting the minimality. The result must therefore be true. □

3.4. Proof of the Theorem.

We can use the previous material to prove the celebrated Lie-Kolchin Theorem, which is an extensive generalization of Lemma 2.13 on commuting operators. This will allow us to prove a theorem on the structure of solvable algebraic groups. We need one definition:

Definition 3.12. If G is a solvable group, then its *derived length* is the smallest d such that $D^d(G) = \{e\}$.

Theorem 3.13 (Lie-Kolchin). *Let G be a closed connected solvable subgroup of $\mathrm{GL}(V)$, with V nonzero and finite-dimensional. Then G has a common eigenvector in V.*

Proof. We use induction on the derived length d of G and the dimension n of V. Note that if $d = 1$, the result is true by Lemma 2.13, and if $n = 1$, the result is obvious, since all linear endomorphisms are scalars.

If V is reducible as a G-representation, there is a nontrivial subspace W stable under G. The map $G \to \mathrm{GL}(W)$ is a morphism, so its image G' is closed, and it is connected and solvable by Fact 3.4, and since W has smaller dimension than V, the induction hypothesis says that G' has an eigenvector in W, hence G has one too.

Next we tackle the case when V is an irreducible representation of G.

Let $A = (G, G)$. This means that A has derived length $d-1$, hence a common eigenvector in V by induction, or in other words a one-dimensional A-submodule of V. The sum W of all such A-modules, hence the sum of all the V_χ for characters $\chi : A \to K^\times$, is acted on diagonally by A by Lemma 3.11. Furthermore, if $g \in G$, $xyx^{-1}y^{-1} \in A$, and $v \in W$ is an eigenvector of A, then $xyx^{-1}y^{-1}(gv) = gg^{-1}(xyx^{-1}y^{-1})g(v) = g(a(v)) = c(gv)$, where $a \in A$ and $c \in K$ by Lemma 3.2. Thus W is stable under G, so $W = V$ by irreducibility, hence G' acts diagonally on V, meaning it is commutative because G' acts faithfully on V.

Since G' is normal in G, xyx^{-1} acts diagonally in a basis in which G' acts diagonally. Since it is conjugate to y, it has the same eigenvalues of y, meaning it is one of $n!$ possibilities. But since the image of G under the morphism $x \mapsto xyx^{-1}$ is finite and connected, it must be equal only to y, meaning that G' acts on V as G-invariant maps, hence by Schur's lemma, as scalars. But all of G' has determinant 1, so G' is finite. Since it is connected by Lemma 3.7, $G' = \{e\}$, meaning that G is commutative, so G has a common eigenvector by the case $d = 1$, and by irreducibility, the space spanned by this eigenvector is all of V.

□

3.5. **Structure of Solvable Groups.** This allows us to prove an important component of the structure theory for algebraic groups.

Corollary 3.14. *If G is a connected solvable algebraic group, then G/G_u is a subgroup of a torus (a product of copies of $K^\times$).*

Proof. As in Lemma 2.13, this means that there is a basis in which G is upper triangular, i.e. G is a closed subgroup of T_n, the group of upper triangular $n \times n$ matrices. The subgroup of T_n consisting of unipotent matrices is closed, normal, and has quotient equal to H^n, where $H = K^\times$, and the intersection of this subgroup with G is equal to G_u. Thus the corollary follows. □

Note that if we ask about non-connected groups, we can take the connected component of the identity. The quotient by this component is a finite solvable group.

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