Goal: Understand \( P(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1 s_1} \) (\( s_i \geq 2 \))

(NB Some conventions \( s_i \mapsto S_{k+1-i} \))

No alg geom or topology

Let \( \sigma_0 = \sigma_1^2, \ldots, \sigma_n > 0, \sigma_{n+1} = 0 \) \( \in R^N \)

Thm

\[
\int \frac{dt_{s_1}}{t_{s_1}} \frac{dt_{s_2}}{1-t_{s_2}} \cdots \frac{dt_{s_k}}{1-t_{s_k}} = \sum (S_1, \ldots, S_k)
\]

\( \boxdot \) Proof if \( k=1 \)

\( S = S_1 \)

\[
\int_0^{t_{s_1}} dt_{s_1} \int_0^{t_{s_1}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \frac{dt_{s_1}}{1-t_{s_1}} \frac{dt}{1-t}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} dt_{s_2} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} dt_{s_2} \cdots \int_0^{t_{s_1}} dt_{s_{n+1}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} dt_{s_{n+1}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} dt_{s_{n+1}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]

\[
= \sum_{n=0}^{\infty} \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt}{t_{s_{n+1}} - t} \int_0^{t_{s_1}} \frac{dt_{s_1}}{t_{s_1}} \frac{dt}{1-t} \int_0^{t_{s_1}} \frac{dt_{s_2}}{t_{s_2}} \cdots \int_0^{t_{s_1}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}} \frac{dt_{s_{n+1}}}{t_{s_{n+1}}}
\]
Good: Integral of alg diff form along something

Problem 1 On isn’t a cycle or κⁿ 2 Not an integral on X

Will see 1 is a cycle modulo \( x_i = 1 \) if \( x_N = 0 \) \( \bigcup_{i=1}^{N-1} (x_i = x_{i+1}) \)

2 is related to \( \pi_1(X) \)

Iterated Integrals

Define Let \( \omega_1, \ldots, \omega_N \) be a sequence of 1-forms on \( X \)

Let \( \alpha : [0,1] \to X \) be a (smooth) path

(smooth mfld) (later: complex var, \( \bar{\mathbb{C}} \))

Define \( \int_{\alpha} \omega_1, \ldots, \omega_N = \int_X \bigwedge_{i=1}^{N-1} \omega_i \wedge \omega_{i+1} \)

of length \( N \)

Fact If \( \dim X = 1 \) and \( \omega_i \) holomorphic [e.g. \( \dim_X = 1 \) and \( \omega_i \) holomorphic]

and \( \omega_i \) closed, this does not depend on path homotopy class of \( \alpha \)

\[ \Rightarrow \omega_1, \ldots, \omega_N \text{ define a map } \pi_1(X,a) \to \mathbb{C} \text{ (a} \in \text{ X}) \]

\[ \iff \text{ linear map } \mathbb{Z} [\pi_1(X,a)] \to \mathbb{C} \]

If \( N = 1 \), \( \int_{\alpha} \omega = \int_{\alpha} \omega \), so \( \int_{\alpha} \omega + \int_{\beta} \omega = \int_{\alpha} \omega \)

\[ \Rightarrow \text{ factors through } \mathbb{Z} [\pi_1^h(X,a) \cong H_1(X;\mathbb{Z})] \]
Let \( \text{Ch}(a, \mathbb{M}) \subseteq \text{Hom}_{\text{Ab}}(\mathbb{Z}[[\pi_1(X,a)]]), C) \)

given by \( C \)-span of \( \# w_1 \cdots w_n \forall\{w_i\} \) holom \( \sum w_i w_{i+1} = 0 \) \( \dim X = 1 \)

\( \text{Ch}^N(a, \mathbb{M}) \subseteq \text{Ch}(a, \mathbb{M}) \) for integrals of length \( \leq N \)

Thm 1 Ch(a, \mathbb{M}) is a (graded) subalgebra of \( \text{Hom}(\mathbb{Z}[[\pi_1(X,a)]])) \)

2 \( \text{Ch}(a, \mathbb{M}) \) is a Hopf alg wrt mult in \( \pi_1(X,a) \)

3 Key If \( \alpha_1, \ldots, \alpha_{NH} \in \pi_1(X,a) \)
\( \text{Ch}^N(a, \mathbb{M}) \) kills \( \sum_{i=1}^{NH} (\alpha_i - 1) \in \mathbb{Z}[[\pi_1(X,a)]] \)

Pf sketch
First two reduce to a formula

2 \( \sum w_1 \cdots w_N = \sum_{j=0}^{N} \sum w_1 \cdots w_j \sum w_{j+1} \cdots w_N \) (sign?)

1 \( \sum_\alpha w_1 \cdots w_N \sum w_{NH} \cdots w_{NH} = \sum_{\sigma \in \text{Sh}(N,M)} w_{\sigma(1)} \cdots w_{\sigma(\text{Sh}(N,M))} \)

3 is trickier. Will discuss proof later. For \( N = 1 \)

\( \sum w - w_1 - w_2 + 0 = 0 \)

\( (\alpha - \Omega(p - 1)) \alpha \rho = \rho \)
\[ u : \mathbb{Z} \left[ \pi_1(X,a) \right] \to \mathbb{C} \quad \alpha \mapsto 1 \quad I = \ker u \]

"augmentation ideal"

\[ I \text{ generated by } \{ \alpha - 1 \}_{\alpha \in \pi_1} \]

\[ \text{Thm (Chen)} \quad \text{Ch}^N(a,X) = \left( \mathbb{Z} \left[ \pi_1(X,a) \right] / I^{N+1} \right)^{\mathbb{C}} \]

Furthermore, one can generate Ch(a,X) using only a basis for \( H_{\text{fr}}(X) \)

\( \S 3 \) Unipotent Completion of \( \Gamma \) (a group, e.g. \( \pi_1(X,a) \))

\[ \text{I is a Hopf ideal} \]

\[ \begin{bmatrix}
\{ c(\alpha - 1) = \alpha \otimes \alpha - 1 \}
= (\alpha - 1) \otimes \alpha + 1 \otimes (\alpha - 1)
\end{bmatrix}
\]

\[ \alpha \in \Gamma \]

\[ \Rightarrow \mathbb{Q}[\Gamma] / I^{N+1} \text{ is a quotient Hopf alg} \]

(co-commutative, not comm.)

\[ \Rightarrow \left( \mathbb{Q}[\Gamma] / I^{N+1} \right)^{\text{op}} \text{ is a commutative Hopf algebra} \]

\[ \text{Hom}_e (\mathbb{Q}[\Gamma] / I^{N+1}, \mathbb{Q}) \]

\[ \text{rank} \text{ Ch}^N(a,X) = \left( \mathbb{Q}[\Gamma] / I^{N+1} \right)^{\text{op}} \otimes \mathbb{C} \]
Spec \left( \mathbb{Q} \left[ \Gamma_{n+1} \right] \right) is an alg. group over \mathbb{Q}, call \Gamma_{n+1}^u.

System \quad \Gamma_{n+1}^u \rightarrow \Gamma_N^u

How to think of this

Defn. A rep \rho \in \text{Rep}(\Gamma) of a \mathbb{K}-vspace V is \textit{unipotent} if \exists 0 = V_0 \subseteq \cdots \subseteq V_{n+1} \subseteq V such that \frac{V_{i+1}}{V_i} is \\& a trivial rep of \Gamma

\iff \text{im}(\Gamma) = \left( \begin{array}{c} 0 \\ \ast \\ \vdots \end{array} \right)

Prop. If \text{dim} V \leq n+1, \rho \text{ unipotent} \iff \rho : \mathbb{Q}[\Delta] \rightarrow \text{End}(V) \text{ kills } I_{\text{N+1}}

Pf. \text{im}(\rho - 1) \text{ in } V_N, (\rho - 1)(\rho - 1) \text{ in } V_{N-1}, etc.

\{ \text{unip reps of } \mathbb{Q} \left[ \Gamma_{\text{dim} \leq n+1} \right] \} \rightarrow \{ \text{unip reps of } \Gamma \text{ of dim } \leq n+1 \}

\{ \text{null reps of } \mathbb{Q} \left[ \Gamma_{\text{dim} > n+1} \right] \} \rightarrow \{ \text{null reps of } \Gamma \text{ for } n+2 \rightarrow \infty \}
Define \[ \Gamma^u = \text{Spec} \lim_{\to} \left( \frac{\mathbb{Q}[[T]]}{I^{N+1}} \right) \] \[ = \text{Spec} ( \mathcal{O}_{\Gamma^u} ) \]

\[ \Gamma^u = \lim_{\to} \Gamma_{N+1}^u \]

\[ \Gamma^u \] is a group scheme.

Group Theoretic Description:

Defn: Let \( Z^0 = \Gamma \), \( Z^{it} = [\Gamma, Z^{it}] \)

descending central series (dcs)

\[ Z^{it} / Z^{it+1} \subseteq \Gamma / Z^{it+1} \text{ is in the center} \]

and \( Z^{it} / Z^{it+1} \) is f.g. abelian group.

Let \( T_i \subseteq \Gamma / Z^{it} \) torsion subgroup

Let \( Z^{it} \) be inverse image of \( T_i \) in \( \Gamma \)

\[ \Gamma = Z^{it} \supseteq Z^{i+1} \supseteq ... \]

so \( \Gamma / Z^{it} \) is nilpotent (i.e. dcs terminates)

\( Z^{it} / Z^{it+1} \) is free f.g. ab. grp

For each \( i \), choose a basis, and lift to an elt of \( \Gamma \).
For given $i$, let $\gamma_1, \ldots, \gamma_m$ be the elements of $\Gamma$ obtained

$$\Delta \in \mathbb{Z}^m \mapsto \gamma_1^\Delta, \ldots, \gamma_m^\Delta$$

is a bijection

$\Rightarrow$ group law given by polynomials in $\gamma_i$

$\Rightarrow$ get an alg group $\mathcal{O}$

as $i \to \infty$, get pro-alg grp

Thm (Quillen) This group is $\Gamma^\text{un}$ (as defined before)

Morally, $\Gamma \approx \Gamma^\text{un}(\mathbb{Z})$

Cohomological Interpretation

Let $Y_0 = (x_i = a), Y_n = (x_i = a)$

$Y_i = (x_i = x_{i+1})$

$\alpha \in \Gamma$ \mapsto $\alpha^N / \sigma_N$

is an isomorphism

Cor $(\mathbb{Z}[\Gamma] / \mathcal{I}^{NH})^\vee \cong H^N(X^N \cup Y_i \cdot \mathbb{Z})$

Fact: commutes with $\gamma_1, \ldots, \gamma_m \cdot \gamma_n \mapsto \gamma_1, \ldots, \gamma_m, \gamma_n$ on $X^N$
Intermediary: $H^N_\bullet$ (complex on $X^N$)

Note that one could construct $\Gamma^{un}$ as follows:
- Take cat. of unipotent reps of $\Gamma$ over $\mathbb{Q}$
- This is Tannakian (w/ obvious fiber functor)
- $\Gamma^{un}$ is the Tannakian group of this category

Local Systems

There is a classical equivalence for smooth mf/vs

\[
\begin{array}{ccc}
\{\text{reps of } \pi_1(X)\} & \rightarrow & \{\text{vector bundles w/ flat connection on } X\} \\
\end{array}
\]

If the rep of $\pi_1(X)$ is unipotent, the coproduct vector bundle is trivial.

We review connections on a trivial vector bundle.

Defn. Let $E$ be a vector bundle on $X$. A connection on $E$ is a map $\nabla: E \rightarrow \mathcal{D}^1 \otimes E$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f \nabla(s)$ on sections, $f$ a section of $\mathcal{O}_X$. 
If $E$ is trivial w/fixed trivialization, i.e.

$$E = \mathbb{R}^n \times X,$$

there is the trivial

(or $\mathbb{C}^n \times \mathbb{O}X$)

Connection given by taking the exterior deriv. of each coord.

Call it $\Delta$

Then $\Delta - d$ is an $\mathfrak{g}_X$-linear map $\mathfrak{g}_X \otimes E \to \mathfrak{g}_X \otimes E$

i.e. $\Delta - d \in \mathfrak{g}_X \otimes \text{End}(E)(X)$

Thus it is given by a matrix of 1-forms.

We say $\Delta$ is unipotent if this matrix is nilpotent.

Let $\alpha : [0,1] \to X$ be a loop at $a$. We can pullback $\Delta$ and $E$ to $[0,1]$.

For a given $v \in \mathbb{R}^n \cong E_{a}$, there is a unique section $s$ of $E_{[0,1]}$ such that $\Delta s = 0$.

(this is linear ODE)

We define $s(1) = T(\alpha)v$ to be $s(1)$.

$T(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map called parallel transport.

If $\Delta$ is flat (i.e. $\Delta^2 = 0$), this depends only on the homotopy class of $\alpha$.

It is in this way that $(E,\Delta)$ gives a rep of $\pi_1(X,a)$ on $E_a$. 
The following form of Baker–Campbell–Hausdorff is key:

Let \( \omega \) be the matrix of 1-forms representing \( \nabla \).

Then \( T(\omega) = I + \omega + \frac{\omega \omega}{2} + \frac{\omega \omega \omega}{3} + \cdots \).

Here, \( \omega \omega \) is an iterated integral, and multiplication comes from matrix multiplication (then iterated integration).

In particular, if \( \omega \) is nilpotent, the sum is finite.

E.g., \( \omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) then \( T(\omega) = \begin{pmatrix} 1 & \omega & \frac{\omega \omega}{2} & \frac{\omega \omega \omega}{3} \\ 0 & 1 & \omega \omega & \frac{\omega \omega \omega \omega}{4} \\ 0 & 0 & 1 & \omega \omega \omega \omega \\ 0 & 0 & 0 & 1 \end{pmatrix} \). (not to scale)

Remark: The identity \( T(\omega) T(\zeta) = T(\omega \zeta) \) gives an easy proof of nilpotence.

Remark: Motivic Theory: One can replace vector bundles with flat connection with holomorphic or algebraic vector bundles with corresponding flat connection. We can then take algebraic vector bundles with flat connection over \( \mathbb{Q} \).

These form a Tannakian category, with group a \( \mathbb{Q} \)-form of \( \text{Pic} \).

Its Hopf algebra comes from taking \( \otimes \) in \( \text{Ch}(\mathbb{Q}, X) \) on algebraic differential forms over \( \mathbb{Q} \).