Research statement
Darij Grinberg

My research belongs to the field of algebraic combinatorics, centered on (but not limited to) symmetric functions and related concepts, such as combinatorial Hopf algebras, Young tableaux and trees. These objects live at the borderlands of algebra and combinatorics, often allowing for viewpoints from both sides and transfer of knowledge from one to the other.

Among my contributions to this discipline are a new generalization of the dual stable Grothendieck polynomials (which themselves generalize the Schur functions), an antipode formula for certain quasisymmetric functions, and a proof of a conjecture on “birational rowmotion”. Details on this and other work can be found below.

Background. The history of symmetric functions goes back at least as far as the 17th Century, when Newton and Girard explored the relations between elementary symmetric polynomials and power sums. The next major steps were Cauchy’s 1815 introduction of what later came to be known as Schur functions, and Jacobi’s and Trudi’s determinantal formulas for them (1841 and 1864). Symmetric functions found various uses in the algebra of the 19th Century, particularly in Galois and invariant theory, as well as in Schubert calculus. However, their combinatorial meaning was not discovered until the 1930s, when Schur and others connected them to Young tableaux and the representation theory of symmetric and general linear groups. This connection opened the floodgates, and the research that followed since the mid-20th Century could easily fill a dozen of tomes. Symmetric functions were found to form a Hopf algebra, which tended to appear in various guises in seemingly unrelated fields such as algebraic topology (as cohomology of some classifying spaces) and number theory (as coordinate ring of the Witt vectors). The multiplication of Schur functions turned out to be governed by a combinatorial rule (the Littlewood-Richardson rule), formulated in 1934 and first proven in 1974, with applications in theoretical physics. The combinatorics of Young tableaux became a subject of its own, bordering on theoretical computer science (Knuth devoted a section in “The Art of Computer Programming” to it). Symmetric functions have been applied in fields as diverse as random matrix theory, K-theory (particularly of Grassmannians), group theory and quantum groups. The description of the representations of symmetric and general linear groups using Schur functions has become a mold in which many other representation theories have been shaped. By now, even as various questions on Schur functions remain unanswered, the focus has broadened to include generalizations and analogues thereof, such as Hall-Littlewood and Macdonald polynomials, factorial Schur functions, Schubert and Grothendieck polynomials, “immaculate” functions, $k$-Schur functions, P-partition enumerators and others.
It is such generalizations that I have been dealing with in much of my research. From a bird’s eye view, their theories follow a certain pattern: a family of power series is defined, and analogues of classical properties of the Schur functions (such as symmetry, determinantal formulas à la Jacobi-Trudi, Littlewood-Richardson rule(s), and antipode formulas) are proven for this family. However, this is rarely ever straightforward, as each generalization comes with its additional complications; consequently, these programs are at rather different stages of completion, and some of them (e.g., a full Littlewood-Richardson rule for Schubert polynomials) appear out of reach today. Additionally, each generalization has its own motivation, sometimes stemming from a totally different field.

Combinatorial Hopf algebras are one (although not the only) place where these generalizations live. They are interesting both in their intrinsic properties (e.g., some of them are free as algebras for non-obvious reasons) and for the special elements they contain (such as the above-mentioned generalizations of symmetric functions). They have found applications to algebraic groups, Lie groups, probability and renormalization theory.

Overview of research. My results so far, as well as my research plans for the future, live in and around the algebro-combinatorial landscape surveyed above. In [Grinbe14], I have proven a conjecture of Mike Zabrocki on a quasisymmetric analogue of Bernstein’s creation-operator approach to the Schur functions. In [Grinbe15a], I have reproven and generalized a formula of Malvenuto and Reutenauer for the antipode of a P-partition enumerator (which itself extends a classical formula for the antipode of a Schur function). In [GaGrLi15], the dual stable Grothendieck polynomials (a recent generalization of Schur functions motivated by K-theory) are refined to include new parameters, and the symmetry of these new power series is proven combinatorially. In [BorGri13] (joint with James Borger), positivity properties of symmetric functions are explored and applied to study Witt vectors over semirings. The lecture notes [GriRei15] (joint with Victor Reiner), to which I have contributed exercises and a section on the freeness of the quasisymmetric functions, are an introduction to both symmetric functions and combinatorial Hopf algebras. Two works of mine which are not directly concerned with symmetric functions are [GriRob14] (joint with Tom Roby), which proves a conjecture by Einstein and Propp on “birational rowmotion” (and other results), and [Grinbe15b], which computes the kernel of a certain operator on the tensor algebra. Nevertheless they are both related to the subject, as [GriRob14] generalizes a property of Young tableaux, and [Grinbe15b] is connected both to Hopf algebras and to the representation theory of the symmetric group.

Current and finished research

Refined dual stable Grothendieck polynomials. Dual stable Grothendieck polynomials first appear in the work of Lam and Pylyavskyy [LamPy10], after having been anticipated by Lenart and Buch. In joint work [GaGrLi15] with
Pavel Galashin and Gaku Liu, I have extended their definition and some of their properties to a more general setup, involving an infinite family of new parameters.

A weak composition means a sequence \((\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty\) (where \(\mathbb{N} = \{0, 1, 2, \ldots\}\)) such that all but finitely many \(i\) satisfy \(\alpha_i = 0\).

Consider a skew partition \(\lambda/\mu\). A reverse plane partition (short: rpp) of shape \(\lambda/\mu\) means a filling of the skew Young diagram of \(\lambda/\mu\) with positive integers which increase weakly along rows and weakly along columns. (Requiring them to increase strictly along columns would instead yield the definition of a semistandard tableau.) For every rpp \(T\), we let \(\text{ircont}_T\) be the weak composition whose \(i\)-th entry is the number of columns of \(T\) which contain the entry \(i\). Moreover, for every rpp \(T\), we let \(\text{ceq}_T\) be the weak composition whose \(i\)-th entry is the number of cells \(c\) in the \(i\)-th row of \(T\) such that the entry of \(T\) in cell \(c\) equals the entry of \(T\) in the cell directly below \(c\) (and, in particular, the latter entry exists). Notice that \(\text{ceq}_T = (0, 0, 0, \ldots)\) if and only if \(T\) is a semistandard tableau.

The refined dual stable Grothendieck polynomial \(\tilde{g}_{\lambda/\mu}\) corresponding to the skew partition \(\lambda/\mu\) is defined to be
\[
\sum_{T \text{ is an rpp of shape } \lambda/\mu} t^{\text{ceq}_T} x^{\text{ircont}_T} \in (\mathbb{Z}[t_1, t_2, t_3, \ldots])[[x_1, x_2, x_3, \ldots]].
\]

Here we are using the notation \(x^\alpha\) for the monomial \(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots\) whenever \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)\) is a weak composition, and similarly the notation \(t^\alpha\) stands for \(t_1^{\alpha_1}t_2^{\alpha_2}t_3^{\alpha_3}\cdots\).

The dual stable Grothendieck polynomials are obtained from the \(\tilde{g}_{\lambda/\mu}\) by setting all \(t_i\) equal to 1, whereas the skew Schur functions are obtained by setting all \(t_i\) equal to 0. Other specializations of \(\tilde{g}_{\lambda/\mu}\) have not been explored so far, but the parameter space is obviously vast.

Various questions suggest themselves now:

- Are the \(\tilde{g}_{\lambda/\mu}\) symmetric (in the \(x_1, x_2, x_3, \ldots\))? The answer is positive, and this is the main result of [GaGrLi15]. Two proofs are given in [GaGrLi15], and yet another can be obtained with the methods of [LamPyl07].

- Do the \(\tilde{g}_{\lambda/\mu}\) satisfy a determinantal formula generalizing (one of) the Jacobi-Trudi identities? The answer appears to be positive, but a proof has not been found so far. Damir Yeliussizov, in very recent work, has found a proof in the \(t_i = 1\) specialization.

- Do the \(\tilde{g}_{\lambda/\mu}\) satisfy a Littlewood-Richardson rule? There are several ways to interpret the question, some of which (e.g., expanding the product \(\tilde{g}_{\lambda/\emptyset}\tilde{g}_{\mu/\emptyset}\)) as a linear combination of the \(\tilde{g}_{\nu/\emptyset}\) appear out of reach. I have proven one Littlewood-Richardson rule (expanding \(s_\nu \tilde{g}_{\lambda/\mu}\) in terms of the \(s_\nu\)) using the
results of [GaGrLi15] and an analogue of Stembridge’s proof of the classical Littlewood-Richardson rule\footnote{See http://web.mit.edu/˜darij/www/algebra/chicago2015.pdf for a statement of this (proof not yet written up) and also of the conjectural Jacobi-Trudi identity.}.

• It appears that a similar refinement can be done to the (non-dual) stable Grothendieck polynomials $G_{\lambda/\mu}$.

[GaGrLi15] is supposed to be the first paper of a series; further research is in progress.

**Antipodes and P-partitions.** We shall use the notation $S$ for the antipode of a Hopf algebra. It is well-known that any skew Schur function $s_{\lambda/\mu}$ (regarded as a symmetric function) satisfies $S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda'/\mu'}$, where $\nu'$ denotes the transpose of a partition $\nu$. Ira Gessel generalized this fact to $P$-partition enumerators. Here, we consider a finite set $E$ with a partial order relation $<_1$ and a total order relation $<_2$. We define an $(E,<_1,<_2)$-partition to be a map $f : E \to \{1,2,3,\ldots\}$ such that

• $f(u) \leq f(v)$ for any $u \in E$ and $v \in E$ with $u<_1 v$;
• $f(u) < f(v)$ for any $u \in E$ and $v \in E$ with $u<_1 v$ and $v<_2 u$.

We define the P-partition enumerator $F_{(E,<_1,<_2)}$ as the formal power series

$$\sum_{f \text{ is an } (E,<_1,<_2)\text{-partition}} x_f \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]], \quad \text{where } x_f = \prod_{e \in E} x_{f(e)}.$$ 

The power series $F_{(E,<_1,<_2)}$ is a quasisymmetric function (although its most important particular cases are the skew Schur functions, which are symmetric). Gessel’s formula [GriRei15, Corollary 5.27] states that the antipode $S$ of the Hopf algebra of quasisymmetric functions satisfies $S(F_{(E,<_1,<_2)}) = (-1)^{|E|} F_{(E,>_1,<_2)}$, where $>_1$ is the opposite relation of $<_1$.

In [Grinbe15a], I prove a generalization of this formula in which the elements $e \in E$ are equipped with (positive integer) weights $w(e)$ (and the monomial $x_f = \prod_{e \in E} x_{f(e)}$ is replaced by $\prod_{e \in E} x_{f(e)}^{w(e)}$), in which the order $<_2$ is no longer required to be total (although not completely arbitrary either), and in which a finite group $G$ is allowed to act on $E$ in a way that preserves $<_1$ and $<_2$. The weights and the partial order $<_2$ are not a new idea (in this generality, the result can be viewed as a restatement of Malvenuto and Reutenauer’s [MalRen98, Theorem 3.1]), but the group action appears to be new; it combines Gessel’s antipode formula with the ideas of Pólya enumeration.

**Dual immaculate creation operators.** In [BBSSZ13a], Berg, Bergeron, Saliola, Serrano and Zabrocki introduced a lift of the Schur functions to the
ring of noncommutative symmetric functions, and also a “dual” version of this lift in the ring of quasisymmetric functions. The dual version is probably the simpler one to define: Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \) be a composition (i.e., a tuple of positive integers), and let \( Y(\alpha) \) be the “Young diagram” of \( \alpha \) (this is defined as the set \( \{(i, j) \in \{1, 2, 3, \ldots\}^2 \mid j \leq \alpha_i\} \), even if \( \alpha \) is not a partition). An immaculate tableau of shape \( \alpha \) is a filling of this Young diagram with positive integers such that each row is weakly increasing, and the first column is strictly increasing. (This can be viewed as an \((E, <_1, <_2)\)-partition, where \((E, <_1)\) is a certain binary tree.) The dual immaculate function \( S^*_\alpha \) corresponding to \( \alpha \) is defined to be the sum of \( x_T \) over all immaculate tableaux \( T \) of shape \(\alpha\), where \( x_T = \prod_{e \text{ is a cell of } T} x_{T(e)} \). (This is a particular case of the P-partition enumerator \( F_{(E, <_1, <_2)} \) defined above.)

Mike Zabrocki has conjectured that an alternative definition of \( S^*_\alpha \), using a certain notion of quasisymmetric “creation operators” (similar to the Bernstein creation operators that generate the Schur functions, but more intricate), results in the same power series. I proved this in [Grinbe14], by introducing a dendriform algebra structure on the ring of quasisymmetric functions.

Birational rowmotion. Given a finite poset \( P \), we can form another poset \( \hat{P} \) by adjoining a global minimum (called 0) and a global maximum (called 1) to \( P \). Given a field (or semifield) \( \mathbb{K} \), we consider the set \( \mathbb{K}^{\hat{P}} \) of all labelings of the elements of \( \hat{P} \) by elements of \( \mathbb{K} \). On this set, David Einstein and James Propp have defined a birational equivalence, which they call birational rowmotion, and which generalizes the notion of rowmotion on the order ideals of \( P \). Einstein and Propp have experimentally observed that, for various special classes of posets \( P \), this birational equivalence has finite order (i.e., a certain power of it is the identity). In [GriRob14], Tom Roby and I prove these observations and some others. The most crucial case is that when \( P \) is a “rectangle” (i.e., a product of two chains with \( p \) and \( q \) elements, respectively); in this case, the order of birational rowmotion is \( p + q \), and this generalizes Schützenberger’s classical result that the promotion operator \( \partial \) on the semistandard Young tableaux of a given rectangular shape with entries in \( \{1, 2, \ldots, n\} \) satisfies \( \partial^n = \text{id} \). (The generalization is not trivial; translating from tableaux to labellings of a rectangle poset \( P \) involves an intermediate step through Gelfand-Tsetlin triangles, and the sidelengths of the rectangular partition are not those of \( P \).) Another class of finite posets \( P \) for which birational rowmotion has finite order are graded forests (i.e., forests where all leaves are at the same depth).

The proof of this result on rectangles is inspired by Volkov’s proof of the type-AA Zamolodchikov conjecture [Volkov06]; other cases are handled either by an inductive argument or by a “folding” reduction to the rectangular case. One case — that of “trapezoidal” posets, which can be seen as a type-B analogue of rectangles — is still unresolved, and a conjecture by Nathan Williams connects it to the rectangular case in a remarkable way [GriRob14, §19], which (if cor-
rect) generalizes a conjecture by Elizalde on noncrossing families of Dyck paths [Elizal14].

Further developments have happened in the last year and still remain to be written up. For one, Max Glick has found a more direct relation between birational rowmotion in the case of a rectangle and the type-AA Zamolodchikov Y-system, whereas my advisor, Alexander Postnikov, has suggested a connection to the octahedral recurrence which still remains to be fully understood. Richard Stanley suggested a generalization of the case of graded forests. James Propp has conjectured some “homomesies” (algebraic identities holding for each orbit under birational rowmotion), some of which I have proven. The finite order of birational rowmotion proven for rectangles seems to hold even if $\mathbb{K}$ is replaced by a (noncommutative) semifield, up to a certain conjugation\(^2\). There should be material for at least three smaller papers here.

**Hopf Algebras in Combinatorics.** The lecture notes [GriRei15] by Victor Reiner and myself are, predominantly, an expository work, providing probably the first introduction into the subject with a strong emphasis on the combinatorics. Nevertheless, a few of my exercises contain minor innovations; these are Exercise 2.80(f) (the “third comultiplication” $\Delta_r$ on the symmetric functions), Exercise 2.87(c), Exercise 6.32, and Exercise 6.33. The first two are concerned with symmetric functions, the latter two with certain extensions of the concept of Lyndon words. The proof of [GriRei15, Theorem 6.44] also appears to be new.

**Sage.** I have contributed some functionality to the open-source Sage computer algebra system (in particular, some of its support for symmetric functions), and reviewed some others’ code.

**Research plans and work in progress**

**The Reiner-Saliola-Welker conjecture.** Let $n \in \mathbb{N}$, and consider the group algebra $\mathbb{C}\mathfrak{S}_n$ of the $n$-th symmetric group $\mathfrak{S}_n$. For any $k \in \{1, 2, \ldots, n\}$, we define an element $\text{RSW}_k$ of $\mathbb{C}\mathfrak{S}_n$ as $\sum_{w \in \mathfrak{S}_n} (\text{noninv}_k w) \cdot w$, where $\text{noninv}_k w$ is the number of all $k$-element subsets $I$ of $\{1, 2, \ldots, n\}$ such that $w | I$ is strictly increasing.

A surprisingly difficult result by Reiner, Saliola and Welker ([ReSaWe11, Theorem 1.1]) states that the elements $\text{RSW}_1, \text{RSW}_2, \ldots, \text{RSW}_n$ commute pairwise. They further conjecture that each of these elements (viewed as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}\mathfrak{S}_n$, given by left multiplication) has integer spectrum (i.e., all its eigenvalues are integers). This suggests the existence of a combinatorially meaningful joint eigenbasis for these operators (similar to, e.g., the seminormal basis for $\mathbb{C}\mathfrak{S}_n$). Indeed, an eigenbasis for $\text{RSW}_1$ has been found recently by Dieker

\(^2\)See the end of my 2014 talk in Vienna (http://web.mit.edu/~darij/www/algebra/vienna2014.pdf) for an example of this noncommutative phenomenon.
and Saliola [DieSal15], which led to a proof of the fact that RSW₁ has integer eigenvalues. The proof is a tour-de-force of tensor algebra, representation theory of symmetric groups and Young tableau theory, and naturally suggests further questions (for instance, it appears to contain homological arguments in disguise).

The element RSW₁ of \( \mathbb{C}S_n \) is known as the “random-to-random operator” on \( \mathfrak{S}_n \), due to the following probabilistic interpretation: Imagine a shelf with \( n \) books labelled by \( 1, 2, \ldots, n \). In one step, we take out a randomly chosen book from the shelf, and put it back at a randomly chosen position\(^3\). The transition matrix of this Markov chain is the representing matrix of RSW₁. Thus, the Dieker-Saliola result claims that this Markov chain has integer eigenvalues; from this viewpoint, it appears surprising that such a simple-looking result has not been proven until 2015, and not without such difficulties!

The Markov chain just described is reminiscent of a simpler and better-known Markov chain: the Tsetlin library. Here, one puts the book back at the beginning of the shelf rather than at a random point. This chain is, indeed, closely related, and has a number of similar properties. It, too, has integer eigenvalues, and also corresponds to the first element of a sequence R2T₁, R2T₂, \ldots, R2Tₙ of pairwise commuting elements of \( \mathbb{C}S_n \)\(^4\). These elements R2Tₖ not only commute, but also (unlike the RSWₖ) span a subalgebra of \( \mathbb{C}S_n \), and their products can be explicitly expanded; this is a particular case of the famous “Solomon’s Mackey formula” for the descent algebra of \( \mathfrak{S}_n \).

The symmetric group algebra \( \mathbb{C}S_n \) is a Hopf algebra (as any group algebra is) and thus has an antipode \( S \). It is not hard to see that RSWₖ = \( \frac{1}{(n-k)!} \) R2Tₖ S (R2Tₖ) for every \( k \).

In [Grinbe15b], I describe the kernel of the action of the random-to-top operator R2T₁ on the tensor algebra (or, more precisely, the kernels of two of its actions – one “unsigned” and one “signed”) over fields of arbitrary characteristic (and, in the “signed” case, even over arbitrary commutative rings). While the methods used do not directly apply to diagonalizing R2Tₖ and RSWₖ (which seems out of reach in positive characteristic), they might provide some valuable insights. I hope to explore the algebra of the RSWₖ and R2Tₖ more thoroughly.

**Carlitz-Witt vectors and function-field symmetric functions.** The semi-mathematical concept of the “field with one element” (\( F_1 \)) is a code for the idea that the ring \( \mathbb{Z} \) has deep similarities with the polynomial rings \( \mathbb{F}_q [T] \) over finite fields; that the combinatorics of sets has analogies with the linear algebra of \( \mathbb{F}_q \)-vector spaces; that the symmetric group is, in some sense, the “\( q = 1 \)” version of the general linear group \( \text{GL}_n (\mathbb{F}_q) \). No fully explicatory mathematical

\(^3\)I.e., we place it in one of the \( n \) gaps (either between two books or at the beginning or at the end of the shelf) with equal probability. We do not model gaps between different books as intervals of different sizes (although that, too, might lead to interesting questions).

\(^4\)The notation R2T stands for “random-to-top”, which relates to how these elements are defined.
foundation for these analogies is known, but they have been highly useful as
heuristics many times, and a vast number of objects have been translated from
one world to the other. Some basic examples can be found in [Cohn04]; another
is the theory of Carlitz polynomials [Conrad15].

The ring of symmetric functions is deeply connected with integer partitions
(e.g., almost all of its well-known bases are indexed by partitions); these corre-
respond to conjugacy classes of permutations in symmetric groups. This raises the
question of finding an “$\mathbb{F}_q$-analogue” of this ring which is similarly connected to
“$\mathbb{F}_q[T]$-partitions” (i.e., sequences $(p_1, p_2, p_3, \ldots)$ of monic polynomials in $\mathbb{F}_q[T]$ such that $\cdots | p_3 | p_2 | p_1$ and such that all but finitely many $p_i$ are $= 1$),
or, equivalently, to conjugacy classes of matrices in $\text{GL}_n(\mathbb{F}_q)$. Partial results to-
wards the construction of such an analogue can be found in my work-in-progress
[Grinbe15c]. My approach to finding such an analogue takes a detour through
the notion of Witt vectors, which are an affine group whose coordinate ring is the
symmetric functions ([Hazewi08, §10]). An $\mathbb{F}_q$-analogue of the Witt vectors is not
hard to construct, and its coordinate ring can then be regarded as an $\mathbb{F}_q$-analogue
of the symmetric functions. However, the combinatorics of this $\mathbb{F}_q$-analogue still
remains to be understood, as the theory of Witt vectors reflects but little of the
combinatorics of symmetric functions.

This is not directly related to Hall algebras (see [Dycker15] for a novel cate-
gorical viewpoint by Dyckerhoff and Kapranov, which I am highly interested in),
which can also be regarded as an $\mathbb{F}_q$-analogue of symmetric functions, but which
is a different sort of $\mathbb{F}_q$-analogue. (Hall algebras, at least in the simplest case,
still have bases indexed by integer partitions; only their structure constants rely
on $\mathbb{F}_q$.)

Other projects. Further work I am currently doing or planning includes:

• an approach to the quasisymmetric Bernstein homomorphism using univer-
sal properties [Grinbe15c], and the questions it suggests;

• (joint with Viviane Pons) an inversion statistic on intervals in the Tamari
poset, and some related maps;

• (joint with Eric Neyman) RSK and dual RSK algorithms for cylindric and
periodic tableaux (continuing Neyman’s work [Neyman14], which resulted
from a PRIMES project under my mentorship).

One other area of interest that I am going to explore in the near future is
formal proving in Coq/ssreflect. Coq is a proof assistant for constructive mathe-
matics that has proven its worth in logic and software formalization, and ssreflect
is a “sublanguage” that makes it particularly suitable to formalization of com-
plex mathematical proofs (such as those of the four-color conjecture and the
Feit-Thompson theorem, developed at the MSR-Inria joint centre by a team un-
der Georges Gonthier). Recent work (mostly) by Florent Hivert (the coq-combi
(project) has succeeded in verifying a noticeable chunk of algebraic combinatorics (Littlewood-Richardson rule, hook-length formula and other results) in ssreflect, and I am intent on joining and contributing to this project in the coming year.

References


A preprint of this paper is also available under the name *On Zamolodchikov’s Periodicity Conjecture* as arXiv:hep-th/0606094v1.