American Mathematical Monthly Problem 11403 by Yaming Yu, Irvine, CA.

For every integer \( n \geq 0 \), define a polynomial \( f_n \in \mathbb{Q}[x] \) by
\[
f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j).
\]

Find \( \text{deg} f_n \) for every \( n > 1 \).

Solution by Darij Grinberg.

We claim that \( \text{deg} f_n = \left\lfloor \frac{n}{2} \right\rfloor \) for every integer \( n > 1 \). The key to the proof is the following recurrence relation for our polynomials:
\[
f_n(x) = (n-1)(f_{n-1}(x) + xf_{n-2}(x)) \tag{11403.1}
\]
for every integer \( n > 3 \).

Proof of (11403.1). We have
\[
f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n} \left( \binom{n-1}{i-1} + \binom{n-1}{i} \right) (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)
\]
\[
= \sum_{i=0}^{n} \binom{n-1}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)
\]

(here we substituted \( i+1 \) for \( i \) in the first sum)
\[
= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)
\]

(here we removed a zero addend from each of the two sums)
\[
= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} (x+i) \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} (-x) \prod_{j=0}^{i-1} (x+j)
\]
\[
= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \underbrace{(x+i) + (-x)}_{=i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} i \binom{n-1}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)
\]
\[
= \sum_{i=0}^{n-1} (n-1) \cdot \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \quad \text{(since } i \binom{n-1}{i} = (n-1) \cdot \binom{n-2}{i-1} \text{)}
\]
\[
= (n-1) \cdot \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j). \tag{11403.2}
\]
But adding up the equalities
\[
f_{n-1}(x) = \sum_{i=0}^{n-1} \left( \frac{n-1}{i} \right) (-x)^{(n-1)-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} \left( \binom{n-2}{i} + \binom{n-2}{i-1} \right) (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)
\]

and
\[
x f_{n-2}(x) = x \sum_{i=0}^{n-2} \left( \frac{n-2}{i} \right) (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j) = - \sum_{i=0}^{n-2} \left( \frac{n-2}{i} \right) (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j)
\]

yields
\[
f_{n-1}(x) + x f_{n-2}(x) = \sum_{i=0}^{n-1} \left( \frac{n-2}{i-1} \right) (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)
\]

(since two of the three sums cancel out), and thus (11403.2) becomes \( f_n(x) = (n-1) \cdot (f_{n-1}(x) + x f_{n-2}(x)) \). This proves (11403.1).

Next, we introduce a notation: For any polynomial \( p \in \mathbb{Q}[x] \), and for any integer \( k \geq 0 \), we denote by \( \text{coeff} (p, k) \) the coefficient of \( p \) before \( x^k \). Then, every polynomial \( p \in \mathbb{Q}[x] \) satisfies \( p(x) = \sum_{k \geq 0} \text{coeff} (p, k) \cdot x^k \).

The recurrence (11403.2) immediately yields the relations
\[
\text{deg } f_n \leq \max \{ \text{deg } f_{n-1}, 1 + \text{deg } f_{n-2} \}
\]

(11403.3)

and
\[
\text{coeff} (f_n, s) = (n-1) \left( \text{coeff} (f_{n-1}, s) + \text{coeff} (f_{n-2}, s-1) \right)
\]

(11404.4)

for every positive integers \( n > 3 \) and \( s \). Now, a straightforward induction (using \( f_2(x) = x \) and \( f_3(x) = 2x \) as the induction base, and (11403.3) and (11404.4) for the induction step) shows the three relations
\[
\text{deg } f_{2u} = \text{deg } f_{2u+1} = u,
\]

(11404.5)

\[
\text{coeff} (f_{2u}, u) > 0,
\]

(11404.6)

\[
\text{coeff} (f_{2u+1}, u) > 0
\]

(11404.7)

for every integer \( u \geq 1 \) (we won’t use the relations (11404.6) and (11404.7) anymore, but we need them to survive in the induction step, since they ensure that the leading terms of the polynomials \( f_{n-1}(x) \) and \( x f_{n-2}(x) \) don’t cancel out each other in (11403.1)). In particular, (11404.5) yields our assertion that \( \text{deg } f_n = \left\lfloor \frac{n}{2} \right\rfloor \) for every integer \( n > 1 \).