Thm (Axi-Grothendieck)

Say we have
\[ f : (x_1, \ldots, x_n) \]

\[ \text{giving } f : \mathbb{C}^n \to \mathbb{C}^n \]

Example
\[ n = 1 \]
\[ f_1(x_1) = x_1^2 \quad \text{surjective, not injective} \]
\[ \text{so converse is false} \]

\[ n = 2 \]
\[ f_1(x_1, x_2) = x_1 \]
\[ f_2(x_1, x_2) = 0 \]
\[ \text{not surj } (1, 1) \text{ not in image} \]
\[ \text{also not injective (does not depend on } x_2) \]

\( n \) invertible n x n matrix \( (f_j \text{ all linear in } x_i) \)

\[ n = 2 \]
\[ f_1 = x_1 - x_2^2 \]
\[ f_2 = x_2 \]

\( C \) is an algebraically closed field \(-\text{add, mult, everything but } 0 \text{ has multiplicative inverse}\)
\( \text{and any polyn. has a } \) \( n \) \( \text{soln} \)
\[ x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]
\( n = 1 \)
$x^3 + x + 1 = 0$

now "$R$ has soln to first two" get things of form $a + b\sqrt{2} + c\sqrt{3}$

but not $x^2 + 1 = 0$ so we just create this thing $i$

and declare $i^2 + 1 = 0$ 

"formally adjoin a root to $x^2 + 1 = 0"$ 

\[ \frac{\mathbb{Z}[x]}{(x^2 + 1)} \]

\[ \text{is a field } p \text{ prime} \]

\[ \text{say } f_p(x_1, \ldots, x_n) \text{ map } \mathbb{F}_p^n \rightarrow \mathbb{F}_p^n \]

\[ \text{if injective... then surjective!} \]

now $\mathbb{F}_p$ not alg. closed.

\[ \text{e.g. } x^2 - 2 = 0 \text{ has no soln in } \mathbb{F}_3 \]

(can cook up higher deg examples)

\[ \text{can do } \mathbb{F}_9 = \{ a + b\sqrt{2} | a, b \in \mathbb{F}_3 \} \]

\[ \mathbb{F}_{27} \text{ ("if you adjoin roots of two irreducible polynomials of same degree get same field"}) \]

\[ \mathbb{F}_5 \text{ as etc. finite so this is true} \]

adjoining all roots of all polynomials get $i \mathbb{F}_p$
If \( p \in \mathbb{F}_p \), so \( 1 + 1 + \ldots + 1 = 0 \) in \( \mathbb{F}_p \) \( p \) times.

And every poly has soln.

Now \( \bar{\mathbb{F}}_p \) is union of \( \mathbb{F}_{p^k} \) over \( \mathbb{N} \).

So \( \bar{\mathbb{F}}_p \) is true for \( \mathbb{F}_p \) in place of \( \mathbb{C} \).

Say \( f_{p^n}(x_1, \ldots, x_n) \)

\( f^n(x_1, \ldots, x_n) \) has all coeff in \( \mathbb{F}_{p^k} \).

Then if \( \mathbb{F}_{p^k} \subseteq \mathbb{F}_{p^k'} \) defines map

\[
\mathbb{F}_{p^k}^n \rightarrow \mathbb{F}_{p^k'}^n
\]

inj as inj for \( \bar{\mathbb{F}}_p \)

so surj as \( \mathbb{F}_{p^k'} \) is finite.

Now every \( (y_1, \ldots, y_n) \in \mathbb{F}_p^n \) lies in \( \mathbb{F}_{p^k'} \) for some \( k' \)

So in image of \( f \), \( \mathbb{F}_{p^k'} \) surjective.

So done for \( \bar{\mathbb{F}}_p \) instead of \( \mathbb{C} \).
"proof" if I counterexample, then I proof of

Counterexample using axioms of alg. closed fields of
char 0

i.e. use 1+1 ≠ 0, 1+1+1 = 0, 1+1+1+1 ≠ 0, etc

"not char 2" "not char 3" "not char 5" "not char 7"

"im writing 1+...+1 b/c in any abstract field we have a unit elt.

guess what? a proof is finite

so uses finitely many of these axioms

so I p sit. does not use the axiom p ≠ 0
so I counterexample for F_p but we already proved...

Model Theory

What is a Mathematical structure?

\( \mathbb{Z} \) = \{ 0, 1, +, \ldots, \lt \}

\( \mathbb{R} \) = \{ 0, 1, +, \ldots, \lt \}

\( \mathbb{Q} \)

\( \mathbb{C} \)

& Defn. A structure is a set

I. funs \( \{ f_i \mid i \in \mathbb{I}_0 \} \) \( f_i : M^i \to M \)

\( \mathbb{I}_0 \geq 1 \) another

II. relations \( \{ R_i \mid i \in \mathbb{I}_1 \} \)

III. "constants" \( \{ c_i \mid i \in \mathbb{I}_2 \} \)

\( c_i \in M \)
Now the theory of groups (some set $S$ with an operation $*$ or operations for rings) and with certain axioms: a model of the theory is a set $S$ with operation $*$ and an operation satisfying some axioms.

So to talk in general:

**Defn.** A language $\mathcal{L}$ is a three sets of symbols:

- $I_0$ \{ $\{ f_i \}$ | $f_i : I_0 \to \mathbb{N}$ \}
- $I_1$ \{ $\{ R_i \}$ | $R_i : I_1 \times \ldots \times I_1 \to \{0,1\}$ \}
- $I_2$ \{ $\{ c_i \}$ \}

**Defn.** If $\mathcal{L}$ is a language, then a $\mathcal{L}$-structure $M$ is a set $M$ along with an $\rho$-function $f_i : M^{I_0} \to M$ for each $i \in I_0$, a relation $R_i \subseteq M^{I_1}$ and an element $c_i \in M$ for each $i \in I_2$.
Formulas

Say \( \mathcal{F} = (\top, \hat{x}, 0, 1) \)

\[
\begin{align*}
x_2 \times x_2 & \quad \text{terms} \\
\hat{1} \times x_1 & \quad \text{atomic formula} \\
\hat{0} \times x_2 & \quad \text{use } \equiv \text{ or an } R; \\
x_2 \times \hat{0} & = x_1 \quad \text{use } \land \forall \rightarrow \\
x_2 + x_2 & = x_1 \quad \text{formula} \\
(x_2 + x_1) \times x_2 & = \hat{0} \quad \text{formula}
\end{align*}
\]

\[
\frac{x_1 \times x_2}{(x_1 = x_2)}
\]

\[
\forall x_1 \exists x_2 \left( x_2 \times x_2 = x_1 \right)
\]

\[
\left( f_i(x_2) = x_1 \right)
\]

\( f_i \) is surjective (\( \forall i : 1 \))

(\( f_i \) same for some language)

\[
\forall x_1 (x_1 = 0 \lor \exists x_2 (x_2 \times x_1 = 1))
\]

a formula \((\exists x_2 \ x_2 \times x_2 = x_1) \lor (x_1 < 0)\)

if we append \( \forall x_1 \), we get a sentence in \( \mathcal{F} = (\top, \hat{x}, 0) \)

if \( \phi \) is an \( \mathcal{F} \)-sentence and \( M \) an \( \mathcal{F} \)-structure

we write \( M \models \phi \) to mean "\( \mathcal{F} \) holds in \( M \)"

else \( M \not\models \phi \)
Defn: An \( \mathcal{L} \)-theory \( T \) is a set of \( \mathcal{L} \) sentences in \( \mathcal{L} \) ("possibly infinite")

We say \( M \models T \) if \( M \) is an \( \mathcal{L} \)-structure s.t. \( M \models \phi \forall \phi \in T \)

We say \( M \) is a model of \( T \)

Example \( \mathcal{L} = \{+, \times, \bar{0}, \bar{1} \} \)

Theory of rings

\[
\forall x_1, x_1 + \bar{0} = x_1
\]

\[
\forall x_1, \forall x_2, x_1 + x_2 = x_2 + x_1
\]

\[
\forall x_1, \forall x_2, \forall x_3, x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3
\]

\[
\forall x_1, \exists x_2, x_1 + x_2 = \bar{0}
\]

id, assoc for mult. \( \forall x_1, x \cdot x_1 = x \)

\[
\forall x_1, \forall x_2, \forall x_3, x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3
\]

So a model of \( T \) "i.e. an \( \mathcal{L} \)-structure in which all sentences in \( T \) are true"

is ... a ring

Say we add these axioms (sentences) too in \( T ")

\[
\forall x_1 (\exists x_2 (x_1 \cdot x_2 = \bar{1})) \lor (x_1 = \bar{0})
\]

\[
\forall x_1 \neg (x_1 = \bar{0}) \rightarrow (\exists x_2 \ x_2 \cdot x_1 = \bar{0})
\]

"\( y_3 \)" \( \forall x_1, \forall x_2, \forall y_3 \exists y \ (y^3 + x_1 \cdot y^2 + x_2 \cdot y + x_3 = \bar{0}) \)

"\( y_n \)" in general
p.8 We get ACF a theory "collection of sentences in $I$"

Now $\phi_3 \land (\bar{1} + \bar{1} + \bar{1} = 0)$

$\phi_2 \land (\bar{1} + \bar{1} = 0)$

etc $\phi_5, \phi_6$, etc.

get ACF$_0$.

Taking $\Gamma \cup \{\phi_3, \phi_2, \phi_5, \phi_6\}$ get ACF$_3$ or ACF$_p$ in general.

Defn A theory $T$ is

satisfiable if $\exists$ a model $M$

consistent if cannot derive a contradiction from $T$

Clearly, satisfiable $\to$ consistent

Thm (Gödel's Completeness) if $T$ consistent, satisfiable

$\forall K \models \text{G}_{T+1} \exists \text{ model } M \text{ s.t. } |M| = K$
so say

$$T_0 = ACF_0 + \{ \phi_n \}_{n \in \mathbb{N}}$$

$$\phi \text{ in Marker}$$

$$\mathbb{Z}_{2102}$$ (see next page)

0 $T$ is consistent. Why? If we could derive a contradiction then it would use finitely many $\phi_p$, so it would be true for $ACF_p$ for some $p$. But we showed this is not the case.

So $T$ is satisfiable (by Gödel completeness) by a model with cardinality $|\mathbb{C}| (= |\mathbb{R}|)$ (so some algebraically closed field of char. 0 w/ # of eltts $|\mathbb{C}|$). Algebra tells us all such things are isom to $\mathbb{C}$ so true for $\mathbb{C}$.
\( \forall a_{0,0} \forall a_{0,1} \forall a_{0,2} \forall a_{1,0} \forall a_{1,1} \forall a_{1,2} \forall b_{0,0} \forall b_{0,1} \forall b_{0,2} \forall b_{1,0} \forall b_{1,1} \forall b_{1,2} \left[ \left( \forall x_1, \forall y_1, \forall x_2, \forall y_2 \left( \exists \alpha_{ij} x_1^{i} y_1^{j} = \alpha_{ij} x_2^{i} y_2^{j} \wedge \exists b_{ij} x_1^{i} y_1^{j} = b_{ij} x_2^{i} y_2^{j} \right) \rightarrow (x_1 = x_2 \wedge y_1 = y_2) \right) \right] \)

\( \rightarrow \forall u \forall v \exists x \exists y \exists a_{ij} x^{i} y^{i} = u \wedge i \exists b_{ij} x^{i} y^{j} = v \)

This is \( \gamma_{2,2} \)

(or \( \phi_{2,2} \) in Marker's notes)