Goal: Understand $\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{\sum n_i^{s_i}}$ $(s_1 \geq 2)$

(<u>NB</u> Some conventions $s_i \longleftrightarrow s_{k+1-i}$)

<u>No</u> alg geom. or topology

Let $\sigma_N = \{1 \geq t_1 \geq \ldots \geq t_N \geq 0\} \subseteq \mathbb{R}^N$

<u>Thm</u>

$$\int_{\sigma_{\Sigma s_i}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \cdots \frac{dt_{s_1-1}}{t_{s_1-1}} \frac{dt_{s_1}}{1-t_{s_1}} \frac{dt_{s_1+1}}{t_{s_1+1}} \cdots \frac{dt_{s_1+s_2-1}}{t_{s_1+s_2-1}} \frac{dt_{s_1+s_2}}{1-t_{s_1+s_2}} \cdots \frac{dt_{\Sigma s_i}}{1-t_{\Sigma s_i}} = \zeta(s_1, \ldots, s_k)$$

<u>Proof if $k=1$</u> $s = s_1$

$$\int_0^1 \frac{dt_1}{t_1} \int \cdots \int_0^{t_{s-2}} \frac{dt_{s-1}}{t_{s-1}} \int_0^{t_{s-1}} \frac{dt}{1-t}$$

$$= \int_0^1 \cdots \int_0^{t_{s-1}} \sum_{n=0}^{\infty} t^n dt = \sum_{n=0}^{\infty} \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \cdots \int_0^{t_{s-1}} t^n \, dt$$

$$= \sum_{n=0} \int_0^1 \cdots \int_0^{t_{s-2}} \frac{dt_{s-1}}{t_{s-1}} \frac{t_{s-1}^{n+1}}{n+1} = \sum_{n=1}^{\infty} \int \cdots \int_0^{t_{s-2}} \frac{dt_{s-1}}{t_{s-1}} \frac{t_{s-1}^{n}}{n}$$

$$= \int \cdots \int_0^{t_{s-3}} \frac{dt_{s-2}}{t_{s-2}} \frac{t_{s-2}^n}{n^2}$$

$$\cdots \sum_{n=1}^{\infty} \int_0^1 \frac{dt_1}{t_1} \frac{t_1^n}{n^{s-1}} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Good: Integral of alg diff form along something

Problem (1) σ_N isn't a cycle on X^N (2) Not an integral on X

Will see (1) is a cycle modulo $(x_1=1) \sqcup (x_N=0) \bigsqcup_{i=1}^{N-1} (x_i = x_{i+1})$

(2) is related to $\pi_1(X)$

Iterated Integrals

Defn Let $\omega_1, \dots, \omega_N$ be a sequence of 1-forms on X (smooth mfld) (later: complex var, $\mathbb{P}^1 - \{0,1,\infty\}$)

Let $\alpha : [0,1] \to X$ be a (smooth) path

Define $$\mathrm{It}\int_X \omega_1 \cdots \omega_N = \int_{\sigma_N} (\alpha^N)^* (\mathrm{pr}_1^* \omega_1 \wedge \dots \wedge \mathrm{pr}_N^* \omega_N)$$

of length N

Fact If ~~$\dim_{\mathbb{C}}$~~ [more generally, $\omega_i \wedge \omega_{i+1} = 0$ on X] ~~and~~

[e.g. $\dim_{\mathbb{C}} X = 1$ and ω_i holomorphic]

and ω_i closed, this does not depend on path htpy class of α

$\Longrightarrow$ $\omega_1 \cdots \omega_N$ define a map $\pi_1(X, a) \to \mathbb{C}$ $(a \in X)$

$\Longleftrightarrow$ linear map $\mathbb{Z}[\pi_1(X,a)] \to \mathbb{C}$

If $N=1$, $\mathrm{It}\int \omega = \int_\alpha \omega$, so $\int_{\alpha\beta} \omega = \int_\alpha \omega + \int_\beta \omega = \int_{\beta\alpha} \omega$

$\Longrightarrow$ factors through $\mathbb{Z}[\pi_1^{ab}(X,a) \cong H_1(X; \mathbb{Z})]$

Let $Ch(a, X) \subseteq \mathrm{Hom}_{Ab}(\mathbb{Z}[\pi_1(X,a)], \mathbb{C})$

given by $\mathbb{C}$ - span of $\int \omega_1 \cdots \omega_N$ $\forall \{\omega_i\}$ ~~closed~~ holom

[+ $\omega_i \wedge \omega_{i+1} = 0$] $\dim X = 1$

$Ch^N(a, X) \subseteq Ch(a, X)$ for integrals of length $\leq N$

Thm (1) $Ch(a, X)$ is a ~~(graded)~~ subalgebra of $\mathrm{Hom}(\mathbb{Z}[\pi_1(X,a)], \mathbb{C})$

(2) $Ch(a, X)$ is a Hopf alg wrt mult in $\pi_1(X,a)$

(3) Key If $\alpha_1, \ldots, \alpha_{N+1} \in \pi_1(X,a)$

$Ch^N(a, X)$ kills $\prod_{i=1}^{N+1} (\alpha_i - 1) \in \mathbb{Z}[\pi_1(X,a)]$

Pf sketch

First two reduce to a formula

(2) $$\int_{\alpha\beta} \omega_1 \cdots \omega_N = \sum_{j=0}^{N} \int \omega_1 \cdots \omega_j \int \omega_{j+1} \cdots \omega_N \quad \text{(sign?)}$$

(1) $$\int_{\alpha} \omega_1 \cdots \omega_N \int \omega_{N+1} \cdots \omega_{N+M} = \sum_{\sigma \in Sh(N,M) \subseteq S_{N+M}} \int \omega_{\sigma(1)} \cdots \omega_{\sigma(N+M)}$$

(3) is trickier. Will discuss proof later. For $N = 1$

$$\int_{(\alpha-1)(\beta-1)} \omega = \int_{\alpha\beta} \omega - \int_{\alpha} \omega - \int_{\beta} \omega + 0 = 0$$

Consequences of §3

p.4 $u : \mathbb{Z}[\pi_1(X,a)] \to \mathbb{C}$ $\qquad I = \operatorname{Ker} u$

$\alpha \longmapsto 1$ $\qquad$ "augmentation ideal"

I generated by $\{\alpha - 1\}_{\alpha \in \pi_1}$

③ $\iff$ $Ch^N(a, X)$ Kills I^{N+1}

$$\Rightarrow Ch^N(a, X) \subseteq \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{Z}[\pi_1(X,a)]/I^{N+1}, \mathbb{C}\right) \subseteq \mathrm{Hom}(\pi_1(X,a), \mathbb{C})$$

$$\cancel{\parallel \ (\mathbb{Z}[\ ]/I^{N+1})^{\vee}}$$

Thm (Chen) $Ch^N(a, X) = \cancel{(\mathbb{Z}[\Gamma]/I^{N+1})^{\vee}}$

Furthermore, one can generate $Ch(a,X)$ using only a basis for $H^1_{dR}(X)$

§3 Unipotent Completion of Γ (a f.g. group, e.g. $\pi_1(X,a)$)

I is a Hopf ideal
$$\left[\begin{array}{l} c(\alpha - 1) = \alpha \otimes \alpha - 1 \otimes 1 \\ \qquad = (\alpha - 1) \otimes \alpha + 1 \otimes (\alpha - 1) \\ \alpha \in \Gamma \end{array}\right]$$

$\Longrightarrow$ $\mathbb{Q}[\Gamma]/I^{N+1}$ is a quotient Hopf alg

(cocommutative, not comm.)

$\Longrightarrow$ $\left(\mathbb{Q}[\Gamma]/I^{N+1}\right)^{\vee}$ is a commutative Hopf algebra

$\parallel$

$\mathrm{Hom}_{\mathbb{C}}\left(\mathbb{Q}[\Gamma]/I^{N+1}, \mathbb{Q}\right)$

[Rmk $Ch^N(a, X) \cong \left(\mathbb{Q}[\Gamma]/I^{N+1}\right)^{\vee} \otimes \mathbb{C}$]

$\Rightarrow$ Spec $\left(\left(\mathbb{Q}[\Gamma]/I^{N+1} \right)^{\vee} \right)$ is ~~a group~~ an alg. group over $\mathbb{Q}$, call Γ^{un}_{N+1}

System $\Gamma^{un}_{N+1} \to \Gamma^{un}_{N}$

How to think of this

Defn A rep ρ of Γ on a K-vspace V is unipotent if $\exists\ 0 = V_0 \subseteq \cdots \subseteq V_{N+1} \subseteq V$ s.t. V_{i+1}/V_i is a trivial rep of Γ

$$\Longleftrightarrow \operatorname{im}(\Gamma) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Prop If $\dim V \leq N+1$

V unipotent $\Longleftrightarrow \rho : \mathbb{Q}[\Gamma] \to \cancel{GL(V)}\ \operatorname{End}(V)$ kills I^{N+1}

Pf $\operatorname{im}(\alpha - 1)$ in V_N, $(\alpha-1)(\beta-1)$ in V_{N-1}, etc

---

~~$\{$f.d. reps of $\mathbb{Q}[\Gamma]/I^{N+1}$ of dim $\leq N+1\}$ $\leftrightarrow$ $\{$unip reps of Γ of dim $\leq N+1\}$~~

~~$\{$f.d. reps of $\mathbb{Q}[\Gamma]$ factoring through I^{N+1} for $N \gg 0\}$ $\to$ $\{$unip reps of $\Gamma\}$~~

~~$\left\{ \text{Comodules over } \left(\mathbb{Q}[\Gamma]/I^{N+1}\right)^\vee \right\} \longleftrightarrow \left\{ \text{f.d reps of alg grp } \Gamma^{un}_{N+1} \right\}$~~

<u>Define</u> $\Gamma^{un} = \mathrm{Spec} \varinjlim_{N} \left(\left(\mathbb{Q}[\Gamma]/I^{N+1} \right)^\vee \right) = \mathrm{Spec}\left(\mathcal{O}_{\Gamma^{un}}\right)$

$\Gamma^{un} = \varprojlim \Gamma^{un}_{N+1}$. Γ^{un} is a group scheme

$\{\text{alg reps of } \Gamma^{un}\} \longleftrightarrow \{\text{unipotent reps of } \Gamma\}$

$\boxed{\text{Group Theoretic Description}}$ ~~Γ f.g. (always~~

~~Thm~~ <u>Defn</u> Let $Z^0 = \Gamma$, $Z^{i+1} = [\Gamma, Z^{i}]$

<u>descending central series</u> (dcs)

$\Longrightarrow$ $Z^{i}/Z^{i+1} \subseteq \Gamma/Z^{i+1}$ is in the <u>center</u>

and Z^{i}/Z^{i+1} is f.g. abelian group

~~$\Rightarrow T_{i+1} \subseteq Z^i/Z^{i+1}$ torsion~~

Let $T_i \subseteq \Gamma/Z^i$ torsion subgroup

Let Z'^i be inverse image of T_i in Γ

$\Rightarrow$ $\Gamma \supseteq Z'^1 \supseteq Z'^2 \supseteq \cdots$

so (A) Γ/Z'^i is <u>nilpotent</u> (i.e. dcs terminates)

(B) Z'^i/Z'^{i+1} is free f.g. ab. grp

For each i, choose a basis, and lift to an elt of Γ.

For given i, let
$\gamma_1, \dots, \gamma_M$ be the elements of Γ obtained

$\underline{n} \in \mathbb{Z}^M \mapsto \gamma_1^{n_1} \cdots \gamma_M^{n_M}$ is a bijection

$\Rightarrow$ group law given by $\mathbb{Q}$-polynomials in n_i

$\Rightarrow$ get an alg group / $\mathbb{Q}$

as $i \to \infty$, get pro-alg grp

Thm (Quillen) This group is Γ^{un} (as defined before)
"Morally, $\Gamma \cong \Gamma^{un}(\mathbb{Z})$"

Cohomological Interp Let $Y_0 = (x_1 = a)$ $Y_N = (x_N = a)$
$Y_i = (x_i = x_{i+1})$

Thm (Beilinson) $\mathbb{Z}[\Gamma]/I^{N+1} \to H_N(X^N, \sqcup Y_i; \mathbb{Z})$

$\alpha \in \Gamma \longmapsto \alpha^N\big|_{\sigma_N}$

is an isomorphism

Cor $(\mathbb{Z}[\Gamma]/I^{N+1})^\vee \cong H^N(X^N, \sqcup Y_i; \mathbb{Z})$

Fact commutes with $\omega_1, \dots, \omega_N \to \omega_1 \wedge \dots \wedge \omega_N$ on X^N
$\hookrightarrow Ch^N(a, X)$

Pf Intermediary: $\mathbb{H}^N$(complex on X^N)

Note that one could construct Γ^{un} as follows:

- Take cat. of unipotent reps of Γ over $\mathbb{Q}$
- This is Tannakian (w/ obvious fiber functor)
- Γ^{un} is the Tannakian group of this category

Local Systems

There is a classical equivalence for smooth mflds

$$\left\{\begin{matrix}\text{reps of} \\ \pi_1(X)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{vector bundles w/} \\ \text{flat connection on } X\end{matrix}\right\}$$

If the rep of $\pi_1(X)$ is unipotent, the ~~cope of~~ vector bundle is trivial

We review ~~trivial = connec~~ connections on a trivial vector bundle

<u>Defn</u> Let E be a vector bundle on X. A connection on E is a map $\nabla: E \to \Omega^1 \otimes E$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla(s)$ on sections, f a section of $\mathcal{O}_X$.

If E is trivial w/ fixed trivialization, i.e.

$E = \mathbb{R}^n \times X$, there is the trivial
(or $\mathbb{C}^n \times X$)

connection given by taking the exterior deriv. of each coord.
Call it ~~dx~~ d

Then $\nabla - d$ is an $\mathcal{O}_X$-linear map $E \to \Omega^1 \otimes E$

i.e. $\nabla - d \in \Omega^1 \otimes \underline{\mathrm{End}}(E)(X)$

Thus it is given by a matrix of 1-forms.

We say ∇ is unipotent if this matrix is nilpotent

Let $\alpha : [0,1] \to X$ be a loop at a. We can pullback ∇ and E to $[0,1]$.

For a given $v \in \mathbb{R}^n \cong E_a$, there is a unique section s of $E|_{[0,1]}$ such that $\nabla s = 0$.
(this is linear ODE)

We define ~~s(t) to be~~ $T(\alpha)v$ to be $s(1)$.

$T(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map called parallel transport

If ∇ is flat (i.e. $\nabla^2 = 0$), this depends only on the homotopy class of α.

It is in this way that (E, ∇) gives a rep of $\pi_1(X, a)$ on E_a.

The following form of Baker – Campbell Hausdorff is key:

Let ω be the <u>matrix</u> of 1-forms representing ∇.

<u>Thm</u> $T(\alpha) = I + \int_\alpha \omega + \int_\alpha \omega\omega + \int_\alpha \omega\omega\omega + \cdots$

Here, $\int_\alpha \omega\omega$ is an iterated integral, and multiplication comes from matrix multiplication (then iterated integration).

In particular, if ω is ~~nilpo~~ nilpotent, the sum is <u>finite</u>

E.g. $\omega = \begin{pmatrix} 0 & \omega_1 & 0 & 0 & \cdots & & \\ & 0 & \omega_2 & 0 & \cdots & & \\ & & 0 & \omega_3 & 0 & & 0 \\ & & & 0 & \omega_4 & 0 & \\ & 0 & & & 0 & 0 & 0 \\ & & & & & 0 & \omega_r \\ & & & & & \cdots & 0 \end{pmatrix}$

then $T(\alpha) = \begin{pmatrix} 1 & \int\omega_1 & \int\omega_1\omega_2 & \int\omega_1\omega_2\omega_3 & \cdots & \int\omega_1\omega_2\cdots\omega_r \\ & 1 & \int\omega_2 & \int\omega_2\omega_3 & & \vdots \\ & & 1 & \int\omega_3 & & \\ 0 & & & \cdots & 1 & \int\omega_{r-1} \\ & & 0 & & & 1 \quad \int\omega_r \\ & & & & 0 & 1 \end{pmatrix}$ (not to scale)

<u>RmK</u> The identity $T(\alpha)T(\beta) = T(\alpha\beta)$ gives an easy proof of nilpotence

---

<u>RmK</u> Motivic Theory: One can replace vector bundles w/ flat connection w/ <u>holomorphic</u> or <u>algebraic</u> vector bundles w/ corresponding flat connection

~~The~~ We can then take algebraic vector bundles w/ flat connection / $\mathbb{Q}$.

These form a Tannakian category, w/ group a $\mathbb{Q}$-form of $\Gamma_{\mathbb{C}}^{un}$.

Its Hopf algebra comes from taking in $Ch(a, X)$ on <u>algebraic</u> diff forms / $\mathbb{Q}$.